Numerical and analytical studies of a matrix model with non-pairwise contracted indices

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Based on a collaboration with Shingo Takeuchi
arXiv:1907.06137

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Introduction

- Quantization of gravity — A fundamental problem in theoretical physics

  Perturbative quantization of general relativity fails due to unrenormalizable ultraviolet divergencies.

- A direction toward resolution

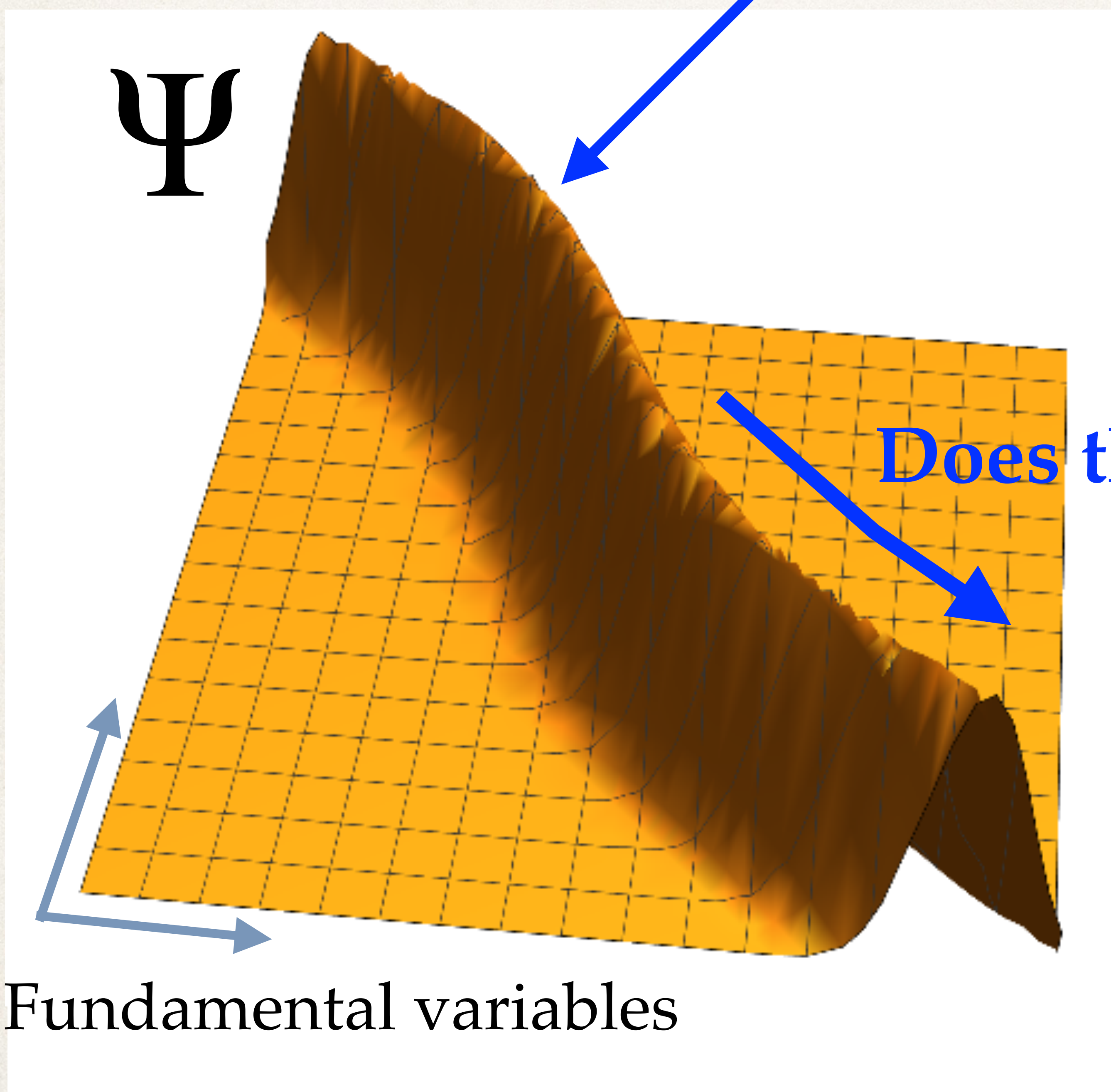
  - Fundamental theory does not contain classical spacetime in its formulation.

  - **Emergent spacetime** — **Spacetime should be generated in the infrared.**

There are various proposals in this direction.

In principle, we can check each proposal by its wave function.

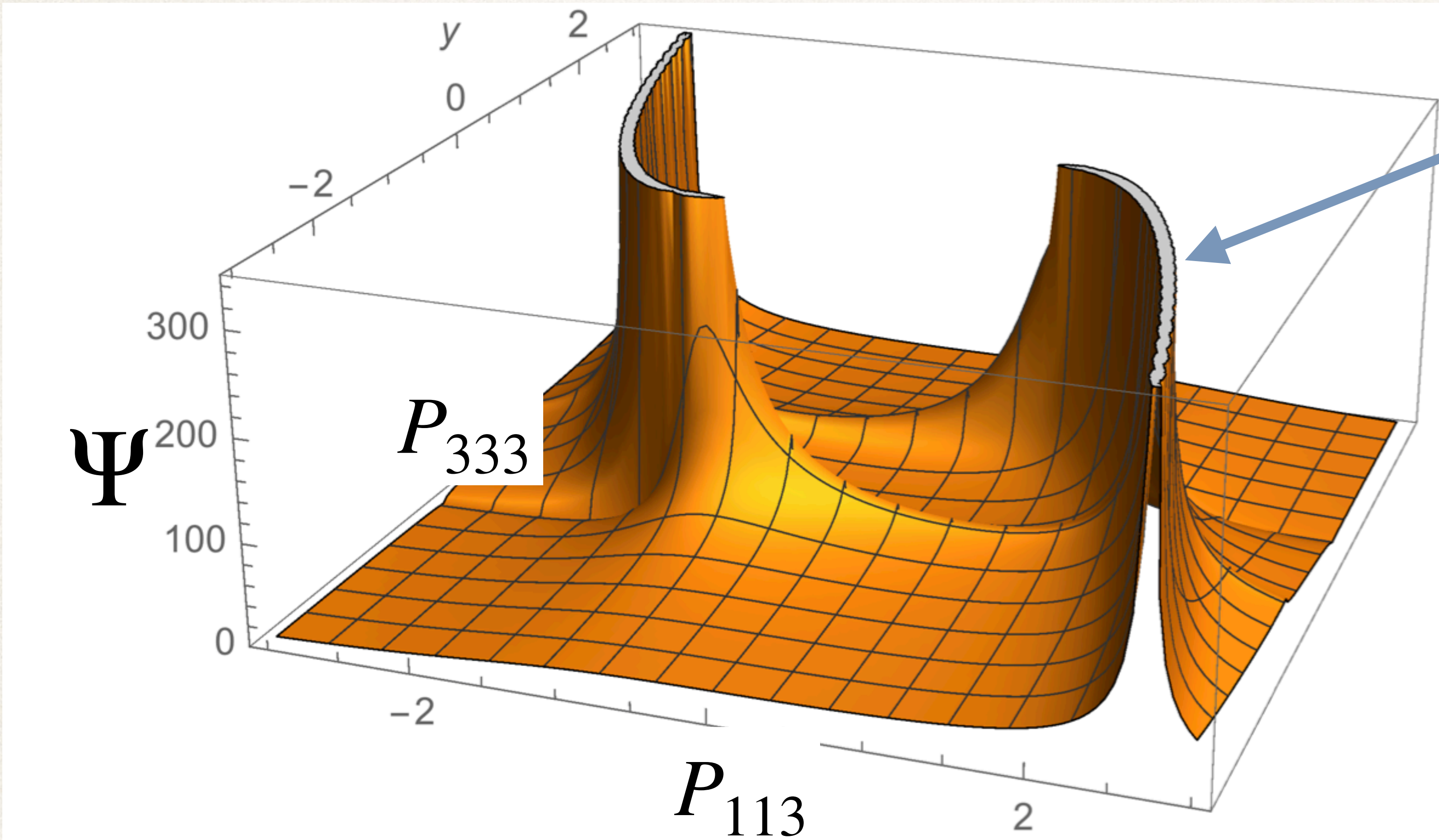
Can be regarded as a classical spacetime ?



Does the trajectory follow GR ?

The wave fn. of a tensor model in the Hamilton formalism (CTM)

Ex. of $N = 3$ Sol. of $\hat{\mathcal{H}}_a|\Psi\rangle = \hat{\mathcal{J}}_{ab}|\Psi\rangle = 0$ $a, b, \cdots = 1,2,\cdots,N$



Peak with $H = SO(2,1)$

Peaks exist at Lie-group symmetric P_{abc}. Obster, NS, arXiv:1710.07449

$$h_a^{a'} h_b^{b'} h_c^{c'} \bar{P}_{a'b'c'} = \bar{P}_{abc},\ h \in H$$ **: Lie group representation (Mixed signature in general)**

This property is interesting, because our spacetime has symmetries.

This phenomenon can be qualitatively explained by the quantum coherence of the integration contained in the wave function.

$$\Psi(P_{abc}) = \varphi(P_{abc})^{R_T/2} \qquad R_T = (N+2)(N+3)/2$$

$$\varphi(P_{abc}) = \int_{\mathbb{R}^{N+1}} d\tilde{\phi} d\phi \, e^{i\overbrace{(P_{abc}\phi_a\phi_b\phi_c - \phi^2\tilde{\phi} + 4\tilde{\phi}^3/27\lambda)}^{\bar{P}_{abc}\bar{\phi}_a\bar{\phi}_b\bar{\phi}_c}}$$

$$\phi \in \mathbb{R}^N \quad \tilde{\phi} \in \mathbb{R}$$

If $h_a^{a'}h_b^{b'}h_c^{c'}\bar{P}_{a'b'c'} = \bar{P}_{abc}$, $h \in {}^{\exists}H$, the integration along the gauge orbit $h_a^{a'}\bar{\phi}_{a'}$ has coherent contributions, and $\varphi(P_{abc})$ takes large values.

However, if not, the integration is a sum of $e^{i\theta}$ with rather random phases θ, and $\varphi(P_{abc})$ tends to be small.

We want to know more about the wave function, enough to take the $N \to \infty$ limit. But presently hard.

We perform the two simplifications to $\Psi(P_{abc}) = \varphi(P_{abc})^{R_T/2}$ for start.

(1) $$\varphi(P_{abc}) = \int_{\mathbb{R}^{N+1}} d\tilde{\phi} d\phi \, e^{i(P_{abc}\phi_a\phi_b\phi_c - \phi^2\tilde{\phi} + 4\tilde{\phi}^3/27\lambda)}$$

Fix $\tilde{\phi}$

The symmetry-peak relation is kept.

(Euclidean)

$$\bar{\varphi}(P_{abc}) = \int_{\mathbb{R}^N} d\phi \, e^{iP_{abc}\phi_a\phi_b\phi_c - k\,\phi^2}$$

(2) Integrate over P_{abc}

$$\int_{-\infty}^{\infty} dP \, e^{-\frac{1}{4\lambda} P^2} \Psi(P)^2 = \int_{-\infty}^{\infty} dP \, e^{-\frac{1}{4\lambda} P^2} \bar{\varphi}(P)^R$$

$$\int_{-\infty}^{\infty} dP\, e^{-\frac{1}{4\lambda}P^2}\, \bar{\varphi}(P)^R \qquad R = R_T = (N+2)(N+3)/2$$

$$= \int_{-\infty}^{\infty} dP\, e^{-\frac{1}{4\lambda}P^2} \left(\int_{\mathbb{R}^N} d\phi\, e^{iP_{abc}\phi_a\phi_b\phi_c - k\,\phi^2} \right)^R$$

The power replaced by replicas

$$= \int_{-\infty}^{\infty} dP \prod_{a,j=1}^{N,R} d\phi_a^j\; e^{-\frac{1}{4\lambda}P^2 + \sum_{j=1}^{R} i\,P_{abc}\phi_a^j\phi_b^j\phi_c^j - k\,\phi_a^j\phi_a^j}$$

$\Big\downarrow$ Gaussian integration over P_{abc}

A model with ϕ_a^i $(a = 1,2,\cdots N,\ i = 1,2,\cdots,R)$

The matrix model

Dynamical variable : $N \times R$ matrix ϕ_a^i

$$Z_{N,R}(\lambda, k) = \int_{\mathbb{R}^{NR}} d\phi \, e^{-\lambda \, U(\phi) - k \, \phi^2}$$

$$d\phi = \prod_{a,i=1}^{N,R} d\phi_a^i \qquad U(\phi) = \underbrace{\sum_{i,j=1}^{R} (\phi_a^i \phi_a^j)^3}_{\geq 0} \qquad \phi^2 = \sum_{i=1}^{R} \phi_a^i \phi_a^i$$

The i, j indices in $U(\phi)$ are triply contracted.
The symmetry of the model is $O(N) \times S_R$.
So ϕ_a^i is not diagonalizable in general.

Our matrix model cannot be solved as the usual matrix models.

This matrix model is similar to that appeared in the replica trick of the spherical p-spin model for spin glasses.

A. Crisanti, H.-J. Sommers, Z. Phys. B 87, 341 (1992).

$$\int_{\substack{\phi_a^i \phi_a^i = 1 \\ \text{each } i}} d\phi \, e^{\lambda \sum_{i,j=1}^{R} (\phi_a^i \phi_a^j)^3}$$

There are some critical differences.

Spherical p-spin model	**The tensor model**
Spherical constraint	Flat space
$R \to 0$	$R_T = (N+2)(N+3)/2$
Opposite signs of the interactions	

It seems necessary to reanalyze our model from the scratch.

We have performed

- Monte Carlo simulations with the Metropolis update method.
- Analytic computations in terms of a perturbative method

Summary of our results

Transition region

$R \sim (N+1)(N+2)/2$

$0 \longrightarrow R$

$\phi_a^i \sim 0$

Good agreement between numerical and analytic results.

The two results deviate

Dimensional transitions of the configurations $(S^1 \to S^2 \to S^3 \to \cdots)$

The tensor model

$R_T = (N+2)(N+3)/2$

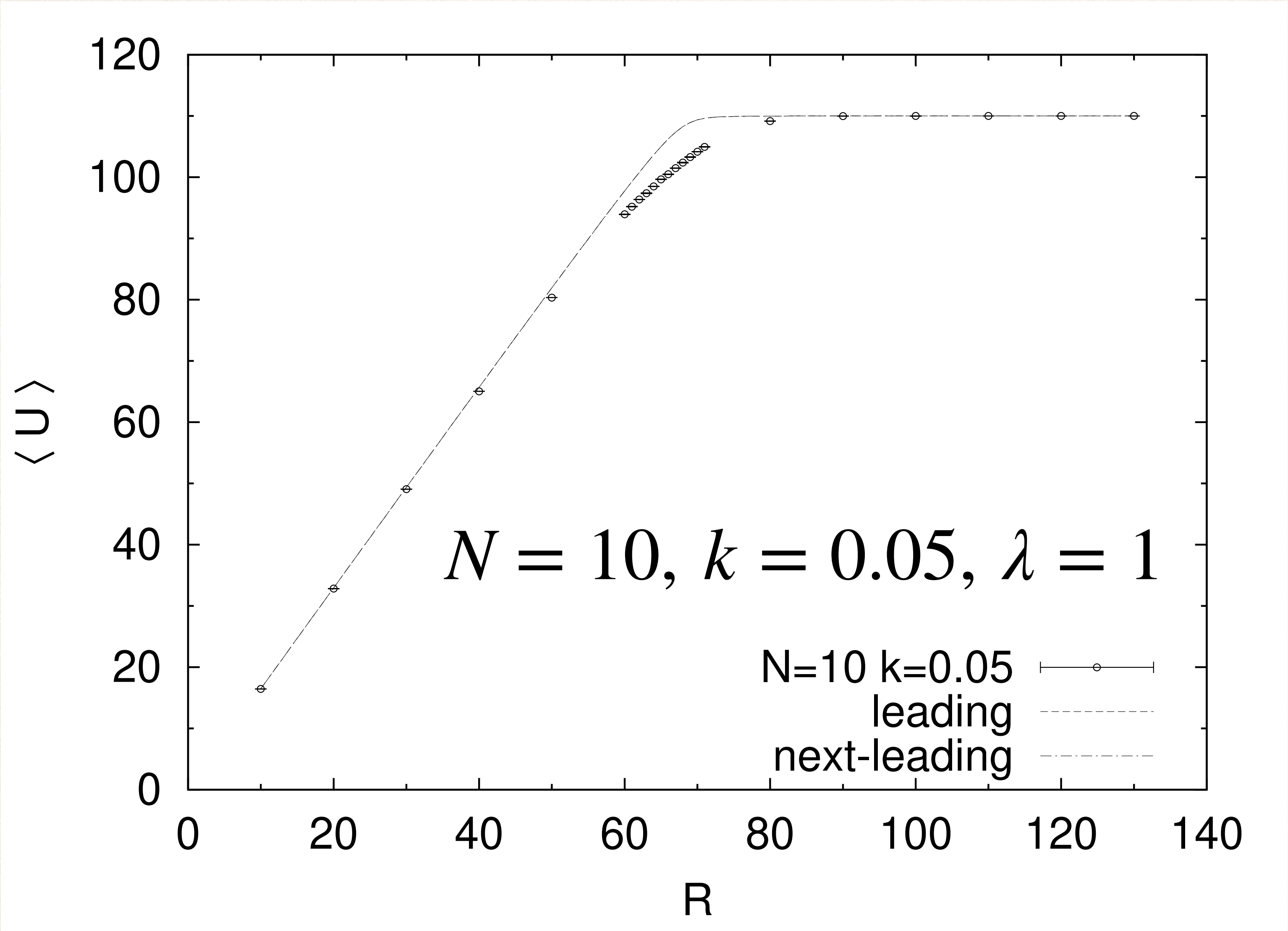
$\phi_a^i \not\sim 0$

Good agreement between numerical and analytic results.

Monte Carlo extremely slow.
High viscosity fluid? Glass?

Results of the Monte Carlo and analytic computations

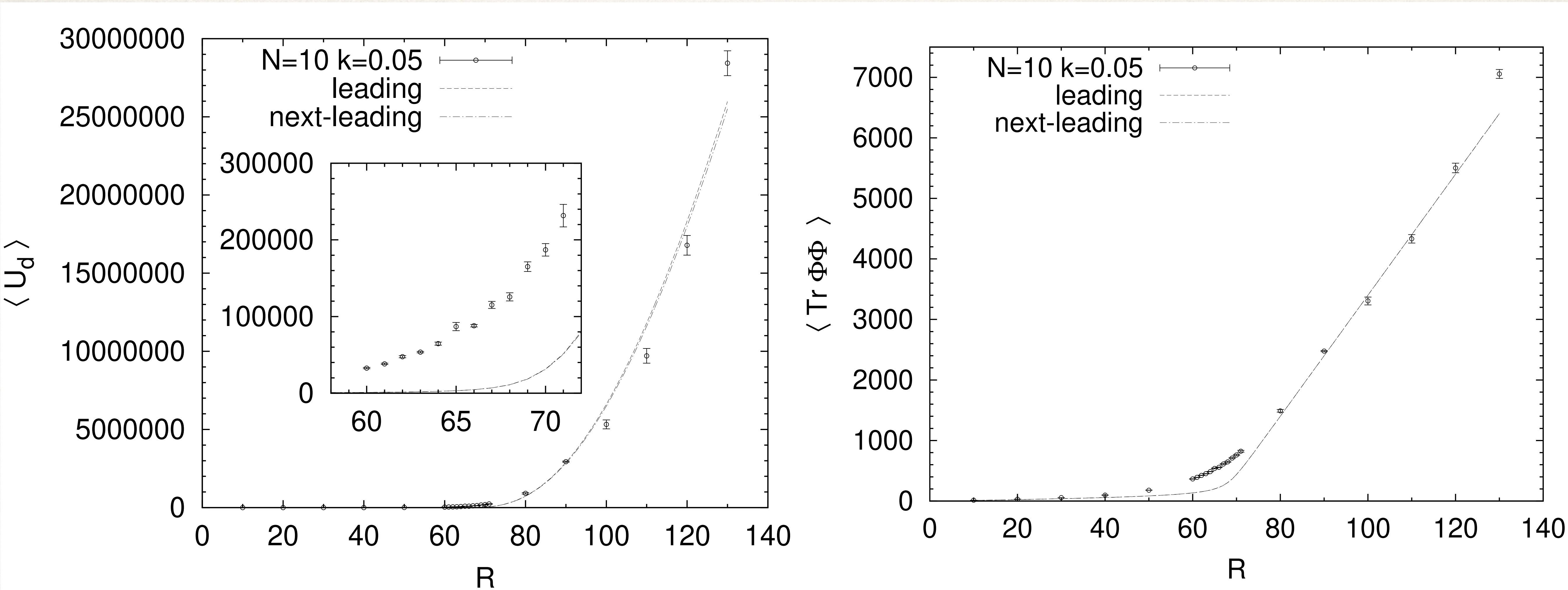
- **Expectation values of observables**



$$U := \sum_{i,j=1}^{R} (\phi_a^i \phi_a^j)^3$$

- **There is a transition region around $R \sim R_c = (N+1)(N+2)/2$.**
- **The analytic computation and the simulation do not agree well in the transition region, while they agree well in the outside region.**

Similar results for other observables.

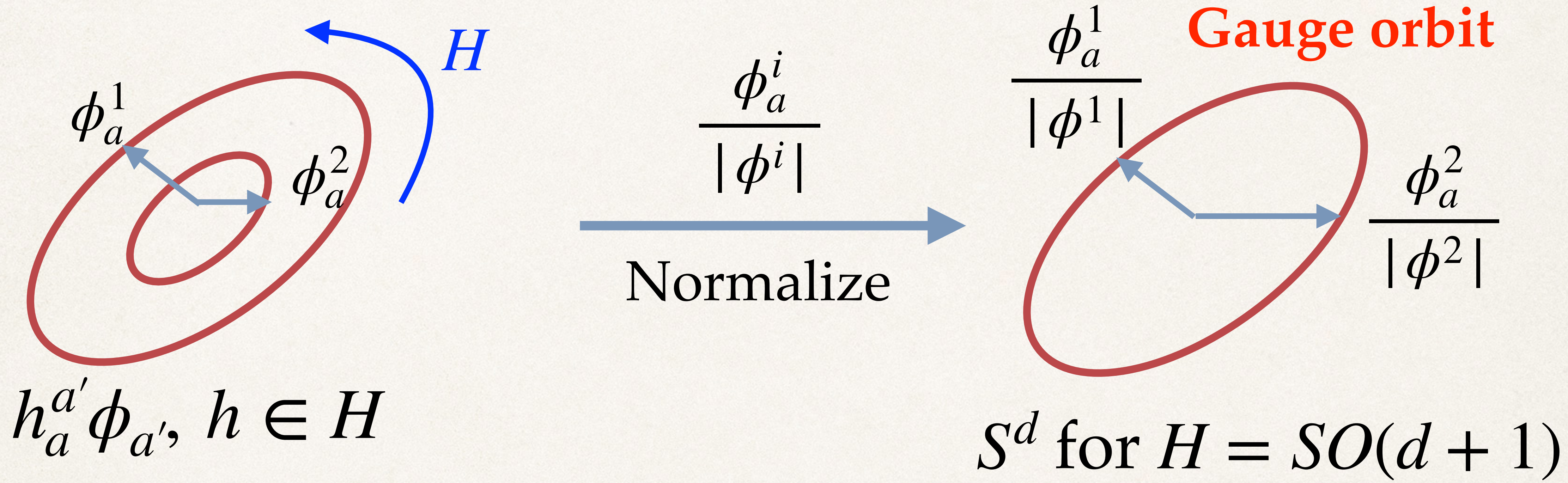


$$U_d := \sum_{i=1}^{R} (\phi_a^i \phi_a^i)^3 \qquad \mathrm{Tr}(\phi^t \phi) := \sum_{i=1}^{R} \phi_a^i \phi_a^i$$

- **Monte Carlo suggests a smoother transition (even a crossover) than the analytic computation. But not conclusive.**

- **Topological properties of the configurations ϕ_a^i**

The matrix model is the integral of a wave function, and its peaks may dominate. In such cases, $\phi_a^i\ (i = 1,2,\ldots,R)$ along the gauge orbits may become the dominant configurations.

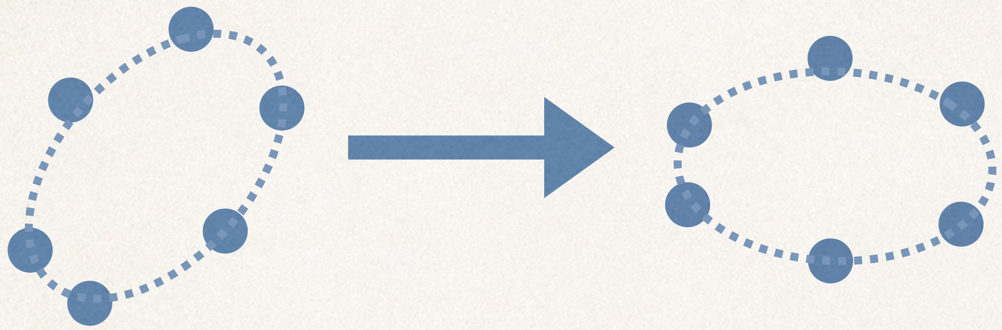


We analyze the topology of $\phi_a^i/|\phi^i|\ (i = 1,2,\ldots,R)$ generated by the Monte Carlo simulations.

Caution : If all the Monte Carlo datas are just piled up, the gauge orbits take arbitrary directions in N-dimensions and non-trivial structures cannot be seen. We have to analyze each data and align them.

We analyzed the topological aspects by three different methods.

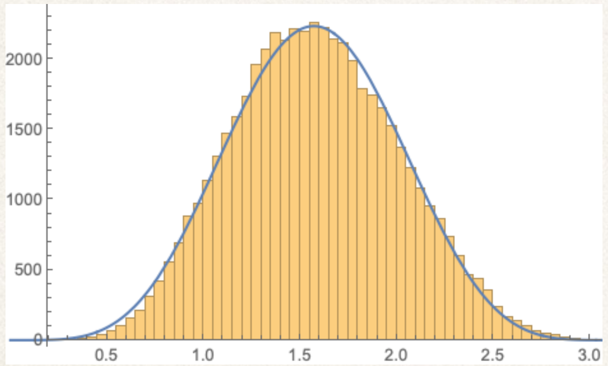
- Perform principal component analysis to $\phi_a^i / |\phi^i|$, and realign.



😁 Clearly seen

😩 Dimensions be lower

- Read dimensions from the distributions of $\cos^{-1}(\phi_a^i \phi_a^j / |\phi^i||\phi^j|)$



$$\propto \sin^{d-1}(\theta)$$

😁 Easy

😩 Indirect

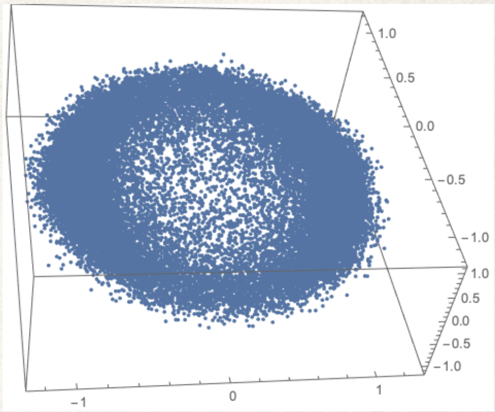
- Use a method in topological data analysis (persistent homology)

😁 Direct method 😩 be subject to noises

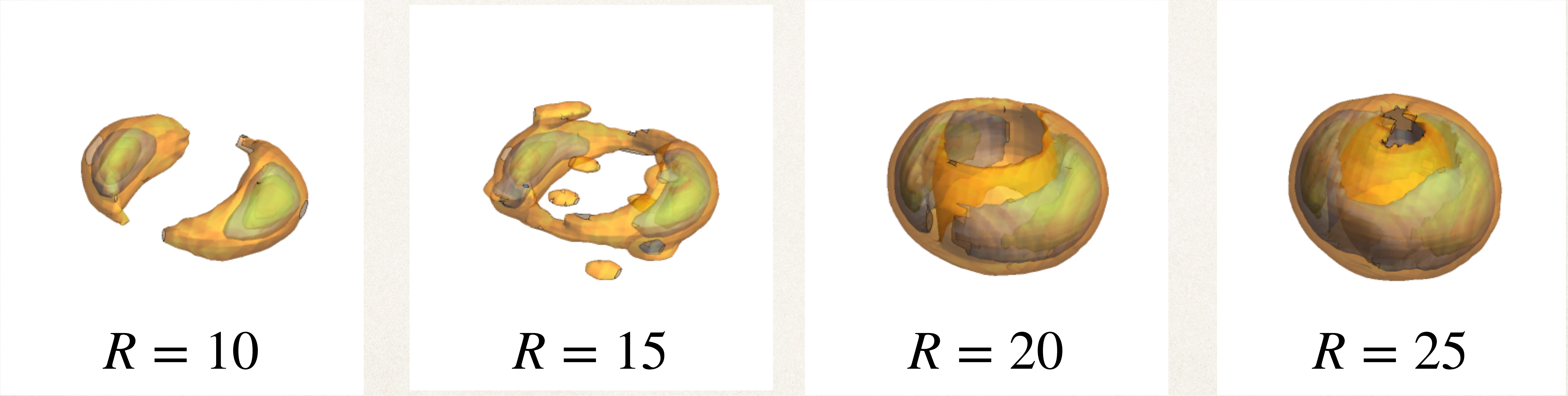
- Principal component analysis for $N = 4,\ k = 0.01,\ \lambda = 1.$

$(R_c = (N+1)(N+2)/2 = 15)$

Each Monte Carlo data $\phi^i_a / |\phi^i|\ (i = 1,2,\ldots,R)$ is analyzed and aligned, and main three dimensional parts are plotted for all data.

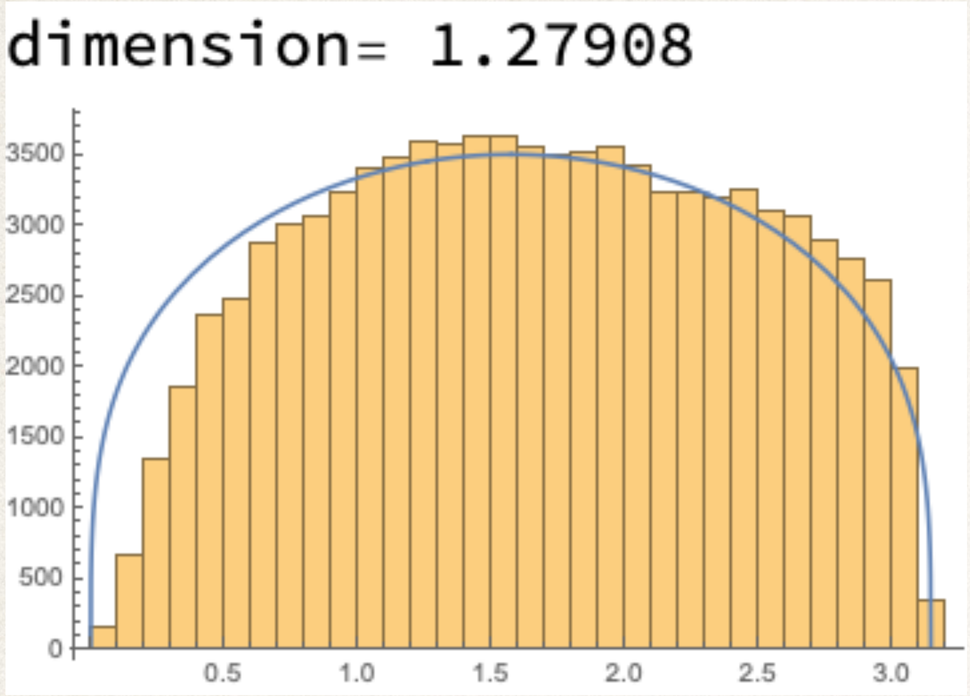


Density plots of the points

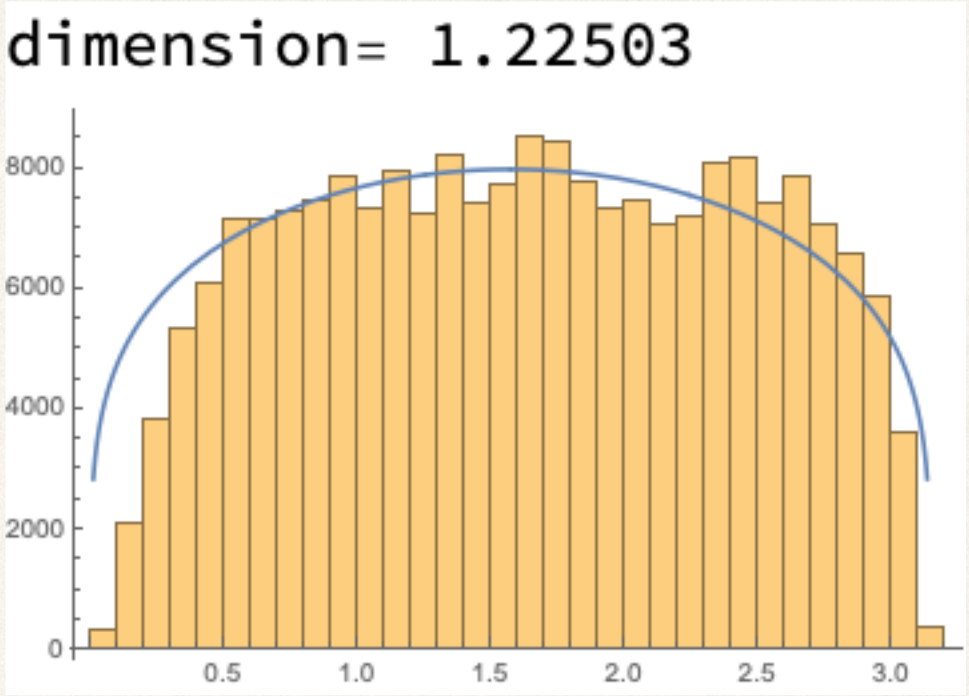


$$S^0 \to S^1 \to S^2$$

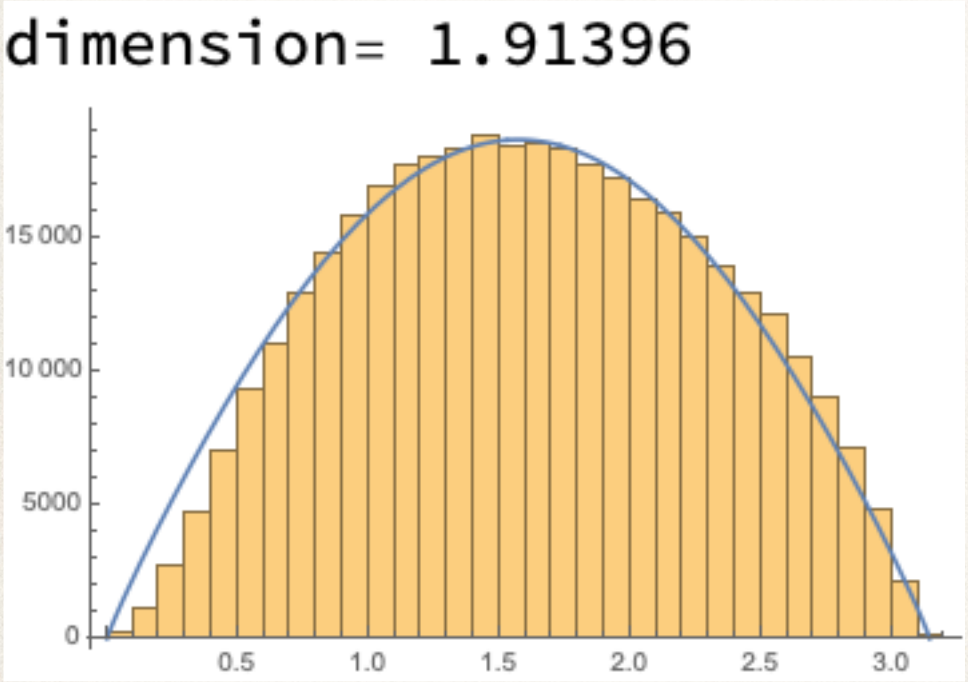
- The fitting of angular distribution $\propto \sin^{d-1}(\theta)$



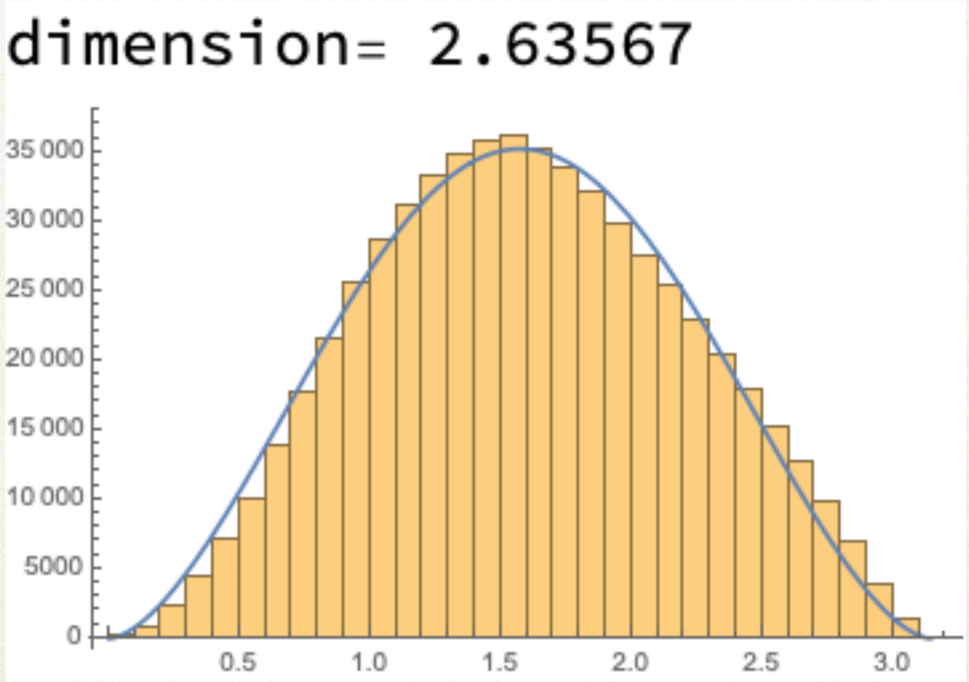
$R = 10$



$R = 15$



$R = 20$



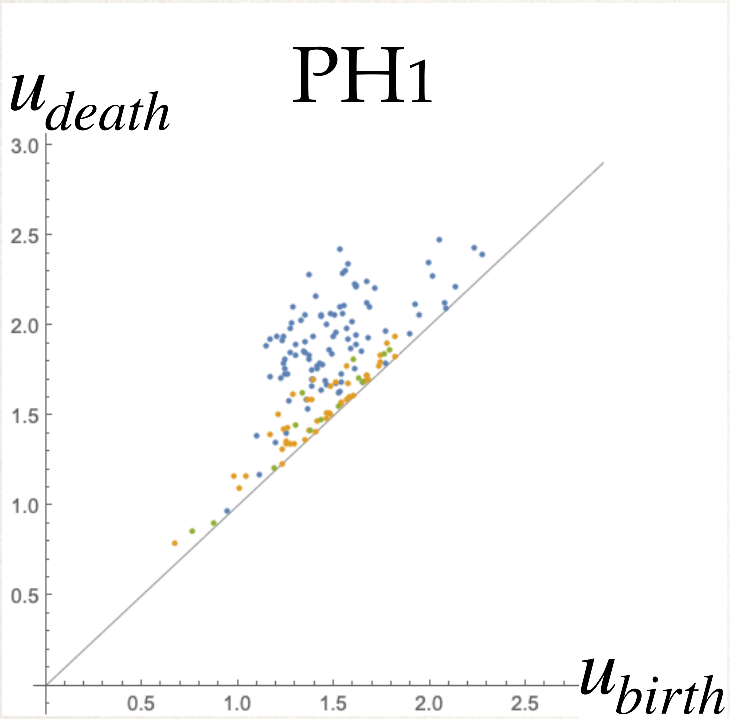
$R = 25$

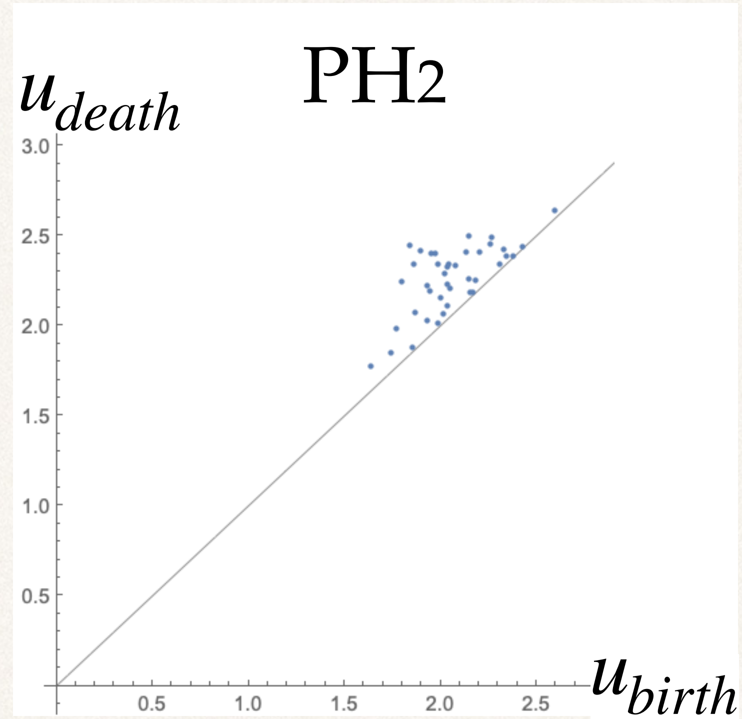
$$S^1 \to S^2 \to S^3$$

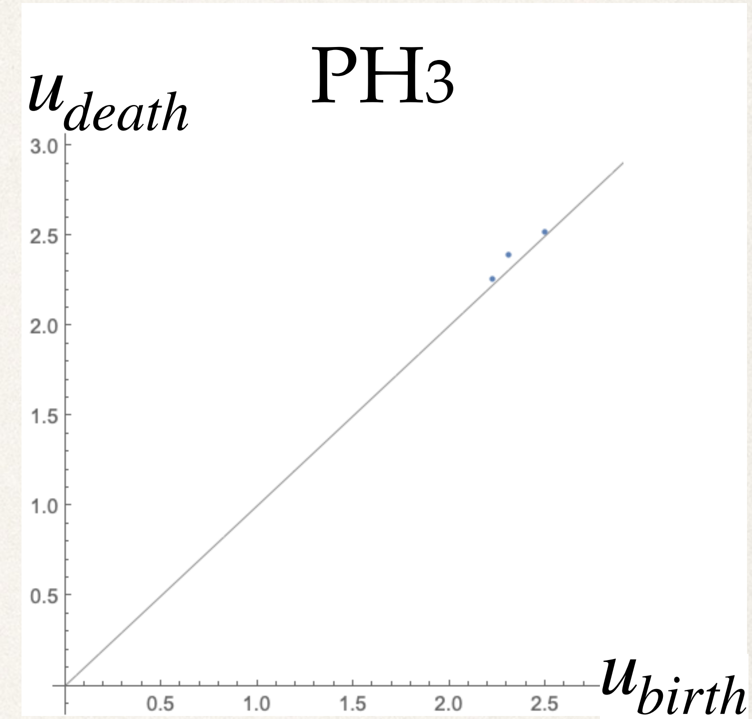
- Analysis of persistent homology

$N = 4,\ k = 0.01,\ \lambda = 1$

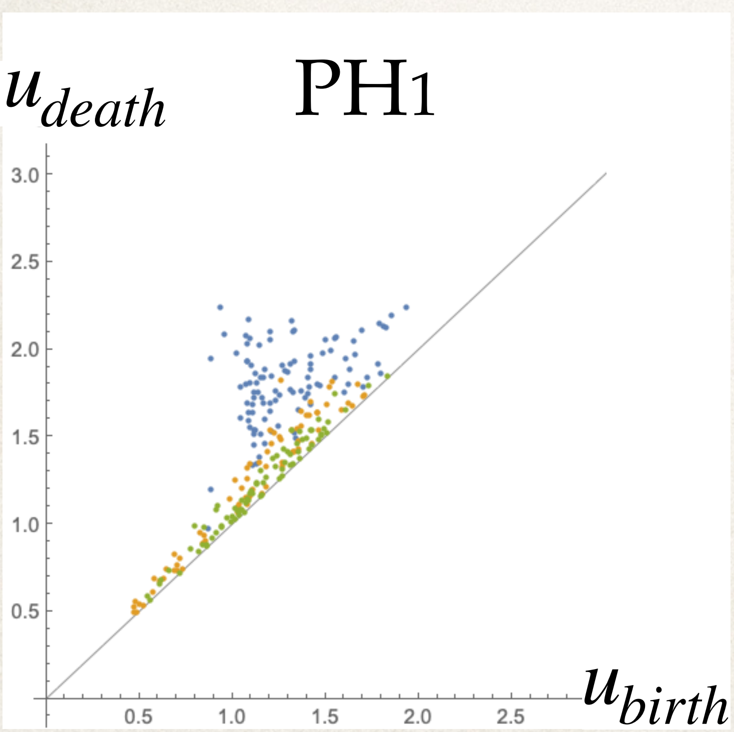
R=10

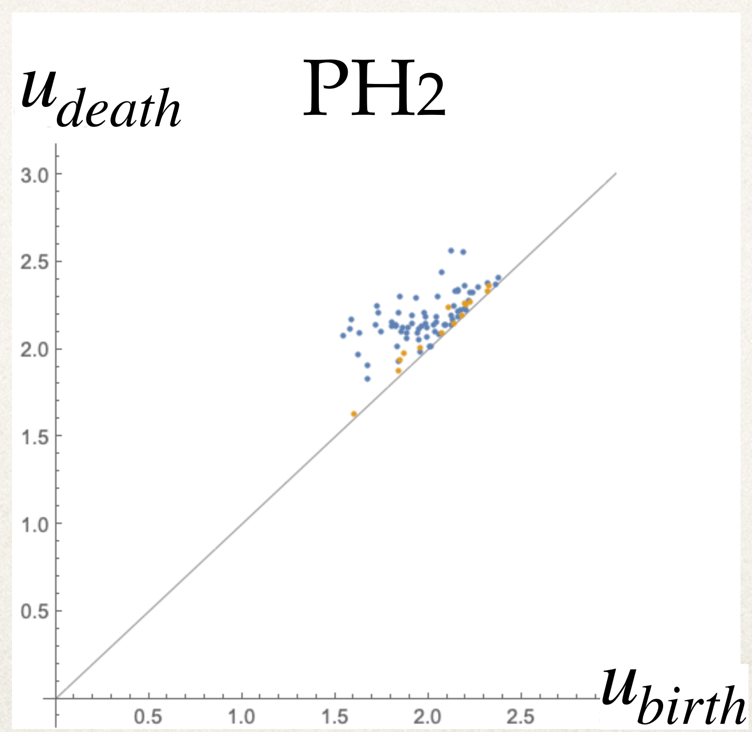


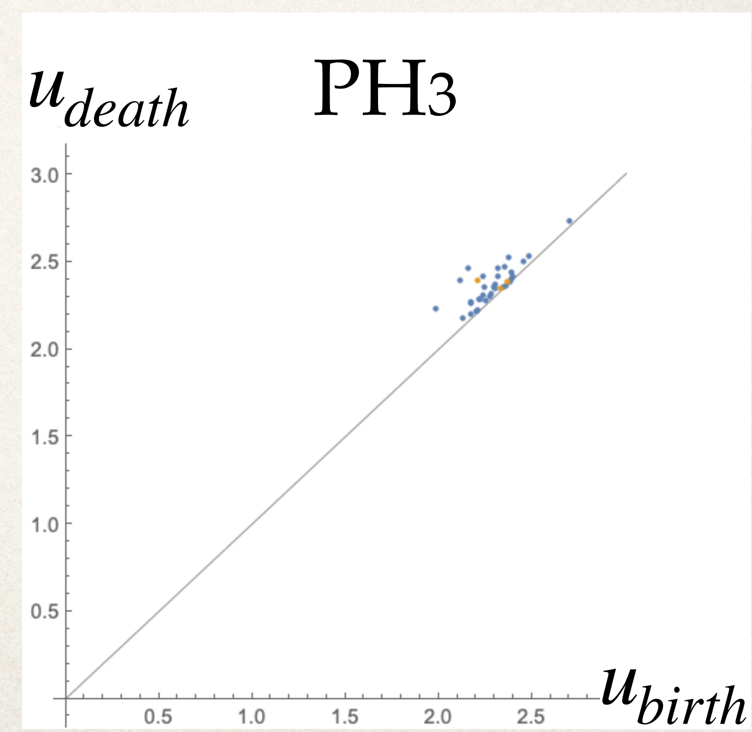




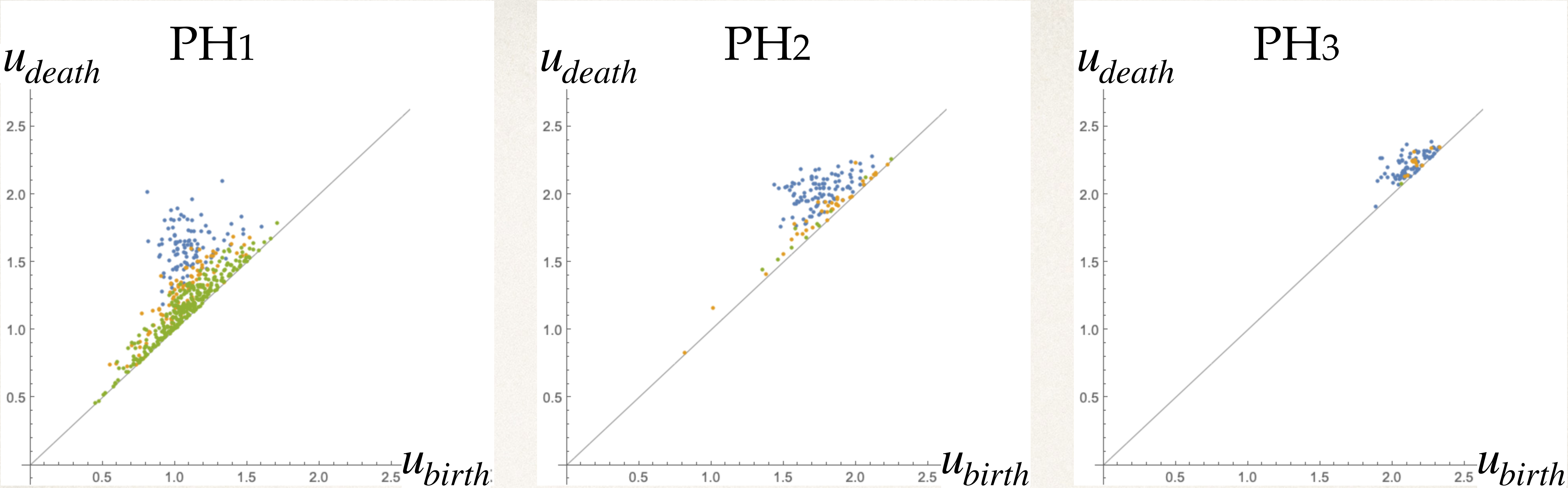
R=15



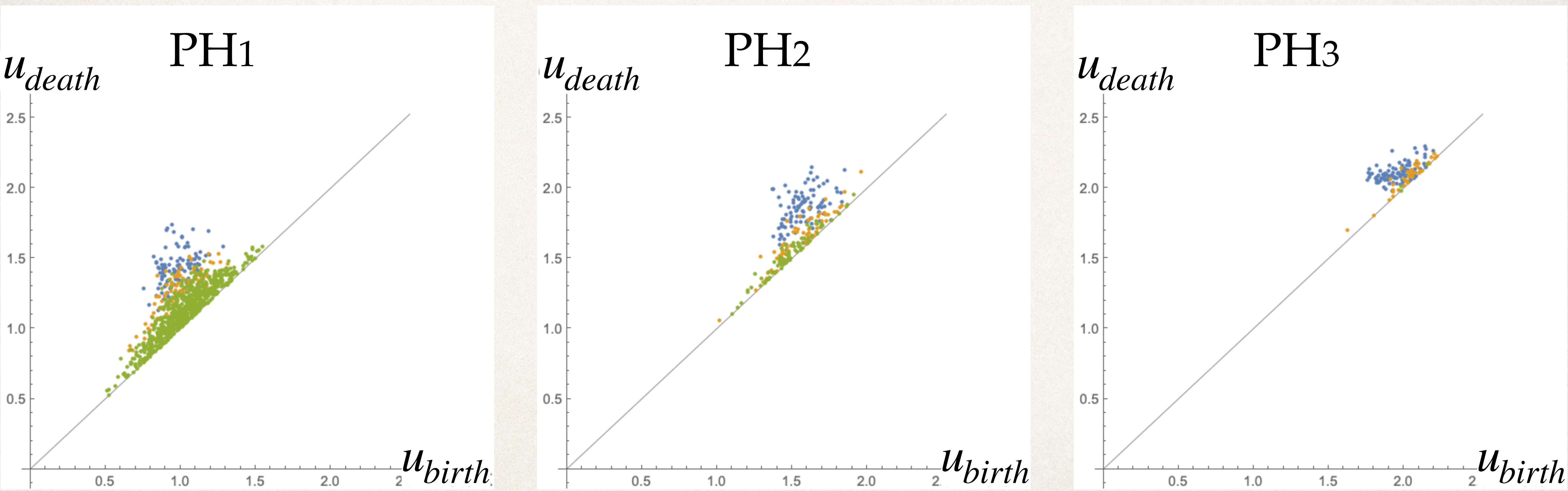




R=20



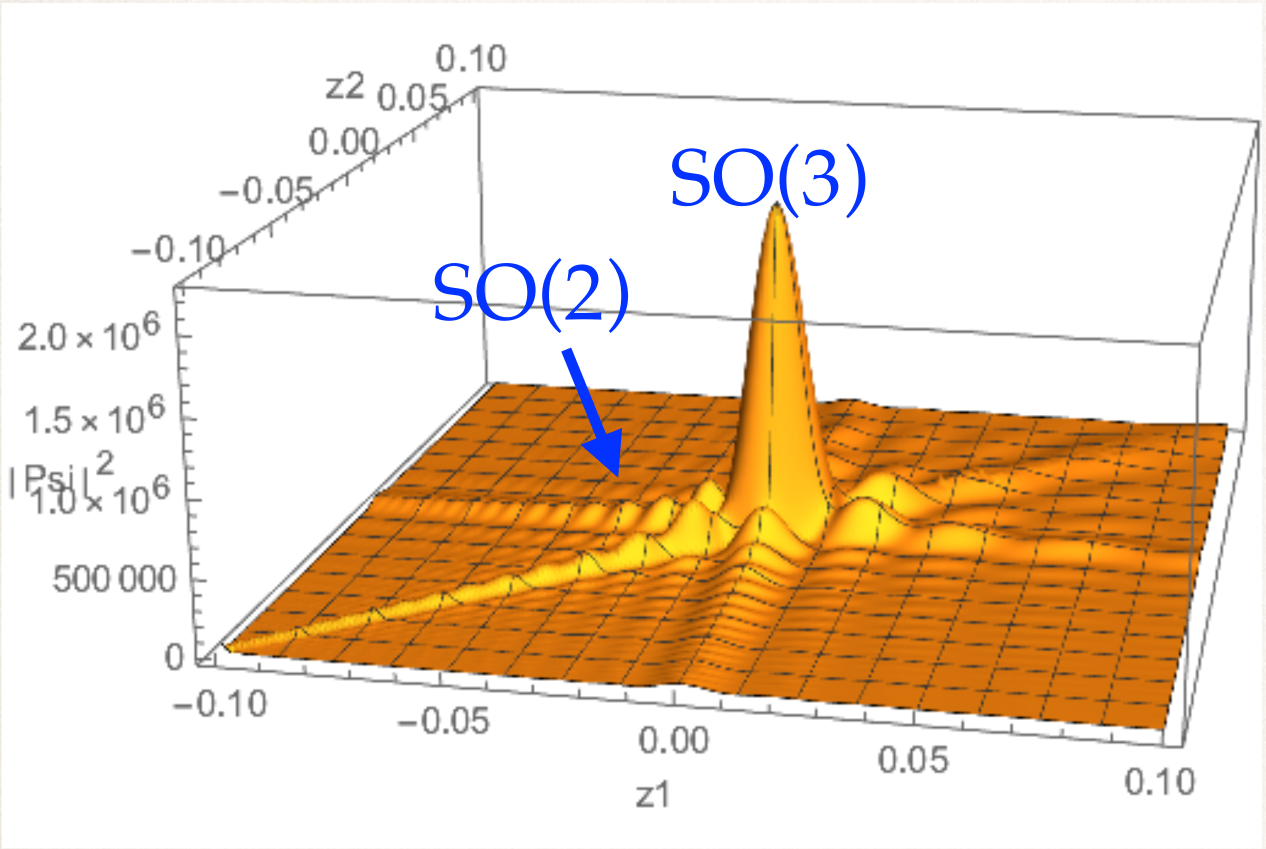
R=25



$$S^1 \to S^2 \to S^3$$

We have observed the transition of dimensions $S^1 \to S^2 \to S^3$ around $R \sim R_c = (N+1)(N+2)/2$, more or less depending on the methods.

This transition is actually consistent with the peak-symmetry relation.



Higher symmetric peaks are high, but there exist less.

Lower symmetric peaks are low, but there exist more.

For larger R, higher symmetric peaks are more enhanced in $Z = \int dP\, \bar{\varphi}(P)^R e^{-P^2/4\lambda}$.

Analytic computation

$$Z_{N,R}(\lambda, k) := \int_{\mathbb{R}^{NR}} d\phi \; e^{-\lambda \sum_{i,j=1}^{R} (\phi_a^i \phi_a^j)^3 - k \sum_{i=1}^{R} \phi_a^i \phi_a^i}$$

$$= \int_0^\infty dr \, f(\lambda r^6) \, e^{-kr^2}$$

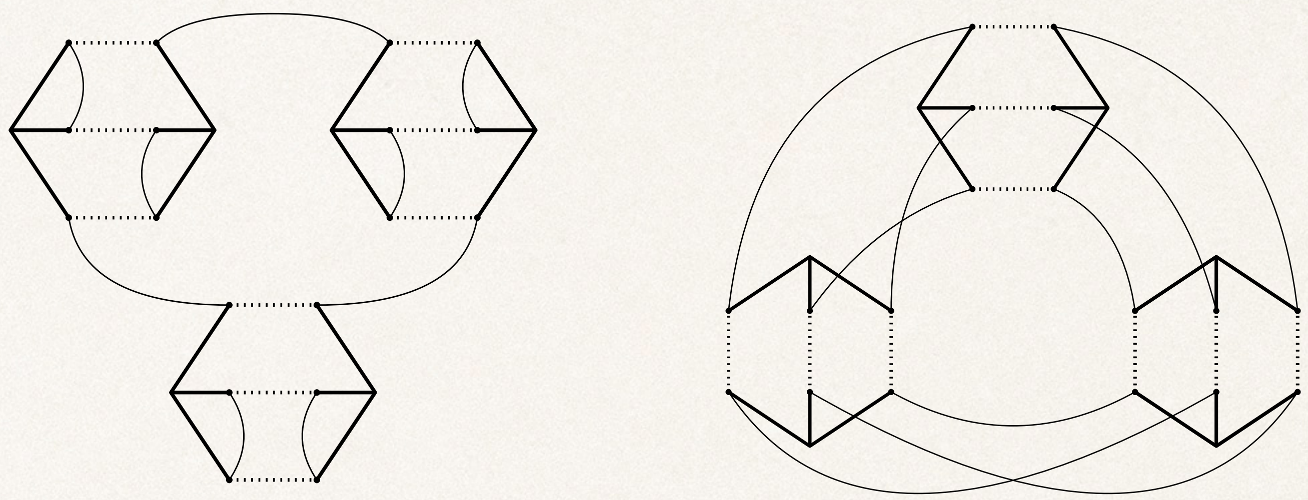
$$f(t) := \int_{S^{NR-1}} d\tilde{\phi} \; e^{-t \sum_{i,j=1}^{R} (\tilde{\phi}_a^i \tilde{\phi}_a^j)^3}$$

$f(t)$ is an entire function: Perturbative series converges for $|t| < \infty$

In the leading order of $1/R$, this is to sum over all the necklace diagrams.

L.Lionni, NS arXiv:1903.05944

Ex. of $n = 3$



Result:

$$f(t) = \left(1 + \frac{12t}{N^3R^2}\right)^{-\frac{N(N-1)(N+4)}{12}} \left(1 + \frac{6(N+4)t}{N^3R^2}\right)^{-\frac{N}{2}}$$

We have also computed the next-leading order, but it does not explain the deviation between the numerical and analytical results.

Summary

Transition region

$R \sim (N+1)(N+2)/2$

$0 \longrightarrow R$

$\phi^i_a \sim 0$

Good agreement between numerical and analytics results.

The two results deviate

Dimensional transitions of the configurations

$(S^1 \to S^2 \to S^3 \to \cdots)$

The tensor model

$R_T = (N+2)(N+3)/2$

$\phi^i_a \nsim 0$

Good agreement between numerical and analytics results.

Monte Carlo extremely slow.
High viscosity fluid? Glass?

Future directions

- Improve the numerical and analytic computations to fully understand. Tempering, Hybrid MC.
- Deal with the matrix model really derived from the tensor model. Have to deal with the sign problem.