# Membranes and Domain Walls in $\mathcal{N}$=1, D=4 SYM

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Based on arXiv:1905.02743 with Igor Bandos and Stefano Lanza

Extension of earlier sugra results with Fotis Farakos and Luca Martucci
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# Main goal - to reveal how $\mathcal{N}$=1, D=4 SYM domain walls look like

- $\mathcal{N}$=1, D=4 SYM was constructed in 1974

  ( *Wess & Zumino, Ferrara & Zumino, Salam & Strathdee*)
- Studied intensively over 45 years
- In particular, since early 80s it is known that pure $SU(N)$ SYM has $N$ degenerate susy vacua destinguished by different vevs of the gluino condensate (related by $Z_N$ R-symmetry transformations)

$$< \mathrm{Tr}\, \lambda^\alpha \lambda_\alpha >= \Lambda^3 e^{2\pi i \frac{n}{N}}, \quad n = 0, 1, \dots N-1$$

- There should exist BPS domain walls interpolating between different vacua and having the following tension (*Dvali & Shifman '97*)

$$T_{\mathrm{DW}} = \frac{N}{8\pi^2} \, | < \lambda\lambda >_n - < \lambda\lambda >_l \, |$$

# Solutions describing the pure SYM domain walls have not been found until recently

- **Reason:** SYM domain walls are not smooth solitonic field configurations. Their existence requires the presence of a source which has been lacking in pure $\mathcal{N}$=1 SYM
  - dynamical membranes
- **Aim of this talk** to show
  - how to couple the membrane to $\mathcal{N}$=1 SYM and its Veneziano-Yankielowicz effective formulation
  - how the membrane creates BPS domain walls and what is their shape

# Review of pure $\mathcal{N}$=1 SU(N) SYM

- Field content - adjoint vector supermultiplet

$$A_m^I,\ \lambda_\alpha^I,\ \bar{\lambda}_{\dot{\alpha}}^I,\ D^I$$

- Building block of the SYM action – chiral spinor superfield

$$\mathcal{W}_\alpha(x,\theta) = -i\lambda_\alpha + \theta_\alpha D - \frac{i}{2}F_{mn}\sigma^{mn}{}_\alpha{}^\beta\theta_\beta + \theta^2\sigma^m_{\alpha\dot{\beta}}\nabla_m\bar{\lambda}^{\dot{\beta}}, \quad \bar{D}_{\dot{\alpha}}\mathcal{W}_\beta = 0.$$

$$\mathcal{L}_{\rm SYM} = \frac{1}{4g^2}\int d^2\theta\,{\rm Tr}\,\mathcal{W}^\alpha\mathcal{W}_\alpha\ +\ {\rm c.c.}$$

# Review of pure $\mathcal{N}$=1 SU(N) SYM

- Special chiral scalar superfield

$$S = \mathrm{Tr}\,\mathcal{W}^\alpha \mathcal{W}_\alpha = -\mathrm{Tr}\lambda^\alpha\lambda_\alpha + \sqrt{2}\theta^\alpha\chi_\alpha + \theta^2 F$$

$$\chi = \sqrt{2}\,\mathrm{Tr}\left(\frac{1}{2}F_{mn}\sigma^{mn}\lambda - i\lambda D\right)$$

$$F = \mathrm{Tr}\left(-2i\lambda\sigma^m\nabla_m\bar{\lambda} - \frac{1}{2}F_{mn}F^{mn} + D^2 - \frac{i}{4}\varepsilon_{mnpl}F^{mn}F^{pl}\right)$$

- Important notice

$$\begin{aligned} \mathrm{Im}\,F = {}^*F_4 \quad &= \quad -{}^*\mathrm{Tr}\,F_2\wedge F_2 - \partial_m \mathrm{Tr}\,\lambda\sigma^m\bar{\lambda} \\ &= \quad -{}^*d\,\mathrm{Tr}\left(AdA + \frac{2i}{3}A^3 + \frac{1}{3!}dx^k dx^n dx^m \epsilon_{mnkl}\mathrm{Tr}\,\lambda\sigma^l\bar{\lambda}\right) \equiv {}^*dC_3 \end{aligned}$$

$$S = -\frac{i}{4}\bar{D}^2\,U, \qquad U(x,\theta,\bar{\theta}) \;-\; \text{real salar superfield}$$ (*Gates '81*)

# Veneziano-Yankielowicz ’82 Lagrangian revisited

- Provides effective description of color-less bound states of the SYM multiplet (glueballs, gluinoballs and their fermionic superpartner), gluino condensate and the N-degeneracy of the SYM vacuum

$$S = s(x) + \sqrt{2}\theta^\alpha \chi_\alpha(x) + \theta^2(\hat{D}(x) + \mathtt{i}\ {}^*dC_3(x))$$

$$^*dC_3 = \partial_m C^m, \quad \text{where} \quad C_1 = {}^*C_3$$

- The form of the VY Lagrangian is (almost) fixed by anomalous superconformal Ward identities of the SU(N) SYM

$$\mathcal{L}_{VY} = \frac{1}{\rho}\int d^2\theta d^2\bar{\theta}(S\bar{S})^{\frac{1}{3}} + \int d^2\theta W(S) + c.c\,,$$

$$W(S) = \frac{N}{16\pi^2} S\left(\ln\frac{S}{\Lambda^3} - 1\right)$$

# Veneziano-Yankielowicz ’82 Lagrangian revisited

- The superpotential and the Lagrangian are not single valued

$$S \to S\, e^{2\pi\, i} \qquad \mathcal{L}_{VY} \to \mathcal{L}_{VY} + \frac{N}{4\pi}\, \partial_m C^m$$

- The special form of the chiral superfield $S = -\frac{i}{4}\bar{D}^2\, U$ requires the variation of the VY Lagrangian with respect to independent real superfield $U$.
- The variation principle is well-defined only with the addition of the boundary (total derivative) term (*Bandos, Lanza, D.S. ’19*)

$$\mathcal{L}_{\rm bd} = -\frac{1}{8}\left(\int d^2\theta \bar{D}^2 - \int d^2\bar{\theta} D^2\right)\left[\left(\frac{1}{12\rho}\bar{D}^2\frac{\bar{S}^{\frac{1}{3}}}{S^{\frac{2}{3}}} + \frac{1}{16\pi^2}\ln\frac{\Lambda^{3N}}{S^N}\right)U\right] + \text{c.c.}$$

# Veneziano-Yankielowicz '82 Lagrangian revisited

- Bosonic part of the Lagrangian

$$\mathcal{L}_{\mathrm{VY}}^{\mathrm{bos}} = K_{s\bar{s}}\left(-\partial_m s \partial^m \bar{s} + (\partial_m C^m)^2 + \hat{D}^2\right) + \left(W_s\left(\hat{D} + i\partial_m C^m\right) + \mathrm{c.c.}\right) + \mathcal{L}_{\mathrm{bd}}^{\mathrm{bos}}$$

- boundary term

$$\mathcal{L}_{\mathrm{bd}}^{\mathrm{bos}} = -2\partial_m\left[C^m\left(K_{s\bar{s}}\partial_n C^n - \mathrm{Im}\, W_s\right)\right], \quad K_{s\bar{s}} \equiv \partial_s \partial_{\bar{s}} K(s,\bar{s}), \quad W_s \equiv \partial_s W(s)$$

- auxiliary field equations of motion

$$K_{s\bar{s}}\hat{D} + \mathrm{Re}\, W_s = 0 \quad \rightarrow \quad \hat{D} = -\frac{\mathrm{Re}\, W_s}{K_{s\bar{s}}}$$

$$\partial_m(K_{s\bar{s}}\partial_n C^n - \mathrm{Im}\, W_s) = 0 \quad \rightarrow \quad \partial_m C^m = \frac{\mathrm{Im}\, W_s - \frac{n}{8\pi}}{K_{s\bar{s}}}$$ <- integration constant

$$F \equiv \hat{D} + \mathrm{i}\partial_m C^m = -\frac{\overline{W}_{\bar{s}} + \mathrm{i}\,\frac{n}{8\pi}}{K_{s\bar{s}}}$$

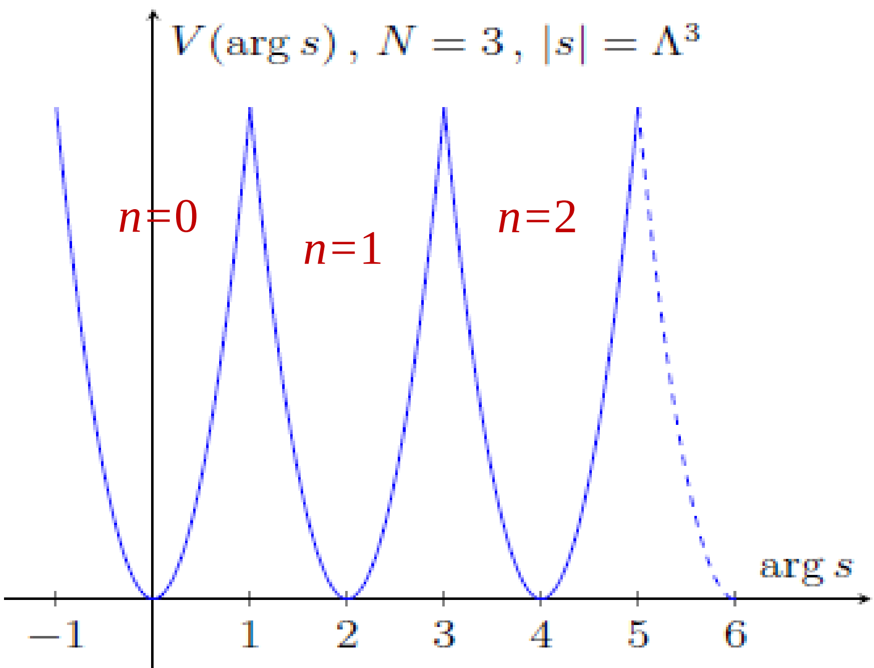
# Veneziano-Yankielowicz ’82 Lagrangian revisited

- Scalar field potential (*Kovner & Shifman ’97*)

$$V(s,\bar{s}) = \frac{9\rho N}{16\pi^2}\,|s|^{\frac{4}{3}}\left(\ln^2\frac{|s|}{\Lambda^3} + (\arg s - 2\pi\frac{n}{N})^2\right)\,, \quad n = 0,1,2,\ldots,N-1$$

Potential is single-valued, multi-branched, has cusps at $\arg s = \frac{\pi n}{N}$

and susy minima at $\langle s\rangle = \Lambda^3 e^{2\pi \mathrm{i}\frac{n}{N}}$ – $SU(N)$ SYM vacua



# Coupling membrane to SYM

- Supersymmetric and kappa-symmetric membrane action (*I. Bandos, S. Lanza, D.S. '19*)

$$\mathcal{S}_{M2+SYM} = -\frac{k}{4\pi}\int d^3\xi\sqrt{-\det h_{ij}}\,|S| + \frac{k}{4\pi}\int \mathcal{C}_3 \quad (k = 0, \pm 1, \pm 2, \ldots)$$

Nambu-Goto        Wess-Zumino

$$S(x,\theta) = \mathrm{Tr}\,\mathcal{W}^\alpha\mathcal{W}_\alpha = -\frac{i}{4}\bar{D}^2\,U\,, \qquad U(x,\theta,\bar\theta)\ -\ \text{real salar superfield}$$

$$E^a = dx^a + i\,\theta\sigma^a\bar\theta + c.c.$$

$$\begin{aligned}\mathcal{C}_3 =& \,\mathrm{i}\,E^a\wedge d\theta^\alpha\wedge d\bar\theta^{\dot\alpha}\sigma_{a\alpha\dot\alpha}U \\ &- \frac{1}{4}E^b\wedge E^a\wedge d\theta^\alpha\sigma_{ab\,\alpha}{}^{\beta}D_\beta U - \frac{1}{4}E^b\wedge E^a\wedge d\bar\theta^{\dot\alpha}\bar\sigma_{ab}{}^{\dot\beta}{}_{\dot\alpha}\bar{D}_{\dot\beta}U \\ &- \frac{1}{48}E^c\wedge E^b\wedge E^a\epsilon_{abcd}\,\bar\sigma^{d\dot\alpha\alpha}[D_\alpha,\bar{D}_{\dot\alpha}]U\,.\end{aligned}$$

$$\mathcal{C}_3|_{\theta=0} = C_3 = \mathrm{Tr}\left(AdA + \frac{2i}{3}A^3\right)$$

# Kappa-symmetry

- Counterpart of local worldvolume supersymmetry

$$\delta\theta^\alpha = \kappa^\alpha(\xi), \qquad \delta x^m = i\kappa\sigma^m\bar{\theta} + c.c.$$

$$\kappa_\alpha = \Gamma_{\alpha\dot{\alpha}}\bar{\kappa}^{\dot{\alpha}}, \qquad \Gamma^2 = 1$$

Gauges away 2 of 4 fermionic modes $\theta^\alpha(\xi), \bar{\theta}^{\dot{\alpha}}(\xi)$ of the membrane

Worldvolume reparametrization gauges away 3 of 4 bosonic modes $x^m(\xi$

$$\left(x^3(\xi), \psi^\alpha(\xi)\right) \quad \text{Goldstone supermultiplet}$$

associated with ½ broken supersymmetry in the 4D bulk, while another ½ of susy remains unbroken allowing for BPS configurations

# BPS domain wall solutions sourced by the membrane

- Consider a static membrane in the Veneziano-Yankielowicz model

$$\theta^\alpha(\xi) = \bar{\theta}^{\dot{\alpha}}(\xi) = 0, \quad \xi^i = x^i \ (i = 0, 1, 2), \quad x^3 = 0$$

- The presence of the membrane modifies the bulk field equations by source terms, in particular the gauge 3-form eq.

$$\partial_m (K_{s\bar{s}} \partial_n C^n - \operatorname{Im} W_s) = -\frac{k}{8\pi} \delta^3_m \delta(x^3)$$

$$\operatorname{Im} F = \partial_m C^m = \frac{8\pi \operatorname{Im} W_s - (n + k\Theta(x^3))}{8\pi K_{s\bar{s}}}$$

$$\langle s \rangle = \Lambda^3 e^{2\pi i \frac{n}{N}} \qquad \Big| \qquad \langle s \rangle = \Lambda^3 e^{2\pi i \frac{n+k}{N}}$$

# BPS domain wall solutions sourced by the membrane

- BPS domain-wall equation dictated by ½ susy conservation $\delta\chi_\alpha = 0$

$$\frac{\partial s(x^3)}{\partial x^3} \equiv \dot{s} = \mathrm{i}e^{\mathrm{i}\alpha}F = -\mathrm{i}e^{\mathrm{i}\alpha}\frac{\overline{W}_{\bar{s}} + \frac{\mathrm{i}}{8\pi}(n + k\Theta(x^3))\, s}{K_{s\bar{s}}}, \qquad e^{\mathrm{i}\alpha} = \frac{|s(0)|}{s(0)} \quad \text{(on M2)}$$

Substituting the form of $W$ and $K$ in the Veneziano-Yankielowiz (VY) model, we get

$$\dot{s} = 9\mathrm{i}\,\rho N|s|^{\frac{4}{3}}e^{\mathrm{i}\,\alpha}\left(\ln\frac{\Lambda^3}{|s|} + \mathrm{i}\,\arg s - \frac{2\pi\mathrm{i}}{N}(n + k\Theta(x^3))\right)$$

BPS value of the on-shell action for the VY model + membrane

$$S_{\mathrm{BPS}} = S_{\mathrm{VY}} + S_{\mathrm{membr}} = -\,2\int d^3\xi\,|W_{+\infty} - W_{-\infty}|$$

$$T_{\mathrm{DW}} = T_s + T_{\mathrm{membr}} = 2\,|W_{+\infty} - W_{-\infty}| = \frac{N}{8\pi^2}\,\Lambda^3\left|e^{2\pi\,i\,\frac{n+k}{N}} - e^{2\pi\,i\,\frac{n}{N}}\right|$$

# Shape of BPS domain walls

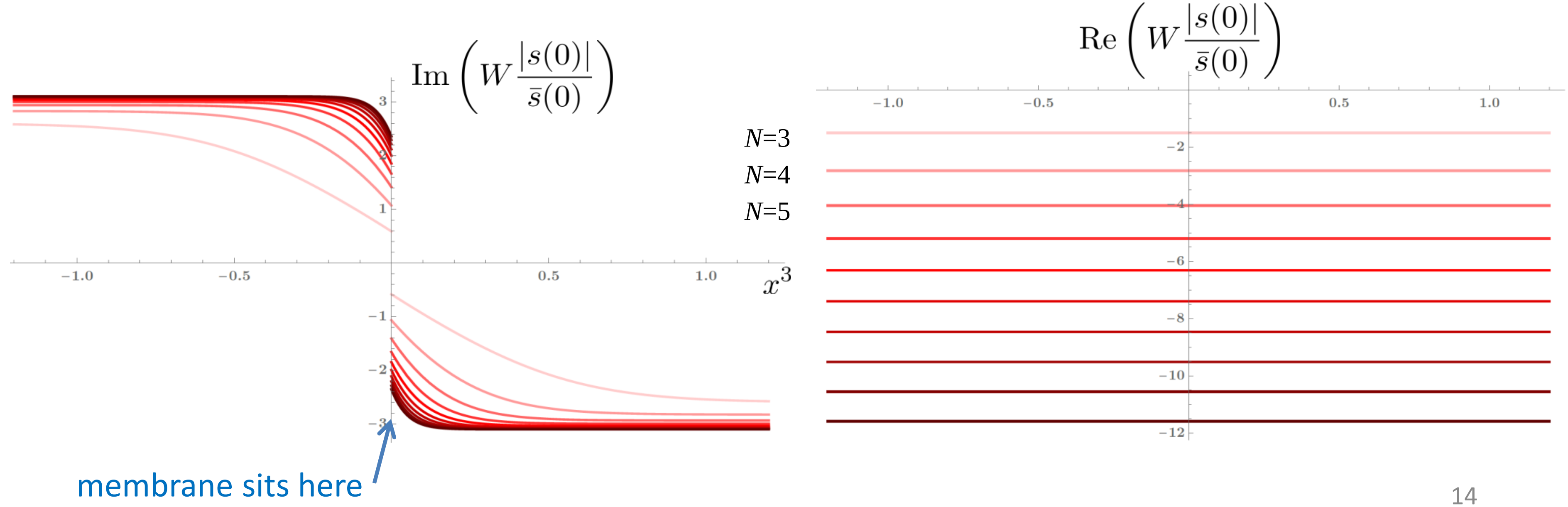
- $s(x)$-continuous domain wall solutions of the BPS equations exist for the membrane charge having the following values

$$|k| \leq \frac{N}{3}, \qquad |k| = N$$

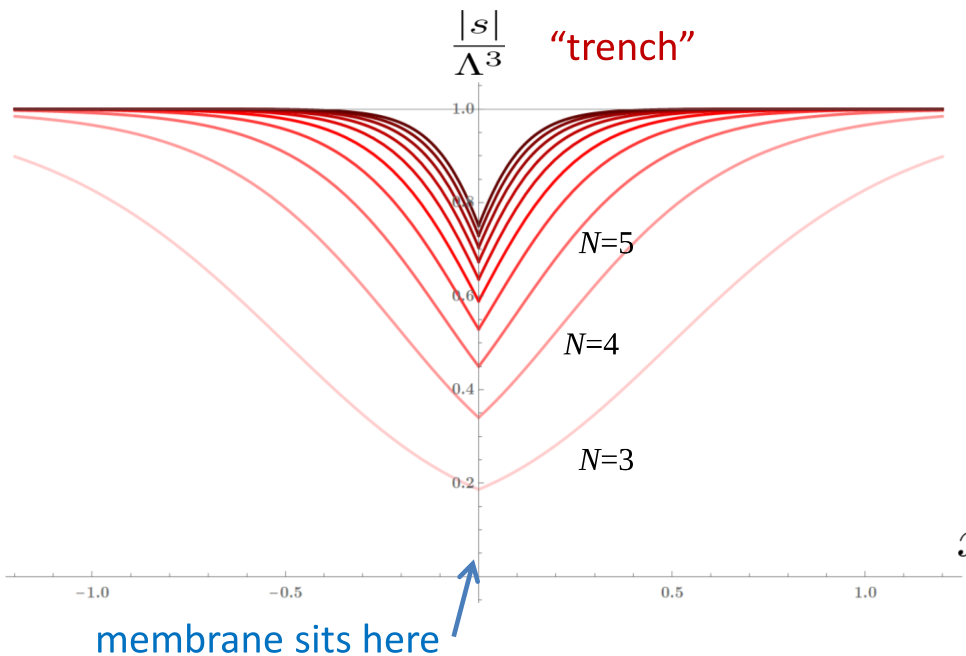
½ BPS domain wall between the same vacuum

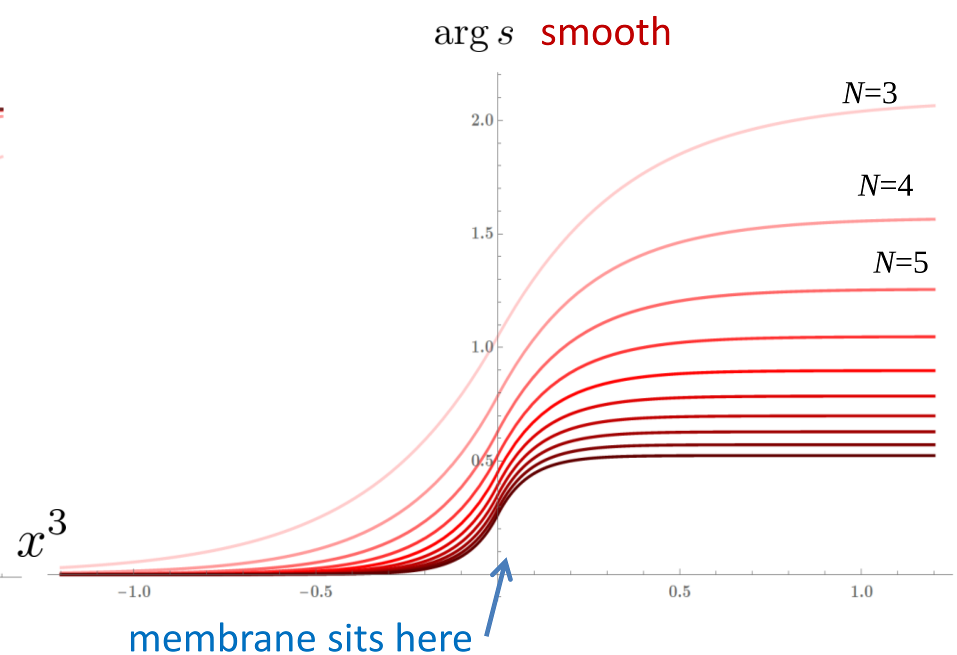
Examples $k = 1, \quad N = 3, 4, 5, 6, 7, \ldots$

Form of the superpotential

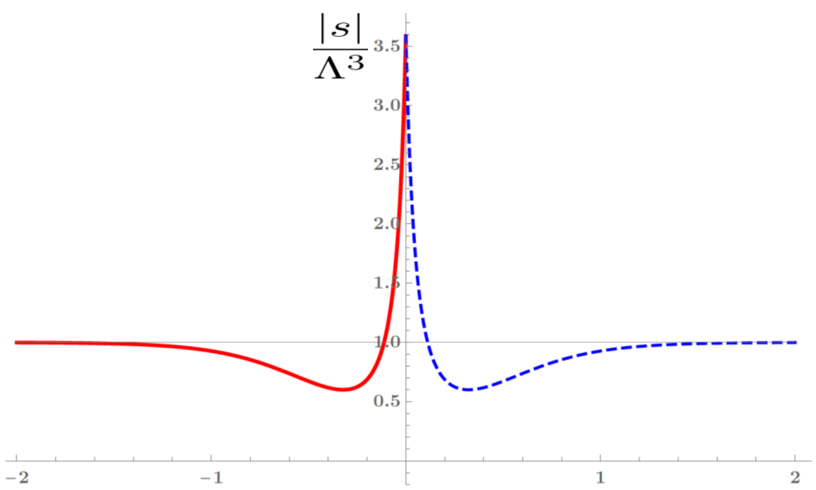


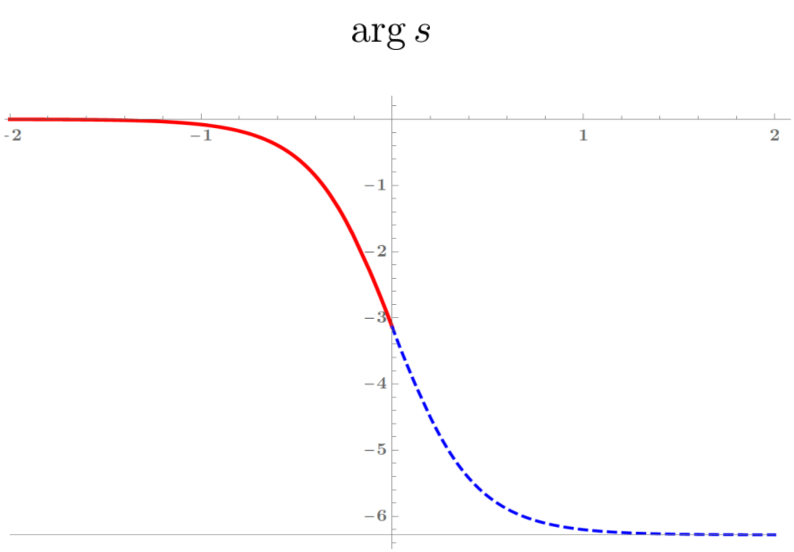
# Shape $s(x)$ of BPS domain walls with $|k| \le \frac{N}{3}$




## $|k|=N$ domain wall



# Conclusions

- We have constructed the supersymmetric and kappa-invariant action describing the coupling of a membrane to $\mathcal{N}$=1, D=4 SYM and its Veneziano-Yankielowicz effective sigma-model

- The membrane of charge $k$ separates two SYM vacua with different phases of the gluino condensate

$$\langle s \rangle = \Lambda^3 e^{2\pi \mathrm{i} \frac{n}{N}} \qquad \Bigg| \qquad \langle s \rangle = \Lambda^3 e^{2\pi \mathrm{i} \frac{n+k}{N}}$$

- and creates BPS domain walls interpolating between these vacua

$$T_{\mathrm{DW}} = \frac{N\Lambda^3}{8\pi^2} \left| e^{2\pi\, i\, \frac{n+k}{N}} - e^{2\pi\, i\, \frac{n}{N}} \right|$$

- Explicit domain wall configurations have been found for $|k| \leq \frac{N}{3}$ and $|k| = N$

- |k|=N ½ BPS domain wall interpolates between the same SYM vacuum and has zero total tension

# Outlook

- To study the relation of this membrane construction with an effective 3d Chern-Simons theory associated with pure $SU(N)$ SYM domain walls by *Acharya and Vafa '01*

- To extend this construction to $\mathcal{N}$=1, D=4 SQCD with flavour multiplets in the fundamental representation and to study domain wall configurations with "jumping" superpotentials in the effective Taylor-Veneziano-Yankielowicz theory.

- To understand how the dynamical membranes fit in $\mathcal{N}$=1, D=4 SQCD domain wall constructions reviewed and generalized recently by *Bashmakov, Benini, Benvenuti & Bertolini '18*