Warped symmetries of the Kerr black hole

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General context

It is known since the 70s and a series of groundbreaking works by Hawking, Bekenstein, Bardeen and Carter that black holes behave like thermal systems:

- they have a temperature T_H (Hawking '75)
- they have an entropy $S_{BH} = \frac{A}{4G}$ (Bekenstein '72 '73), $dS \geq 0$ (Hawking '71 '72)
- they obey the laws of BH mechanics (Bardeen, Carter, Hawking '73)

// laws of thermodynamics

A longstanding question since then has been to provide a microscopic explanation of the macroscopic "thermodynamic" Bekenstein-Hawking entropy

$$S_{BH} = \log \Omega$$

?

Progress on this front has been tightly connected to

$\boxed{\text{two-dimensional conformal field theories } (CFT_2)}$

which is one manifestation of a broader framework, the holographic principle.

A canonical example of the relationship between BH thermodynamics and 2d CFT is provided by

$$\boxed{AdS_3 \text{ gravity} = \begin{array}{l}\text{Einstein-Hilbert action with } \Lambda = -\frac{1}{\ell^2} < 0 \\ \text{and Brown-Henneaux boundary conditions}\end{array}}$$

In this model, the asymptotic symmetries (= set of non-trivial "large diffeomorphisms" acting on the phase space) organize into the 2d conformal algebra:

$$\boxed{[L_n^\pm, L_m^\pm] = (n-m)L^\pm_{n+m} + \frac{c^\pm}{12}(n^3-n)\delta_{n+m}} \qquad c^\pm = \frac{3\ell}{2G}$$

Virasoro $\oplus$ Virasoro

This suggested, upon quantization, a description of AdS_3 gravity in terms of a 2d CFT.

2d CFTs have a Cardy formula, allowing to count the asymptotic density of states at large charges / high temperatures:

$$\boxed{\begin{aligned} S_{\text{Cardy}} &= 2\pi\sqrt{\frac{c^+ L_0^+}{6}} + 2\pi\sqrt{\frac{c^- L_0^-}{6}} & \text{Microcanonical} \\ &= \frac{\pi^2}{3}\left(c^+ T^+ + c^- T^-\right) & \text{Canonical} \end{aligned}}$$

Remarkably, the Bekenstein-Hawking entropy of black holes in AdS_3 gravity (BTZ black holes) is found to match the Cardy entropy (Strominger '97)

$$\boxed{S_{BTZ} = S_{CFT}}$$

These results from AdS_3 gravity can also be exploited to interpret the microscopic entropy of higher-dimensional black holes.

For instance, the extremal limit of the 3-charges BH of 5d supergravity has a near-horizon decoupled geometry including an asymptotically AdS_3 factor.

The Brown-Henneaux analysis can be transposed to this setting, the large Virasoro diffeos now acting on the outer boundary of this decoupled region.

A reasoning similar to the one applied to BTZ using the Cardy formula and a suitable identification of the parameters allows to reproduce the Bekenstein-Hawking entropy:

$$\boxed{S_{BH}^{3ch,\,extremal} = S_{CFT}}$$

3- charges BH of N=4 or 8 5d sugra (metric, $C_{(2)}$, $A^{(1)}$, $A^{(2)}$, 2 scalars)

6 charges : $M, J_L, J_R, Q_1, Q_2, Q_0$

Extremal BPS limit : $\begin{cases} J_R \to 0 \\ J_L = J \text{ finite} \\ Q_i \text{ finite} \end{cases}$ $\qquad M = Q_1 + Q_2 + Q_0$

$$\boxed{S_{\text{Bek.-Haw}} = \frac{\pi^2}{G_5}\sqrt{Q_1 Q_2 Q_0 - \frac{1}{4}J^2}}$$

Uplift solution to $\underbrace{6d}_{\text{radius } R_y}$, take NH limit $\to$ get BTZ $\times S^3$ with parameters

$$l_{AdS_3} = (Q_1 Q_2)^{1/4}, \quad L_0^\pm = \frac{l_{AdS_3}}{8G_3}M_3 \pm J_3 = \begin{cases} \left(Q_0 - \frac{J^2}{4Q_1Q_2}\right)\frac{\pi R_y}{4G_5} & L_0^+ \\ 0 & L_0^- \end{cases}$$

$$G_5\, 2\pi R_y = G_3\, 2\pi^2 l_{AdS}^3$$

Then :

$$\boxed{S_{\text{Bek.-Haw}} = 2\pi\sqrt{\frac{c^+L_0^+}{6}} + 2\pi\sqrt{\frac{c^-L_0^-}{6}}} \qquad \text{w/ } c^\pm = \frac{3\, l_{AdS_3}}{2G_3}$$

This all applies to rather restricted classes of BHs (lower/higher-d, charged, susy)

Real black holes do not have an AdS_3 factor in their near-horizon geometry.

In the last 10 years however, hints have appeared that the relevance of 2d conformal symmetry could be more general and extend to Kerr black holes in (3+1) dimensions, with entropy

$$\boxed{S_{Kerr} = 2\pi\left(M^2 + \sqrt{M^4 - J^2}\right)} = 2\pi J$$

extreme Kerr, $J = M^2$

I will review 3 evidences for this.

1) The Kerr/CFT correspondance (Guica, Hartman, Song, Strominger - 0809.4266)

Extreme Kerr black holes ($J = M^2$) exhibit a near-horizon geometry ("NHEK") sharing certain similarities with $AdS_3 \times S^3$.

It was observed that the asymptotic symmetries of NHEK, in the spirit of Brown-Henneaux, consist in a chiral Virasoro algebra with

$$\boxed{c = 12J}$$

On the other hand, extreme black holes are endowed with a non-zero left-moving Frolov-Thorne temperature (even though $T_H = 0$)

$$\boxed{T_L = \frac{1}{2\pi}}$$

Applying the Cardy formula

$$\boxed{S = \frac{\pi^2}{3} c T_L = 2\pi J}$$

reproduces extreme Kerr entropy

2) Hidden conformal symmetry of Kerr (Castro, Maloney, Strominger - 1004.0996)

The solution space (or phase space) of the wave equation for fields propagating on a Kerr does exhibit 2d conformal symmetry in an appropriate limit.

For a massless scalar field on Kerr in Boyer-Lindquist coordinates:

$$\Box\, \Phi(t, r, \theta, \phi) = 0, \quad \Phi = e^{-i\omega t + i m \phi}\, \phi(r, \theta)$$

the wave equation can be separated (Teukolsky, Carter)

$$\phi(r, \theta) = R(r)\, S(\theta)$$

into an angular and a radial equation.

The radial equation reads as:

$$\boxed{\left[\partial_r (r-r_-)(r-r_+)\partial_r + \frac{\omega-\Omega_+ m}{4K_+^2}\frac{(r_+-r_-)}{(r-r_+)} - \frac{\omega-\Omega_- m}{4K_-^2}\frac{(r_+-r_-)}{r-r_-} + (r^2+2Mr+4M^2)\omega^2\right] R(r) = K_\ell R(r)}$$

$\sim \ell(\ell+1)R(r)$

$K^\pm = \frac{r_+-r_-}{4Mr_\pm}$, $\Omega_\pm = \frac{a}{2Mr_\pm}$, $r_\pm = M \pm \sqrt{M^2-a^2}$, $a = J/M$

Heun equation

considering a "near region" in phase space (not in space-time) defined as

$$\boxed{\omega M \ll 1 \text{ (low frequency modes)}; \quad r\omega \ll 1 \text{ (near region of geometry)}}$$

The radial wave equation reduces to a hypergeometric equation.

The $Sl(2,R)\times Sl(2,R)$ symmetry can be made more manifest introducing conformal coordinates and $Sl(2,R)$ vector fields:

$$\begin{cases} w^+ = \sqrt{\frac{r-r_+}{r-r_-}}\, e^{2\pi T_R \phi} \\ w^- = \sqrt{\frac{r-r_+}{r-r_-}}\, e^{2\pi T_L \phi - \frac{t}{2M}} \\ y = \sqrt{\frac{r_+-r_-}{r-r_-}}\, e^{\pi(T_L+T_R)\phi - \frac{t}{4M}} \end{cases}$$

$T_{R/L} = \frac{r_+ \mp r_-}{4\pi a}$

$$\begin{cases} H_1 = i\partial_+ \\ H_0 = i\left(w^+\partial_+ + \frac{1}{2}y\partial_y\right) \\ H_{-1} = i\left(w^{+2}\partial_+ + w^+ y\partial_y - y^2\partial_-\right) \end{cases} \qquad \begin{cases} \bar{H}_1 = i\partial_- \\ \bar{H}_0 = \dots \\ \bar{H}_{-1} = \dots \end{cases}$$

The radial wave equation in the near region reduces to

$$\boxed{\mathcal{H}^2\Phi = \bar{\mathcal{H}}^2\Phi = \ell(\ell+1)\Phi}$$

$\mathcal{H}^2, \bar{\mathcal{H}}^2$ = quadratic Casimir of $Sl(2,R)$

3) The HHPS proposal (Haco, Hawking, Perry, Strominger - 1810.01847)

They derived an explicit set of Vir ⊕ Vir conformal generators acting non-trivially on the horizon of a generic non-extreme Kerr black hole.

In conformal coordinates $(w^{\pm}, y, \theta)$ there are

$$\boxed{\zeta^{\pm} = \varepsilon(w^{\pm})\,\partial_{\pm} + \frac{1}{2}\,\partial_{\pm}\varepsilon(w^{\pm})\, y\, \partial_y}$$

→ modes $\zeta^{\pm}_n$ satisfy Vir ⊕ Vir

Assuming the existence of a well-defined phase space, the central charges associated with these generators turned out to be

$$\boxed{c^{+} = c^{-} = 12J}$$

Returning to the conformal coordinates (at fixed r), one has

$$\begin{cases} w^+ \sim e^{2\pi T_R \phi} =: e^{t^+} \\ w^- \sim e^{2\pi T_L \phi - \frac{t}{2M}} =: e^{-t^-} \end{cases}$$

$t^\pm$ are naturally interpreted as CFT coordinates on the torus.

Since in the original Kerr coordinates

$$(t, \phi) \sim \underbrace{(t, \phi + 2\pi)}_{\text{spatial}} \sim \underbrace{(t + \beta_H, \phi + i\beta_H \Omega_H)}_{\text{thermal}}$$

(t^+, t^-) have periodicities

$$(t^+, t^-) \sim \underbrace{(t^+ + 2\pi l^+, t^- + 2\pi l^-)}_{\text{thermal}} \sim \underbrace{(t^+ + 2\pi \tau^+, t^- + 2\pi \tau^-)}_{\text{spatial}}$$

with $l^\pm = 2\pi \overbrace{\left(\frac{r_+ \pm r_-}{4\pi a}\right)}^{= T^\pm}$, $\tau^\pm = \pm 1$ (cycles "swapped")

The Cardy entropy for a CFT on such a torus is

$$S = \frac{\pi}{6}\left(\frac{l^-}{\tau^-} c^- + \frac{l^+}{\tau^+} c^+\right)$$

plugging the above values

$$= 2\pi\left(M^2 + \sqrt{M^4 - J^2}\right) = \frac{A_{Kerr}}{4} \; !$$

This all seems to indicate that 2d conformal symmetry, beyond its relevance for stringy black holes, could also underlie real-world Kerr black holes.

I will however now return to the 3 evidences

1) Kerr/CFT

2) Hidden conformal symmetry

3) HHPS conformal symmetry

and look at them with more scrutiny.

1) Kerr/"CFT"?

The NHEK geometry, playing a central role in Kerr/CFT is

$$\boxed{ds^2 = 2GJ\Omega^2\left[\left(-r^2dt^2+\frac{dr^2}{r^2}\right)+\Lambda^2(d\phi+r\,dt)^2+d\theta^2\right]}$$

$\Omega^2 = \frac{1+\cos^2\theta}{2}$, $\Lambda = \frac{2\sin\theta}{1+\cos^2\theta}$

θ = cst slices:

- $\Lambda^2 = 1$: AdS_3
- $\Lambda^2 \neq 1$: not AdS_3, not even asymptotically AdS_3 ($g_{tt} = (\Lambda^2-1)r^2+\ldots$ at $r\to\infty$)

↳ Warped AdS_3 (WAdS$_3$)

~~Brown-Henneaux~~

Should we be scared?

Not really. After all, in recent years, we have learned to get acquainted with holographic models involving non-AdS geometries and non-CFTs. (Schrödinger, Lifshitz e.g.)

Interesting features of $WAdS_3$ spaces have appeared in the last 10 years:

- $SL(2,\mathbb{R}) \times U(1)$ isometries
- $Vir \ltimes \widehat{u(1)}$ asymptotic symmetries

$$\begin{cases} [L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m} \\ [L_n, P_m] = -m P_{n+m} \\ [P_n, P_m] = \frac{k}{2} n \delta_{n+m} \end{cases}$$

0701039

0808.1911

- Allow for BH solutions (Clément et al 0303042, 0706.0263, Anninos et al. 0807.3040)
- Suggested the existence of Warped conformal field theories (WCFT) 1210.0539, 1107.2917
  * $SL(2,\mathbb{R}) \times U(1)$ global symmetry
  * $Vir \ltimes \widehat{u(1)}$ local ∞^{el} symmetry
  * Modular properties
  * Warped analog of Cardy formula: S_{WCFT}
- Observation:

$$S_{WAdS_3\,BH} = S_{WCFT} \qquad (\,//\, S_{BTZ} = S_{CFT})$$

2) Hidden symmetry

Remember the radial wave equation:

$$\left[\partial_r (r-r_-)(r-r_+)\partial_r + \frac{(\omega-\Omega_+ m)^2}{4K_+^2}\frac{r_+-r_-}{r-r_+} - \frac{(\omega-\Omega_- m)^2}{4K_-^2}\frac{r_+-r_-}{r-r_-} + r^2\omega^2 + 2Mr\omega^2 + 4M^2\omega^2\right]R(r) = K_\ell R(r)$$

The l.h.s. could be written as

$$\mathcal{H}^2 R(r) = \bar{\mathcal{H}}^2 R(r) \longrightarrow \text{quadratic Casimir of } SL(2,\mathbb{R})$$

in the limit $\omega M \ll 1$, $r\omega \ll 1$, neglecting the red and green terms.

However, one can be less stringent on the long wavelength approximation. This would amount to discard the red terms compared to the green one.

The l.h.s. is then modified to

$$(\mathcal{H}^2 - \bar{H}_0^2)\, R(r)$$

This is the $SL(2,\mathbb{R}) \times U(1)$ Casimir (= global symmetries of a WCFT)

5/ Revisiting HHPS

We now have indications that a WCFT description might be equally appropriate as a CFT one. Does it allow to reproduce the Kerr entropy ?

Instead of picking conformal generators, take near-horizon diffeos satisfying a WCFT algebra :

$$\boxed{\begin{aligned} \xi &= \epsilon(w^+)\,\partial_+ + \frac{1}{2}\,\partial_+\epsilon(w^+)\,y\,\partial_y \\ \mu &= \hat{\epsilon}(w^+)\,\left(w^-\partial_- + \frac{y}{2}\,\partial_y\right) \end{aligned}}$$

$\to$ modes ξ_n, μ_n satisfy $\mathrm{Vir} \ltimes \hat{u}(1)$

Central extensions can be computed and give

$$\boxed{c = 12J \qquad , \quad k = -2J}$$

A WCFT on a torus

$$(t^+, t^-) \sim (t^+ - 2\pi l^+, t^- + 2\pi l^-) \sim (x^+ - 2\pi z^+, x^- + 2\pi z^-)$$

has entropy (1210.0539, Castro-Hofman-Iqbal 1508.06302)

$$S = 2\pi i \left(\frac{l^+ z^-}{z^+} - l^- \right) P_0^{vac} + 4\pi i \frac{l^+}{z^+} L_0^{vac} \qquad (*)$$

Remember from the "conformal" coordinates that

$$l^\pm = 2\pi \left(\frac{r_+ \mp r_-}{4\pi a} \right), \quad z^\pm = \pm 1, \quad r_\pm = M \pm \sqrt{M^2 - a^2}, \ a = J/M$$

Plugging this and the vacuum values in (*) gives

$$S = 2\pi \left(M^2 + \sqrt{M^4 - J^2} \right) = \frac{A_{Kerr}}{4} \ !$$

Outlook

We have argued that a WCFT description of Kerr black holes could possibly challenge the more conventional CFT one.

WCFTs, however, are much less known than their CFT relative, and usually stand on less firm grounds.

On the other hand, they present some advantages over the CFT description.

An important step would be to properly define the phase space of the theory at the horizon.

We have our work cut out for us!

Thanks!