THE SNYDER MODEL AND ITS GENERALIZATIONS

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SUMMARY

- Snyder model
- Geometry of the model
- Generalizations
- Noncommutative geometry and Hopf algebra
- Snyder Quantum Field Theory
- UV/IR mixing
- Conclusions

• Since the origin of quantum field theory (QFT) there have been proposal to add a new scale of length to the theory in order to solve the problems connected to UV divergences.

• Later, also attempts to build a theory of quantum gravity have proved the necessity of introducing a length scale, that has been identified with the Planck length $L_p = \sqrt{\frac{\hbar G}{c^3}} \sim 1.6 \cdot 10^{-35}$ m.

• A naive application of this idea, like lattice field theory, would however break Lorentz invariance.

• A way to reconcile discreteness of spacetime with Lorentz invariance was proposed by Snyder (Snyder 1947) a long time ago.

• This was the first proposal of a noncommutative geometry: the length scale should enter the theory through the commutators of spacetime coordinates.

- Several models of noncommutative geometries also admit a sort of dual representation on momentum space in theories of doubly special relativity (DSR) (Amelino-Camelia 2001). Here a fundamental mass length is introduced, that causes the curvature of momentum space, and the deformation of both the Poincaré group and the dispersion relations of the particles.
- The Snyder model can be seen as a DSR model, where the Poincaré invariance and the dispersion relations are undeformed.
- Snyder's idea was however almost abandoned with the introduction of renormalization techniques, with the exception of some Russian authors in the sixties (Gol'fand 1960, Kadyshevsky 1962, Mir-Kasimov 1966).
- It revived more recently, when noncommutative geometry became an important topic (Majid-Ruegg 1994).
- The issue of the finiteness of Snyder field theory has not been established up to now.

THE SNYDER MODEL

• The Snyder model is defined on the full relativistic phase space and is based on the Snyder algebra, generated by position x_μ, momentum p_μ and Lorentz generators $J_{\mu\nu}$, that obey:
a) Poincaré algebra commutation relations

$$[J_{\mu\nu}, J_{\rho\sigma}] = i\big(\eta_{\mu\rho}J_{\nu\sigma} - \eta_{\mu\sigma}J_{\nu\rho} + \eta_{\nu\rho}J_{\mu\sigma} - \eta_{\nu\sigma}J_{\mu\rho}\big),$$
$$[p_\mu, p_\nu] = 0, \qquad [J_{\mu\nu}, p_\lambda] = i\left(\eta_{\mu\lambda}p_\nu - \eta_{\lambda\nu}p_\mu\right),$$

b) standard Lorentz action on positions

$$[J_{\mu\nu}, x_\lambda] = i\left(\eta_{\mu\lambda}x_\nu - \eta_{\nu\lambda}x_\mu\right),$$

c) deformation of the Heisenberg algebra (preserving Jacobi identities),

$$[x_\mu, x_\nu] = i\beta J_{\mu\nu}, \qquad [x_\mu, p_\mu] = i(\eta_{\mu\nu} + \beta p_\mu p_\nu).$$

β is a parameter of order L_{Pl}^2 and $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$.

• The generators $J_{\mu\nu}$ can be realized in the standard way on phase space, $J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$.

• In contrast with the most common models of noncommutative geometry and DSR, the commutators are functions of the phase space variables: this allows them to be compatible with a linear action of the Lorentz symmetry, so that the Poincaré algebra is not deformed. However, translations (generated by the p_μ) act in a nontrivial way on position variables.

• Depending on the sign of the coupling constant β, two rather different models are available:

$\beta > 0$	Snyder model
$\beta < 0$	anti-Snyder model

They have very different properties. For example, only in the first case spatial coordinates have a discrete spectrum.

Geometry of the Snyder model

• The subalgebra generated by $J_{\mu\nu}$ and x_μ is isomorphic to the de Sitter/anti-de Sitter algebra, and the Snyder/anti-Snyder momentum spaces have the same geometry as de Sitter/anti-de Sitter spacetime respectively (curved momentum space).
• They are the coset space $SO(1,4)/SO(1,3)$ for Snyder (or $SO(2,3)/SO(1,3)$ for anti-Snyder)

• It follows that the momentum space of the Snyder model can be represented as a hyperboloid $\mathcal{H}$ of equation ($\beta > 0$)

$$\zeta_A^2 = \frac{1}{\beta}$$

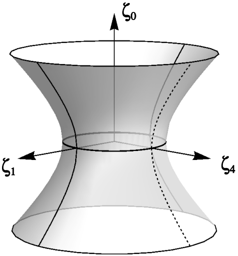
embedded in a 5D space of coordinates ζ_A with signature $(-,+,+,+,+)$.

• Snyder commutation relations are recovered through the choice of isotropic (Beltrami) coordinates on $\mathcal{H}$

$$p_\mu = \frac{\zeta_\mu}{\beta\zeta_4} = \frac{\zeta_\mu}{\sqrt{1-\beta\zeta_\mu^2}}$$

and the identification (M_{AB} are the 5D Lorentz generators)

$$x_\mu = M_{\mu 4}, \qquad J_{\mu\nu} = M_{\mu\nu}.$$



• Note that this implies $m^2 < 1/\beta$! There is a maximal mass!
• This is a common feature in models with curved momentum space (DSR).

• NOTE: One may obtain different noncommutative models with identical position commutation relations but different $[x_\mu, p_\nu]$ by choosing different isotropic parametrizations of the momentum space and maintaining the identification $x_\mu = M_{\mu 4}$.
• For example, choosing $p_\mu = \zeta_\mu$, one obtains

$$[x_\mu, x_\nu] = i\beta J_{\mu\nu}, \qquad [x_\mu, p_\nu] = i\sqrt{1+\beta p^2}\,\eta_{\mu\nu}$$

• The most general choice that preserves the Poincaré invariance is

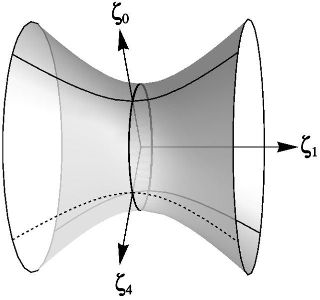
$$p_\mu = f(\zeta^2)\zeta_\mu, \qquad x_\mu = g(\zeta^2)M_{4\mu}$$

Geometry of the anti-Snyder model

• The momentum space of the anti-Snyder model can be represented analogously to that of Snyder as a hyperboloid of equation ($\beta < 0$)

$$\zeta_A^2 = \frac{1}{\beta}$$

embedded in a 5D space of coordinates ζ_A with signature $(-, +, +, +, -)$. No maximal mass appears in this case.



GENERALIZATIONS

a) Generalized Snyder models (Meljanac, SM, Štrajn 2016)

• Generalizations of the Snyder model that preserve the Poincaré invariance and the standard action of Lorentz transformation on the coordinates, can be obtained by modifying the Heisenberg algebra, so that

$$[x_\mu, x_\nu] = i\beta J_{\mu\nu}\, \psi(\beta p^2),$$

$$[x_\mu, p_\nu] = i\big[\eta_{\mu\nu}\phi_1(\beta p^2) + \beta p_\mu p_\nu \phi_2(\beta p^2)\big].$$

The function ϕ_1 and ϕ_2 are arbitrary, but the Jacobi identity implies

$$\psi = \phi_1\phi_2 - 2(\phi_1 + \beta p^2 \phi_2)\, \frac{d\phi_1}{d(\beta p^2)}.$$

• These models correspond to arbitrary isotropic parametrizations of the hyperboloid.

b) Snyder-de Sitter (SdS) model or Triply Special Relativity

• The Snyder model can also be generalized to a curved spacetime (de Sitter) background (Kowalski-Glikman, Smolin 2004), assuming

$$[p_\mu, p_\nu] = i\alpha J_{\mu\nu}$$

with $\alpha \sim \Lambda$ (cosmological constant).

• The other commutation relations are unchanged, except that now

$$[x_\mu, p_\nu] = i(\eta_{\mu\nu} + \alpha x_\mu x_\nu + \beta p_\mu p_\nu + \sqrt{\alpha\beta}(x_\mu p_\nu + p_\mu x_\nu))$$

• This model depends on two invariant scales, that can be identified with the Planck scale and the cosmological constant.

• There are indications from quantum gravity theories that the introduction of α might be necessary.

• The SdS model is dual for the exchange $\alpha x \leftrightarrow \beta p$. (Guo, Huang, Wu 2008)

• The phase space of SdS can be embedded in a 6D space as $\frac{SO(1,5)}{SO(1,3)\times O(2)}$ if $\alpha, \beta > 0$, or as $\frac{SO(2,4)}{SO(1,3)\times O(2)}$ if $\alpha, \beta < 0$. (SM 2006)

• Alternatively, the SdS algebra can be obtained directly from that of Snyder by the nonunitary transformation

$$x_\mu = \hat{x}_\mu + \lambda\frac{\beta}{\alpha}\hat{p}_\mu, \qquad p_\mu = (1-\lambda)\hat{p}_\mu - \frac{\alpha}{\beta}\hat{x}_\mu,$$

where $\hat{x}_\mu$, $\hat{p}_\mu$ are generators of the Snyder algebra and λ a free parameter.

HOPF ALGEBRA

• In the study of noncommutative models an important tool is given by the Hopf algebra.

• Since in noncommutative geometry spacetime coordinates are noncommuting operators, the composition of plane waves $e^{ip\cdot x}$, $e^{iq\cdot x}$ gives rise to nontrivial addition rules for the momenta, denoted by $p \oplus q$, that are described by the coproduct structure of the Hopf algebra, $\Delta(p, q)$.

• Analogously, the opposite of the momentum is determined by the antipode, $S(p)$, such that $p \oplus S(p) = S(p) \oplus p = 0$.

• The Hopf algebra associated to the Snyder model can be calculated (classically) using the previous geometric representation of the momentum space as a coset space and calculating the action of the group multiplication on it. (Girelli, Livine 2011)

Algebraic realizations

- Alternatively, one can use the algebraic formalism of realizations (Battisti, Meljanac 2010).
- This will be useful in the definition of QFT.
- We define a realization of the noncommutative coordinates x_μ in terms of coordinates ξ_μ, p_μ that satisfy canonical commutation relations

$$[\xi_\mu, \xi_\nu] = [\xi_\mu, \xi_\nu] = 0, \qquad [\xi_\mu, p_\nu] = \eta_{\mu\nu}$$

by assigning a function $x_\mu(\xi_\mu, p_\mu)$ that satisfies the Snyder commutation relations.

• The x_μ and p_μ are now interpreted as operators acting on function of ξ_μ, as

$$\xi_\mu \triangleright f(\xi) = \xi_\mu f(\xi), \qquad p_\mu \triangleright f(\xi) = -i\partial f(\xi)/\partial\xi_\mu.$$

• The realization of the Snyder model is given by

$$x_\mu = \xi_\mu + \beta\, \xi\cdot p\, p_\mu + \beta p_\mu \chi(\beta p^2).$$

• The function χ is arbitrary and does not contribute to the commutators, but takes into account ambiguities arising from operator ordering of ξ_μ and p_μ.

- In general, the action of noncommutative plane waves is

$$e^{ik\cdot x} \triangleright e^{iq\cdot\xi} = e^{i\mathcal{P}(k,q)\cdot\xi + i\mathcal{Q}(k,q)},$$

$$e^{ik\cdot x} \triangleright 1 = e^{i\mathcal{K}(k)\cdot\xi + i\mathcal{J}(k)},$$

with $\mathcal{K}_\mu(k) \equiv \mathcal{P}_\mu(k,0)$ and $\mathcal{J}(k) \equiv \mathcal{Q}(k,0)$.

- It can be shown that $\mathcal{P}_\mu$ and $\mathcal{Q}$ can be determined from the knowledge of the deformed Heisenberg algebra.
- The generalized addition of momenta is then given by

$$k_\mu \oplus q_\mu = \mathcal{D}_\mu(k,q), \qquad \text{where} \qquad \mathcal{D}_\mu(k,q) = \mathcal{P}_\mu(\mathcal{K}^{-1}(k), q),$$

and the coproduct is given simply by

$$\Delta p_\mu = \mathcal{D}_\mu(p\otimes 1, 1\otimes p).$$

- The antipode $S(p_\mu)$, is $-p_\mu$ for all (generalized) Snyder models.

IMPORTANT: The Snyder addition law turns out to be nonassociative (and noncommutative). Hence the algebra is noncoassociative, so strictly not a Hopf algebra.

• For the calculations, it is useful to define also a star product, that gives a representation of the product of functions of the noncommutative coordinates x in terms of a deformation of a product of functions of commuting coordinates ξ.

• In particular, from the previous results one can calculate the star product of two plane waves:

$$e^{ik\cdot\xi} \star e^{iq\cdot\xi} = e^{i\mathcal{D}(k,q)\cdot\xi + i\mathcal{G}(k,q)},$$

where

$$\mathcal{G}(k,q) = \mathcal{Q}(\mathcal{K}^{-1}(k), q) - \mathcal{Q}(\mathcal{K}^{-1}(k), 0).$$

Star product for the Snyder model

• We consider now a Hermitean realization of Snyder commutation relations

$$x^\mu = \xi^\mu + \frac{\beta}{2}\left(\xi\cdot p\, p^\mu + p^\mu p\cdot\xi\right) = \xi^\mu + \beta\, \xi\cdot p\, p^\mu - \frac{5i}{2}\beta\, p^\mu,$$

• The Hermiticity will be important for field theory.

• We get

$$\mathcal{D}_\mu(k,q) = \frac{1}{1-\beta k\cdot q}\left[\left(1+\frac{\beta\, k\cdot q}{1+\sqrt{1+\beta p^2}}\right)k_\mu + \sqrt{1+\beta p^2}\, q_\mu\right],$$

$$\mathcal{G}(k,q) = \frac{5i}{2}\ln\left[1-\beta\, k\cdot q\right].$$

and hence

$$e^{ik\cdot\xi} \star e^{iq\cdot\xi} = \frac{e^{i\mathcal{D}(k,q)\cdot\xi}}{(1-\beta\, k\cdot q)^{5/2}}.$$

FIELD THEORY IN NONCOMMUTATIVE SPACES

• Let us consider a QFT for a scalar field ϕ on a Snyder space.
• Usually, field theory in noncommutative spaces are constructed by continuing to Euclidean signature and writing the action in terms of the star product.
• In most cases, a phenomenon called UV/IR mixing occurs: the counterterms needed for the UV regularization diverge for vanishing incoming momenta, inducing an IR divergence.
• A model that avoids this problem in Moyal theory was proposed by Grosse et al. (Grosse, Wulkenhaar 2014).
• Besides the kinetic term $\phi\,\partial^2\phi$ and the interaction term, its action also contains a term proportional to $\phi\, x^2\phi$.

Free field theory in Snyder space

The action functional for a free massive real scalar field $\phi(x)$ can be defined through the star product (Meljanac, SM, Štrajn 2016)

$$S_{\rm free}[\phi] = \frac{1}{2}\int d^4\xi \left(\partial_\mu\phi \star \partial^\mu\phi + m^2\phi\star\phi\right)$$

The star product of two real scalar fields $\phi(\xi)$ and $\psi(\xi)$ can be computed by expanding them in Fourier series,

$$\phi(\xi) = \int d^4k\, \tilde\phi(k) e^{ik\cdot\xi}.$$

Then

$$\int d^4\xi\ \psi(\xi)\star\phi(\xi) = \int d^4\xi \int d^4k\, d^4q\ \tilde\psi(k)\,\tilde\phi(q)\ e^{ik\cdot\xi}\star e^{iq\cdot\xi}$$
$$= \int d^4k\, d^4q\ \tilde\psi(k)\,\tilde\phi(q)\ \frac{\delta^{(4)}\left(\mathcal{D}(k,q)\right)}{(1-\beta\, k\cdot q)^{5/2}}.$$

But

$$\delta^{(4)}(\mathcal{D}(k,q)) = \frac{\delta^{(4)}(q+k)}{\left|\det\left(\frac{\partial \mathcal{D}_\mu(k,q)}{\partial q_\nu}\right)\right|_{q=-k}} = (1+\beta k^2)^{5/2}\delta^{(4)}(q+k).$$

- The two $(1+\beta k^2)^{5/2}$ factors cancel and then,

$$\int d^4x\ \psi(\xi)\star\phi(\xi) = \int d^4x\ \psi(\xi)\,\phi(\xi),$$

namely the star product obeys the cyclicity property, as in other noncommutative models, and hence the free theory is identical to the commutative one, (this property holds only for the hermitian representation)

$$S_{\mathrm{free}}[\phi] = \frac{1}{2}\int d^4\xi\left(\partial_\mu\phi\,\partial_\mu\phi + m^2\phi^2\right).$$

- The propagator is therefore the standard one

$$G(k) = \frac{1}{k^2+m^2}.$$

Interacting Snyder field theory (Meljanac, SM, Trampetić, You 2018)

• The interacting theory is much more difficult to investigate. There are several problems:

- The addition law of momenta is noncommutative and nonassociative, therefore one must define some ordering for the lines entering a vertex and then take an average.
- The conservation law of momentum is deformed at vertices, so loop effects may lead to nonconservation of momentum in a propagator.

• For example, let us consider the simplest case, a ϕ^4 theory with interaction

$$S_{\rm int} = \lambda \int d^4x \; \phi \star (\phi \star (\phi \star \phi))$$

• The parentheses are necessary because the star product is nonassociative. Our definition fixes this ambiguity, but other choices are possible.

With this choice, the 4-point vertex function turns out to be

$$G^{(0)}(p_1, p_2, p_3, p_4)$$
$$= (2\pi)^4 \sum_{\sigma \in S_4} \delta\Big(\mathcal{D}_4(\sigma(p_1, p_2, p_3, p_4))\Big)\ g_3(\sigma(p_1, p_2, p_3, p_4)),$$

where

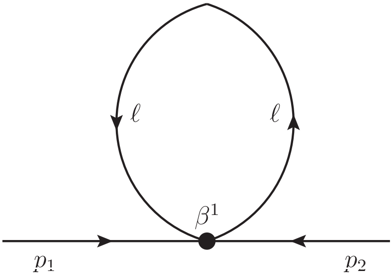
$$\mathcal{D}_4(q_1, q_2, q_3, q_4) = q_1 + \mathcal{D}(q_2, \mathcal{D}(q_3, q_4))$$
$$g_3(q_1, q_2, q_3, q_4) = e^{i\mathcal{G}(q_2, \mathcal{D}(q_3, q_4))} e^{i\mathcal{G}(q_3, q_4)}$$

and σ denotes all possible permutations of the momenta entering the vertex.

• With the expressions of the propagator and the vertex one can finally compute Feynman diagrams.

For example, the one-loop two-point function is given by

$$
\begin{aligned}
&G^{(1)}(x_1, x_2) \\
&\quad = -\frac{1}{2}\frac{\lambda}{4!}\int \frac{d^4 p_1}{(2\pi)^4}\frac{d^4 p_2}{(2\pi)^4}\frac{d^4 \ell}{(2\pi)^4}\frac{e^{ip_1 x_1}}{p_1^2+m^2}\frac{e^{ip_2 x_2}}{p_2^2+m^2}\frac{(2\pi)^4}{\ell^2+m^2} \\
&\qquad \sum_{\sigma\in P_4}\delta\Big(D_4(\sigma(p_1, p_2, \ell, -\ell))\Big)\; g_3\Big(\sigma(p_1, p_2, \ell, -\ell)\Big).
\end{aligned}
$$



- To evaluate the diagram, one must consider the 24 permutations of the momenta entering the vertex.
- Among these, only 8 conserve the momentum (i.e. $p_1 = -p_2$), while the remaining 16 do not.
- At the linear level in β the calculation can be done explicitly, showing stronger divergences than in the commutative theory. However the effect of momentum nonconservation are cancelled.
- At nonlinear level, not all diagrams can be explicitly calculated. However, there are indications that they might be finite, at least for our choice of the interaction term.
- The phenomenon of UV/IR mixing could however still be present.

• This problem might be avoided defining the theory in a curved background (SdS model) by a mechanism similar to that of the GW model (Franchino-Viñas, SM 2019).

• Using the previous relation between SdS and Snyder algebra (with $\lambda = 0$) and the realization of the Snyder algebra, the de Sitter-invariant free action can be reduced to (up to first order in α, β)

$$S_{free} = \int dx^4 \phi \Big[p^2 + \frac{\alpha}{\beta} x^2 + (m^2 - 4\alpha) + \alpha \left(\frac{3}{2} x^2 p^2 + x \cdot p \; p \cdot x \right) + \frac{\alpha^2}{\beta} x^4 \Big] \phi,$$

• The order zero part is identical to the GW model.

• One may therefore hope that also in this case the IR divergences are suppressed and one can obtain a finite theory.

CONCLUSIONS

- The Snyder model was proposed with the aim of avoiding UV divergences in field theory.
- It has several nice properties, like undeformed Lorentz invariance.
- Unfortunately, QFT on a Snyder background can be studied only partially, due to computational problems.
- Although there are hints that the theory could be UV finite, one cannot exclude the presence of the phenomenon UV/IR mixing.
- However, this should disappear at least in the SdS case. This topic is presently under study.