4D conformal quantum fields from TFT2 and polynomial rings

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Talk at Corfu workshop on Quantum Geometry, Field Theory and Gravity.

Based on "CFT4 as SO(4,2)-invariant TFT2"
R. de Mello Koch and S. Ramgoolam arXiv:1403.6646 [hep-th] (NPB890)
"Free field primaries in general dimensions: Counting and construction with rings and modules,"
R. de Mello Koch and S. Ramgoolam arXiv:1806.01085 [hep-th] (JHEP1808)

Introduction : CFT in dimensions $d > 2$

CFT in dimensions $d > 2$ have been actively studied in recent years. Motivations include

- The AdS/CFT correspondence, e.g. $N = 4$ SYM in $\mathbb{R}^{3,1}$ dual to 10 D string theory. How does the 10 D quantum gravity theory emerge from local operators and correlators of CFT ?

- Exotic CFTs without conventional Lagrangian descriptions, e.g. Argyres-Douglas fixed points in 4D, $(0, 2)$ theories in 6D.

- Bootstrap program revived: Use OPE associativity to determine CFT data.

CFT2s and algebras

CFT2's very widely studied in 80's and 90's and since.

Motivations from critical string theory:
CFT2 plus ghost system = string.

Features:

Infinite dimensional Lie algebras controlling the spectrum and correlators: Virasoro, Current algebras.

Rep theory of these algebras, extended by considerations of modular transformations of characters.

Rational conformal field theories, with finitely many primary fields for these algebras.

"Vertex operator algebras" - mathematical constructions for field operators, operator product expansions.

CFT2s (and CFTds) : two kinds of algebras

In the mathematics of CFT2s, there are two kinds of algebras:

the symmetry algebras : infinite dimensional Lie algebras - Virasoro, current etc.

Algebra of quantum fields - formalized through vertex operator algebras.

Analogous to constructions in non-commutative geometry: a quantum space (e.g. a q-sphere, which is an associative q-deformed coordinate algebra) and a Hopf algebra (e.g. $U_q su(2)$) acting as a symmetry of the quantum space.

Expect similar in CFTd, except finite dimensional symmetry algebra (e.g. SO(d,2) instead of Virasoro) and large multiplicities of irreps coming from the fields/quantum states.

TFT3, TFT2 and algebras

Rational CFT2s were related to Chern Simons Topological field theory.

Topological : no dependence on the metric, e.g. action uses wedge products of forms but not the metric.

Mathematical (Axiomatic) definitions of topological field theories (Atiyah). Some of the simplest TFTs were two dimensional TFT (TFT2) – essentially algebras – associative finite dimensional algebras (with an additional non-degeneracy condition) – Frobenius algebras.

CFT operators : Algebras and quantum states

And important property of CFT (any d) is the Operator-state correspondence in radial quantization.

$$\lim_{x \to 0} \mathcal{O}_a(x)|0> = |\mathcal{O}_a>$$

In AdS/CFT, e.g.

$$\text{Strings in } AdS_5 \times S^5 \leftrightarrow \mathcal{N} = 4 \quad SYM$$

a good understanding of the quantum states (and associated physics) in the quantum gravity on AdS requires good undeerstanding of the CFT operators, and associated algebraic structures.

Quantum states in AdS

In string theory on $AdS_5 \times S^5$, quantum states come from a variety of constructions:

Gravitons - Fluctuations of the metric field (+ form fields, fermions etc.) around the $AdS_5 \times S^5$ solution of the stringy Einstein equations. These are in short (half-BPS) multiplets of supersymmetry.

Brane configurations (e.g. half-BPS giant graviton branes) and their quantum fluctuations.

Moduli spaces of supergravity solutions with $AdS_5 \times S^5$ asymptotics.

Non-overlapping regimes of validity.

Quantum states in CFT4

All these different types of states comes from local operators in CFT4.
CFT4 is $\mathcal{N} = 4$ SYM with $U(N)$ gauge group, with N arbitrary.
Given local operators $\mathcal{O}_a$ of scaling dimension Δ_a and a correlator

$$\langle \mathcal{O}_{a_1} \mathcal{O}_{a_2} \mathcal{O}_{a_3} \rangle$$

We can study different regimes e.g.

$$\begin{aligned} \Delta_a \sim 1 \text{ as } N \to \infty \quad &\Longrightarrow \quad \text{gravitons} \\ \Delta_a \sim N \text{ as } N \to \infty \quad &\Longrightarrow \quad \text{Branes} \\ \Delta_a \sim N^2 \text{ as } \text{ as } N \to \infty \quad &\Longrightarrow \quad \text{geometries} \end{aligned}$$

Combinatorics of correlators captured by TFT2

The interpretation and separation of these different regimes relies on detailed understanding of the dependence on correlators on N, Δ_a – an active and ongoing field ...

A sequence of many papers, led to the understanding that this combinatorics of dependences of correlators on Δ_a, N can be expressed in terms of TFT2 based on symmetric groups S_n where n is related to the Δ_a.

E.g. Corley, Jevicki, Ramgoolam (2001) ; Berenstein(2004); de Mello Koch, Smolic, Smolic (2006) ; Brown, Heslop, Ramgoolam (2007), Bhattacharyya, Collins, de Mello Koch (2008); Mattioli,Ramgoolam (2014); Kimura (2014) and others ... see short review in proceedings of Corfu2015 - arXiv:1605.00843 “Permutations and the combinatorics of gauge invariants for general N”

Can TFT2 capture the space-time dependence of correlators in CFT4 (and CFTd $d > 4$?

"CFT4 as SO(4,2)-invariant TFT2"
R. de Mello Koch and S. Ramgoolam arXiv:1403.6646 [hep-th] (NPB890)

We gave a positive answer for the case of free scalar field, along with free vector and matrix fields in 4D.

OUTLINE OF TALK

- The CFT4/TFT2 construction for free scalar field: $U(so(4,2))$ equivariant TFT2 and quantum field as a vertex operator.
- Ring structure in the TFT2 and construction of primaries.
- Further directions: conformal quantum field emergent from algebras and representation theory.

TFT2 - Axiomatic Approach

- Associate a vector space $\mathcal{H}$ to a circle - for explicit formulae choose basis e_A.
- Associate tensor products of $\mathcal{H}$ to disjoint unions.

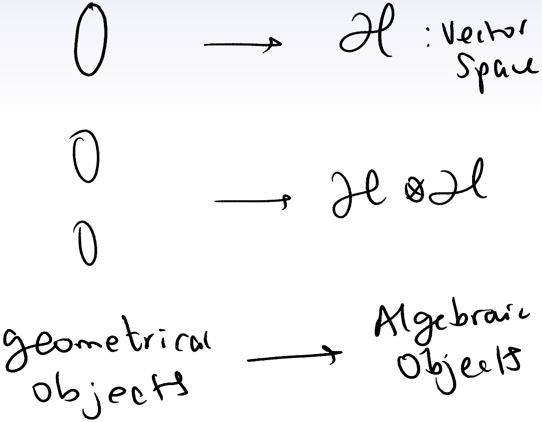

Figure: Geometrical and Algebraic Objects in TFT2

- Interpolating surfaces between circles (cobordisms) are associated with linear maps between the vector spaces.

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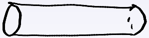


e_A e_B : $\delta_A{}^B$

$\delta : \mathcal{H} \to \mathcal{H}$

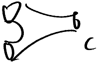
e_A

e_B

: η_{AB}

η = pairing

$\eta : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$

A
B


: $\underbrace{C_{AB}{}^{C}}_{\text{product}}$

$C : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$

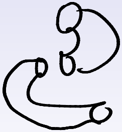
In math language, the circles are objects and interpolating surfaces (cobordisms) are morphisms in a geometrical category.

The vector spaces are objects, and linear maps are morphisms in an algebraic category.

The correspondence is a functor.

All relations in the geometrical side should be mirrored in the algebraic side.

: $\tilde{\eta}^{BC}$



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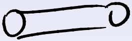
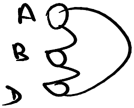
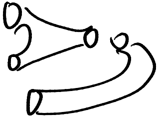


$$\eta_{AB}\,\tilde{\eta}^{BC} = \delta_A{}^C$$

Non-degeneracy.

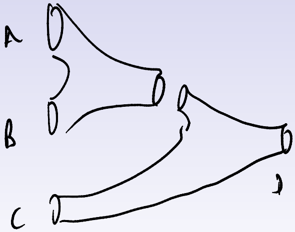


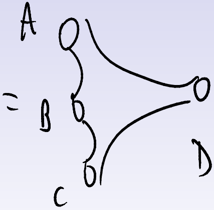
$= C_{ABD} =$

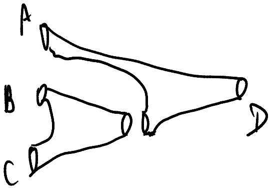


$$= C_{AB}{}^{C}\,\eta_{CD}$$

Figure: Non-degeneracy







2 gluings to give
same cobordism

$\Downarrow$

$$C_{AB}^{E}\, C_{EC}^{D} = C_{BC}^{E}\, C_{EA}^{D}$$

Associativity

Figure: Associativity

To summarise, TFT2's correspond to commutative, associative, non-degenerate algebras - known asFrobenius algebras.

TFT2 with global symmetry group G defined by Moore-Segal.

• The state space is a representation of a group G - which will be $SO(4,2)$ in our application.

• The linear maps are G-equvariant linear maps.

• State space is infinite dimensional : amplitudes are defined for surfaces without handles. This is a genus restricted TFT2.

$$\begin{tikzcd} V \arrow[r, "\eta"] \arrow[d, "g"'] & W \arrow[d, "g"] \\ V \arrow[r, "\eta"'] & W \end{tikzcd}$$

$$g \circ \eta = \eta \circ g$$

Special case : $W = \mathbb{C}$

- trivial rep:
- invariant rep

$\Rightarrow$ $g \circ \eta = \eta$

The map η is invariant.

η is an invt. map $V \otimes V \to \mathbb{C}$

Figure: G-equivariance

Invariant linear maps and the basic CFT4 2-point function

- Free massless scalar field theory in four dimensions.

- The basic two-point function

$$\langle \phi(x_1)\phi(x_2) \rangle = \frac{1}{(x_1 - x_2)^2}$$

- All correlators of composite operators are constructed from this using Wick contractions.

Ths theory has $SO(4,2)$ symmetry - here starting point. Lie algebra spanned by $D, P_\mu, M_{\mu\nu}, K_\mu$ - Scaling operator, translations, $SO(4)$ rotations, and special conformal transformations.

• In radial quantization, we choose a point, say origin of Euclidean R^4, and we

$$
\begin{aligned}
&Lim_{x\to 0}\phi(x)|0>= v^+\\
&Lim_{x\to 0}\partial_\mu\phi(x)|0>= P_\mu v^+\\
&\vdots
\end{aligned}
$$

• The state v^+ is the lowest energy state, in a lowest-weight representation.

$$
\begin{aligned}
&Dv^+ = v^+\\
&K_\mu v^+ = 0\\
&M_{\mu\nu} v^+ = 0
\end{aligned}
$$

Higher energy states are generated by $S_l^{\mu_1\cdots\mu_l} P_{\mu_1}\cdots P_{\mu_l} v^+$, where S_l is a symmetric traceless tensor of $SO(4)$.

There is a dual representation V_-, which is a representation with negative scaling dimensions.

$$\begin{aligned} Dv^- &= -v^- \\ K_\mu v^- &= 0 \\ M_{\mu\nu} v^- &= 0 \end{aligned}$$

Other states are generated by acting with $K \cdots K$.

There is an invariant map $\eta : V_+ \otimes V_- \to \mathbb{C}$.

$$\eta(\mathcal{L}_a v, w) + \eta(v, \mathcal{L}_a w) = 0$$

$$\eta(v^+, v^-) = 1$$

The invariance condition determines η, e.g

$$\eta(P_\mu v^+, K_\nu v^-) = -\eta(v^+, P_\mu K_\nu v^-)$$
$$= \eta(v^+, (-2D\delta_{\mu\nu} + 2M_{\mu\nu})v^-) = 2\delta_{\mu\nu}$$

Using invariance conditions once finds that $\eta(P_\mu P_\mu v^+, v)$ is a null state. Setting this state to zero (imposing EOM), defines a quotient of a bigger representation $\tilde{V}_+$ which is the irreducible V_+. And makes η non-degenerate : no null vectors.

So we see that η is teh kind of thing we need for TFT2 with $SO(4,2)$ symmetry. It has the non-degeneracy property and the invariance property.

Before relating this to the 2-point function, let us define a closely related quantity by taking the second field to the frame at infinity.

$$x_2' = \frac{x_2}{x_2^2}$$

$$\langle \phi(x_1)\phi'(x_2') \rangle = x_2^2 \langle \phi(x_1)\phi(x_2) \rangle = \frac{1}{(1 - 2x_1 \cdot x_2' + x_1^2 x_2'^2)} \equiv F(x_1, x_2')$$

Now to link CFT4 to TFT2, calculate

$$\eta(e^{-iP\cdot x} v^+, e^{iK\cdot x_2} v^-)$$

by using invariance and commutation relations as outlined above.

and find

$$\eta(e^{-iP\cdot x} v^+, e^{iK\cdot x_2'} v^-) = F(x_1, x_2)$$

So there is an invariant in $V_+ \otimes V_-$ and thus in $V_- \otimes V_+$, but not in $V_+ \otimes V_+$ or $V_- \otimes V_-$. It is useful to introduce $V = V_+ \oplus V_-$ and define $\eta : V \otimes V \to \mathbb{C}$.

$$\eta = \begin{pmatrix} 0 & \eta_{+-} \\ \eta_{-+} & 0 \end{pmatrix}$$

In V we have a state (THE QUANTUM FIELD)

$$\Phi(x) = \frac{1}{\sqrt{2}} e^{-iP\cdot x} v^+ + x'^2 e^{iK\cdot x'} v^-$$

so that

$$\eta(\Phi(x_1), \Phi(x_2)) = \frac{1}{(x_1 - x_2)^2}$$

This is the basic free field 2-point function, now constructed from the invariant map $\eta : V \otimes V \to \mathbb{C}$. The factor of 2 because $\eta(-,+)$ and $\eta(+,-)$ both contribute the same answer.

The TFT2 state space and amplitudes for CFT4 correlators

To get ALL CORRELATORS, we must set up a state space, which knows about composite operators.

- The states obtained by the standard operator state correspondence from general local operators are of the form

$$P_{\mu_1} \cdots P_{\mu_{n_1}} \phi \;\; P_{\mu_1} \cdots P_{\mu_{n_2}} \phi \;\; \cdots P_{\mu_1} \cdots P_{\mu_{n_m}} \phi$$

- Particular linear combinations of these are primary fields which are lowest weight states that generate irreducible representations (irreps) of $SO(4,2)$ through action of the raising operators.

• The list of primary fields in the *n*-field sector is obtained by decomposing into irreps the space

$$Sym(V_+^{\otimes n})$$

For the state space $\mathcal{H}$ of the TFT2 - which we associate to a circle in TFT2, we take

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} Sym(V^{\otimes n})$$

where $V = V_+ \oplus V_-$.

This state space is

- big enough to accommodate all the composite operators
- and admit an invariant pairing,
- small enough for the invariant pairing to be non-degenerate

Recall

$$\Phi(x) = \frac{1}{\sqrt{2}}(e^{-iP\cdot x}v^{+} + x'^{2}e^{iK\cdot x'}v^{-})$$

The state space contains

$$\Phi(x) \otimes \Phi(x) \otimes \cdots \Phi(x)$$

which is used to construct composite operators in the TFT2 set-up.

• The pairing $\boldsymbol{\eta} : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ is constructed so as to be able to reproduce all the 2-point functions of arbitrary composite operators.

• The $\mathcal{H}$ is built from tensor products of V.

• The $\boldsymbol{\eta}$ is built from products of the elementary η, using Wick contraction sums.

$$V = V^{(0)} + V^{(1)} + V^{(2)} + V^{(3)} + \cdots$$

$$\qquad\qquad\qquad\qquad V^{(1)} \in V, \quad V^{(2)} \in \mathrm{Sym}(V^{\otimes 2}), \quad V^{(3)} \in \mathrm{Sym}(V^{\otimes 3})$$

$$\eta(v^{(i)}, v^{(j)}) \propto \delta^{ij}$$

For $\eta(v^{(1)}, v^{(1)})$ use $\hat{\eta}$ defined before;

$$\eta(v^{(2)}, v^{(2)}) :$$

$$= \eta(v_1 \otimes v_2, v_3 \otimes v_4)$$

$$= \hat{\eta}(v_1, v_3)\, \hat{\eta}(v_2, v_4) + \hat{\eta}(v_1, v_4)\, \hat{\eta}(v_2, v_3)$$

i.e. Apply Wick Contraction sums to generate products of the elementary $\hat{\eta}$.

Figure: Wick patterns for pairing

This defines the pairing η_{AB} where A, B take values in the space $\mathcal{H}$ - sum of all n-fold symmetric prducts of $V = V_+ \oplus V_-$.

The building blocks are invariant maps, so the product of these invariant maps is also invariant.

This is shown to be non-degenerate. Basically if you have a non-degenerate pairing $V \otimes V \to \mathbb{C}$, it extends to a non-degenerate pairing on $\mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ – by using he sum over Wick patterns.

Hence

$$\eta_{AB}\tilde{\eta}^{BC} = \delta_A^C$$

The snake-cylinder equation.

Similarly can define 3-point functions

$$C_{ABC}$$

and higher

$$C_{ABC\ldots}$$

using Wick pattern products of the basic η's

By writing explicit formulae for these sums over Wick patterns, we can show that the associativity equations are satisfied.

The C_{ABC} give 3-point functions. The $C_{AB}^{C} = C_{ABD}\tilde{\eta}^{DC}$ give OPE-coefficients. And the associativity equations of the TFT2 are the crossing equations of CFT4 - which are obtained by equating expressions for a 4-point correlator obtained by doing OPEs in two different ways.

An important property of the OPE in this language, illustrated

$$Sym^2(V) \otimes Sym^2(V) \rightarrow Sym^4(V) \oplus Sym^2(V) \oplus \mathbb{C}$$

which corresponds, schematically, to

$$\phi^2(x)\phi^2(0) \rightarrow \phi^4 \oplus \phi^2 \oplus 1$$

Presence of V_+, V_- important in order to construct this in rep theory.

PART 2: The ring structures in the state space of the TFT2

The state space in radial quantization is

$$\bigoplus_{n=0}^{\infty} Sym^n(V_+)$$

V_+ is isomorphic to a space of polynomials in variables x_μ, quotiented by the ideal generated by $x_\mu x_\mu$.

The decomposition of this into irreducible representations is usefully done by recognizing that this is a problem about rings (as we saw in Robert's talk). We will do a quick recap and compare to the correlator discussion we just had.

Rings

Consider the polynomial ring $\mathbb{C}[x^l_\mu]$ in $n \times d$ variables, with

$$1 \leq \mu \leq d$$
$$1 \leq l \leq n$$

This is a ring : has addition and multiplication. It is also a vector space over $\mathbb{C}$.

Define "Lowest weight polynomials" as those $f \in \mathbb{C}[x^l_\mu]$, obeying

$$\sum_{l=1}^{n} \frac{\partial f}{\partial x^l_\mu} = 0 \quad \text{for all } \mu$$
$$\sum_{\mu=1}^{d} \frac{\partial^2 f}{\partial x^l_\mu x^l_\mu} = 0 \quad \text{for all } l$$

The ring $\mathbb{C}[x_\mu^I]$ is a representation of S_n.

$$\sigma : \{1, 2, \cdots, n\} \to \{\ \sigma(1), \sigma(2), \cdots, \sigma(n)\ \}$$

Polynomials transform as

$$f(x_\mu^I) \to f(x_\mu^{\sigma(I)})$$

It will be interesting to consider S_n invariant LWPs - which obey the LWP equations and

$$f(x_\mu^{\sigma(I)}) = f(x_\mu^I)$$

Problem: Counting and explicitly constructing of primary operators for general d, n.

Has been understood for a while for $n = 2$, and has found applications in higher spin theory (esp. $d = 3$).

Result Primary fields for general n are in 1-1 correspondence with S_n invariant lowest weight polynomials.

This correspondence implies several counting and construction algorithms for primary fields.

Ring structure of LWPs

The space of lowest weight polynomials is isomorphic, as a graded vector space, to a quotient ring:

$$\mathbb{C}[x_\mu^I]/\langle \sum_I x_\mu^I, \sum_\mu x_\mu^I x_\mu^I \rangle$$

This is a quotient ring

$$\mathcal{R}/\mathcal{I}$$

where $\mathcal{I}$ is the ideal generated by quadratic polynomials $q_1, \cdots, q_n$ and linear $\ell_1, \cdots, \ell_d$. $\mathcal{I}$ is the set

$$\sum_{I=1}^{n} r_I q_I + \sum_{\mu=1}^{d} \tilde{r}_\mu \ell_\mu$$

where $r_I, \tilde{r}_\mu \in \mathcal{R}$.

If we drop the linear constraint, we have a ring which corresponds to all the states in $\mathrm{Sym}^n(V_+)$.

Ring structure versus OPE algebra

In the ring structure described above, we have a product which closes on fixed number of fields. Exists at fixed n in $Sym^n(V_+)$. The product is a map

$$Sym^n(V_+) \otimes Sym^n(V_+) \to Sym^n(V_+)$$

In the operator product expansion, we have

$$Sym^{n_1}(V) \otimes Sym^{n_2}(V) \to \otimes Sym^{n_1+n_2}(V) \oplus Sym^{n_1+n_2}(V) \oplus \cdots$$

There are TWO products on the space

$$\bigoplus_{n=0}^{\infty} Sym^n(V)$$

PART 3: Future directions and open problems.

Will be interesting to explore the role of these two products. In the combinatoric aspects of correlators (the dependences on Δ_a, N), observables are related to properties of

$$\bigoplus_{n=0}^{\infty} \mathbb{C}(S_n)$$

In this case, again there are two products. One within each n. Another "outer product"

$$\mathbb{C}(S_{n_1}) \otimes \mathbb{C}(S_{n_2}) \rightarrow \mathbb{C}(S_{n_1+n_2})$$

These two products, denoted as $\sigma_1.\sigma_2$ and $\sigma_1 \circ \sigma_2$, are used to express the dependence of correlators on the N, Δ_a.

And there are "coherence relations" of the form

$$(\sigma_1 \circ \sigma_2).(\sigma_3 \circ \sigma_4) = (\sigma_1.\sigma_2) \circ (\sigma_2 \circ \sigma_4)$$

In the CFT4/TFT2 of primaries :
- are there any analogous coherence relations ? which would be useful in organising the correlators of the operators constructed with the help of of the ring structure.

Beyond free fields, can we extend the CFT4/TFT2 correspondence can be extended to perturbative CFTs – i.e. CFT in weak coupling expansions.

Preliminary results in ongoing work (with Robert) suggest : Yes.

Strongly coupled CFTs ??

Another longer term goal for the CFT4/TFT2 program ... Can we get the same TFT2 as $\mathcal{N} = 4$ SYM from string theory on AdS_5. Could be an avenue for better understanding of AdS/CFT.