Principal Chiral Model: T-dualities and Doubling

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with V. E. Marotta and P. Vitale

[arXiv:1903.01243 [hep-th]]

JHEP **1808**(2018)185 [arXiv:1804.00744 [hep-th]]

and F. Bascone

[arXiv:1904.03727 [hep-th]] + work in progress

Workshop on Recent Developments in Strings and Gravity - Corfu

September 15, 2019

Plan of the talk

1. Introduction and Motivation

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2. Principal Chiral Model
3. A Drinfel'd Double

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- By exchanging momentum and winding modes, it implies that the short distance behavior is governed by long distance behavior in the dual torus $\tilde{T}^d$.
- The indefinite orthogonal group $O(d, d)$ naturally appears in the Hamiltonian description of the bosonic string in the target space M with two peculiar structures, the generalized metric $\mathcal{H}$ and the $O(d, d)$ invariant metric η.

Generalized Metric and $O(d,d)$ Invariant Metric

- Generalized vector A_P in $TM \bigoplus T^*M$

$$A_P(X) = \partial_\sigma X^a \frac{\partial}{\partial x^a} + 2\pi\alpha' P_a dx^a \qquad P_a = \frac{\partial L}{\partial(\partial_\tau X^a)}$$

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$$H = \frac{1}{4\pi\alpha'} \begin{pmatrix} \partial_\sigma X \\ 2\pi\alpha' P \end{pmatrix}^t \mathcal{H}(G,B) \begin{pmatrix} \partial_\sigma X \\ 2\pi\alpha' P \end{pmatrix}$$

where the *generalized metric* is introduced:

$$\mathcal{H}(G,B) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}$$

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Constraints and $O(d,d)$ invariant Metric

- In terms of A_P the constraints

$$G_{ab}(\dot{X}^a\dot{X}^b + X'^a X'^b) = 0 \quad ; \quad G_{ab}\dot{X}^a X'^b = 0$$

become $\quad A_P^t \mathcal{H} A_P = 0 \quad A_P^t \eta A_P = 0$.

- The first sets H to zero ; the second completely determines the dynamics, rewritten in terms of the $O(d,d)$ invariant metric: $\eta = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$.

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- All the admissible generalized vectors satisfying $A_P^t\eta A_P = 0$ are related by an $O(d,d)$ transformation via $A'_P = \mathcal{T} A_P$.
- For A'_P to solve the first constraint, a compensating transformation $\mathcal{T}^{-1}$ has to be applied to $\mathcal{H}$. , i.e. $\mathcal{H}' = (\mathcal{T}^{-1})^t\mathcal{H}(\mathcal{T}^{-1})$.

- $\mathcal{H}$ and its inverse $\mathcal{H}^{-1}$ can be rewritten in products:

$$\mathcal{H}(G,B) = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}$$

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- This indeed shows that the background B can be created from the G-background through a B-transformation.

$O(D, D)$ AND CONSTANT BACKGROUNDS

- On T^d (G_{ab} and B_{ab} constant) with its isometries $U(1)^d$, the e.o.m.'s for the string coordinates are a set of conservation laws on the world-sheet:

$$\partial_\alpha J^\alpha_a = 0 \qquad J^\alpha_a = h^{\alpha\beta} G_{ab} \partial_\beta X^b + \epsilon^{\alpha\beta} B_{ab} \partial_\beta X^b h \equiv \epsilon^{\alpha\beta} \partial_\beta \tilde{X}_a$$

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- Through the use of auxiliary fields, one gets the dual Polyakov action $\tilde{S}[\tilde{X}; \tilde{G}, \tilde{B}]$ on $\tilde{T}^d$ with string coordinates $\tilde{X}_a$ and connected to $S[X; G, B]$ by $X^a \to \tilde{X}_a$ and suitable transformations of (G, B) through the Büscher rules.

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- *Abelian T-duality*: based on the presence of global Abelian isometries in the target spaces of both the paired sigma models.

T-dual invariant bosonic string formulation

- *Doubling* the coordinates, i.e. putting *both* the coordinates X^a *and* the dual ones $\tilde{X}_a$ in a *generalized* vector $\mathbb{X}^A = (X^a, \tilde{X}_a)$ $(A = 1, \ldots 2d), (a = 1, \ldots d)$ it is natural to replace the standard notation in string theory based on G and B by η and $\mathcal{H}$.
- Tseytlin action for constant backgrounds (A. A. Tseytlin, 1990 and 1991) highlights the role of the generalized vector $\mathbb{X}$ and of the two metrics:

$$S = -\frac{T}{2} \int d\tau d\sigma \left[\partial_\tau \mathbb{X}^A \partial_\sigma \mathbb{X}^B \eta_{AB} - \partial_\sigma \mathbb{X}^A \partial_\sigma \mathbb{X}^B \mathcal{H}_{AB} \right]$$

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- First example of *Born geometry*.

What about more general settings?

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- The *Poisson-Lie T-duality* [Klimčik and Severa, 1996] generalizes the previous definitions to all the other cases.

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- Also it allows to establish more connections with Generalized Geometry since tangent and cotangent vector fields of G may be respectively related to the span of the Lie algebra $\mathfrak{g} anditsdual \tilde{\mathfrak{g}}$.
- The Doubled Geometry may play a role in describing the generalized dynamics on the tangent bundle $TD \simeq D \times \mathfrak{d}$ used to describe within a single action both dually related models.

Drinfel'd double

- The Drinfel'd double of G: Lie group D, with dimension twice the one of G. Its Lie algebra $\mathfrak{d}$ can be decomposed into a pair of maximally isotropic sub-algebras, $\mathfrak{g}, \tilde{\mathfrak{g}}$ with respect to a non-degenerate invariant bilinear form on $\mathfrak{d}$.

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- The set of all decompositions $(\mathfrak{d}, \mathfrak{g}, \tilde{\mathfrak{g}})$, plays the role of the modular space of sigma models mutually connected by an $O(d,d)$ transformation.
- In particular, for the manifest Abelian T-duality of the string model on the d-torus, the Drinfel'd double is $D = U(1)^{2d}$ and its modular space, is in one-to-one correspondence with $O(d,d;\mathbb{Z})$.

SU(2) Principal Chiral Model

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- Action written in terms of $\mathrm{g}^{-1}\mathrm{d}g$, the Lie algebra valued left-invariant one-forms with $g \in SU(2)$:

$$S = \frac{1}{4}\int_{\mathbb{R}^2} \mathrm{d}t\mathrm{d}\sigma \; Tr\left[(g^{-1}\partial_t g)^2 - (g^{-1}\partial_\sigma g)^2\right]$$

trace $\rightarrow$ scalar product on the Lie algebra $\mathfrak{su}(2)$, non-degenerate and invariant.

$$S = \frac{1}{2}\int_{\mathbb{R}^2} \mathrm{d}t\mathrm{d}\sigma \; (A^i\delta_{ij}A^j - J^i\delta_{ij}J^j)$$

currents A^i and J^i: $g^{-1}\partial_t g = 2A^i e_i, g^{-1}\partial_\sigma g = 2J^i e_i$, $e_i = \sigma_i/2$.

$$A^i = \mathrm{Tr}\left[(g^{-1}\partial_t g)e_i\right] \qquad J^i = \mathrm{Tr}\left[(g^{-1}\partial_\sigma g)e_i\right]$$

- Equations of motion:

$$\partial_t A = \partial_\sigma J,$$
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- At fixed t, all g's constant at infinity form an infinite dimensional Lie group $SU(2)(\mathbb{R}) \equiv \mathrm{Map}(\mathbb{R}, SU(2))$, given by smooth maps $g : \sigma \in \mathbb{R} \rightarrow g(\sigma) \in SU(2)$.
- At fixed time, the currents J and A take values in the Lie algebra $\mathfrak{su}(2)(\mathbb{R})$ of the group $SU(2)(\mathbb{R})$.

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- Infinitesimal generators of the Lie algebra $\mathfrak{su}(2)(\mathbb{R})$:

$$X_i(\sigma) = X_i^a(\sigma)\frac{\delta}{\delta g^a(\sigma)},$$

with Lie brackets:

$$[X_i(\sigma), X_j(\sigma')] = c_{ij}{}^k X_k(\sigma)\delta(\sigma - \sigma')$$

defining the *current algebra* $\mathfrak{su}(2)(\mathbb{R}) \simeq \mathfrak{su}(2) \otimes C^\infty(\mathbb{R})$.

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- (J^i, I_i) conjugate variables with J configuration space coordinates and I left generalized momenta:

$$I_i = \frac{\delta L}{\delta \left(g^{-1}\partial_t g\right)^i} = \delta_{ij}(g^{-1}\partial_t g)^j = \delta_{ij} A^j \quad .$$

- Hamiltonian:

$$H = \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}\sigma (I_i I_j \delta^{ij} + J^i J^j \delta_{ij}) = \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}\sigma \; I_I \; (\mathcal{H}_0^{-1})^{IJ} \; I_J$$

  $I_I = (I_i, J^i)$ components of the current 1-form on $T^*SU(2)$ and

$$(\mathcal{H}_0^{-1})^{IJ} = \begin{pmatrix} \delta^{ij} & 0 \\ 0 & \delta_{ij} \end{pmatrix}$$

  is a Riemannian metric on $T^*SU(2)$.

- The Hamiltonian description of the Principal Chiral Model on $SU(2)$ naturally involves the Riemannian *generalized* metric $\mathcal{H}_0^{-1}$ on the cotangent bundle.

- The first-order Lagrangian, together with the canonical one-form and the symplectic form, allows to determine the equal-time Poisson brackets :

$$
\begin{aligned}
\{I_i(\sigma), I_j(\sigma')\} &= \epsilon_{ij}{}^{k} I_k(\sigma)\delta(\sigma-\sigma') \\
\{I_i(\sigma), J^j(\sigma')\} &= \epsilon_{ki}{}^{j} J^k(\sigma)\delta(\sigma-\sigma') - \delta_i^j \delta'(\sigma-\sigma') \\
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$$

- The I's generators of the Lie algebra $\mathfrak{su}(2)(\mathbb{R})$.
- The J's span an Abelian algebra $\mathfrak{a}$.
- I and J span the infinite-dimensional current algebra $\mathfrak{c}_1 = \mathfrak{su}(2)(\mathbb{R}) \ltimes \mathfrak{a}$.

- Equations of motion:

$$\partial_t I_j(\sigma) = \{H, I_j(\sigma)\} = \partial_\sigma J^k \delta_{kj}(\sigma),$$

and

$$\partial_t J^j(\sigma) = \{H, J^j(\sigma)\} = \partial_\sigma I_k \delta^{kj}(\sigma) - \epsilon^{\,jl}{}_k I_l J^k(\sigma).$$

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- It is possible to give an equivalent description of the dynamics in terms of a new one-parameter family of Poisson algebras [Rajeev, 1989] and modified Hamiltonians, with the currents playing a symmetric role.
- Deformed brackets by parameter τ (imaginary):

$$\begin{aligned}
\{I_i(\sigma), I_j(\sigma')\} =& (1-\tau^2)\epsilon_{ij}{}^k I_k(\sigma)\delta(\sigma-\sigma') \\
\{I_i(\sigma), J^j(\sigma')\} =& (1-\tau^2)\big(J^k(\sigma)\epsilon_{ki}{}^j\delta(\sigma-\sigma') - (1-\tau^2)^2\delta_i^j\delta'(\sigma-\sigma')\big) \\
\{J^i(\sigma), J^j(\sigma')\} =& (1-\tau^2)\tau^2\epsilon^{ij}{}_k I_k(\sigma)\delta(\sigma-\sigma').
\end{aligned}$$

- The new brackets correspond to the infinite-dimensional Lie algebra $\mathfrak{c}_2 \simeq \mathfrak{sl}(2,\mathbb{C})(\mathbb{R})$ isomorphic to the current algebra modelled on the Lorentz algebra $\mathfrak{sl}(2,\mathbb{C})$.

- Modified Hamiltonian:

$$H_\tau = \frac{1}{2(1-\tau^2)^2}\int_{\mathbb{R}} \mathrm{d}\sigma \, \left(I_i I_j \delta^{ij} + J^i J^j \delta_{ij}\right).$$

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- The previous equations of motion remain unmodified.
- Alternative description of one and the same dynamics even considering deformed algebras of $SL(2,\mathbb{C})$.
- Rescale the fields according to:

$$\frac{I_i}{1-\tau^2} \to I_i \quad \frac{J^i}{1-\tau^2} \to J^i.$$

The rescaled Hamiltonian H_τ becomes identical to the undeformed one H, while the Poisson algebra acquires the form:

$$\begin{aligned}
\{I_i(\sigma), I_j(\sigma')\} &= \epsilon_{ij}{}^k I_k(\sigma)\delta(\sigma-\sigma'), \\
\{I_i(\sigma), J^j(\sigma')\} &= J^k(\sigma)\epsilon_{ki}{}^j \delta(\sigma-\sigma') - \delta_i^j \delta'(\sigma-\sigma'), \\
\{J^i(\sigma), J^j(\sigma')\} &= \tau^2 \epsilon^{ij}{}_k I_k(\sigma)\delta(\sigma-\sigma').
\end{aligned}$$

- Introduce new generators showing the bi-algebra structure of $\mathfrak{sl}(2,\mathbb{C})(\mathbb{R})$.
- Keeping the generators of $\mathfrak{su}(2)(\mathbb{R})$ unmodified, consider the linear combination:

$$K^i(\sigma) = J^i(\sigma) - i\tau\epsilon^{li3} I_l(\sigma).$$

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together with

$$\begin{aligned}\{I_i(\sigma), I_j(\sigma')\} &= \epsilon_{ij}{}^k I_k(\sigma)\delta(\sigma-\sigma') \\ \{I_i(\sigma), K^j(\sigma')\} &= \left(K^k(\sigma')\epsilon_{ki}{}^j + i\tau f^{jk}{}_i I_k(\sigma')\right)\delta(\sigma-\sigma') - \delta_i^j\delta'(\sigma-\sigma')\end{aligned}$$

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- The Lie algebra $\mathfrak{c}_2 \equiv \mathfrak{sl}(2,\mathbb{C})(\mathbb{R})$ has been expressed as

$SL(2,\mathbb{C})$ as a Drinfel'd Double

- The Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is defined by the Lie brackets:

$$[e_i, e_j] = i\epsilon_{ij}{}^k e_k \quad [e_i, b_j] = i\epsilon_{ij}{}^k b_k \quad [b_i, b_j] = -i\epsilon_{ij}{}^k e_k .$$

with

$$e_1 = \frac{\sigma_1}{2}, \quad e_2 = \frac{\sigma_2}{2}, \quad e_3 = \frac{\sigma_3}{2} \quad \text{generators of } \mathfrak{su}(2)$$

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- It is equipped with two non-degenerate invariant scalar products:

$$\langle u, v\rangle = 2\mathrm{Im}(\mathrm{Tr}(uv)) \quad \forall u, v \in \mathfrak{sl}(2,\mathbb{C})$$

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- $\langle u, v\rangle$ defines two maximally isotropic subalgebras
$$[e_i, e_j] = i\epsilon_{ij}{}^k e_k, \quad [\tilde{e}^i, e_j] = i\epsilon_{jk}{}^i \tilde{e}^k + ie_k f^{ki}{}_j, \quad [\tilde{e}^i, \tilde{e}^j] = if^{ij}{}_k \tilde{e}^k$$
spanned by $\{e_i\}$ and the linear combination $\tilde{e}^i = b_i - \epsilon_{ij3} e_j$.

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- $\mathfrak{su}(2)$ and $\mathfrak{sb}(2,\mathbb{C})$ maximally isotropic with respect to $<,>$:

Manin Triple

- Each subalgebra acts on the other one non-trivially, by co-adjoint action:

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- $SL(2,\mathbb{C})$ is Drinfel'd double of $SU(2)$ and $SB(2,\mathbb{C})$ with polarization $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \bowtie \mathfrak{sb}(2,\mathbb{C})$ and $(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{su}(2), \mathfrak{sb}(2,\mathbb{C}))$ is a Manin triple.
- $SU(2)$ and $SB(2,\mathbb{C})$ are then dual groups.

- *Doubled* notation:

$$e_I = \begin{pmatrix} e_i \\ \tilde{e}^i \end{pmatrix}, \qquad e_i \in \mathfrak{su}(2), \quad \tilde{e}^i \in \mathfrak{sb}(2,\mathbb{C}) \ .$$

The first scalar product then becomes:

$$(e_I, e_J) = \eta_{IJ} = \begin{pmatrix} 0 & \delta_i^j \\ \delta_j^i & 0 \end{pmatrix}$$

which is $O(3,3)$ invariant by construction. The second scalar product yields:

$$(e_I, e_J) = \begin{pmatrix} \delta_{ij} & \epsilon_{ip3}\delta^{pj} \\ \delta^{ip}\epsilon_{jp3} & \delta^{ij} - \epsilon^{ik3}\delta_{kl}\epsilon^{jl3} \end{pmatrix} \ .$$

- The splitting $\mathfrak{d} = C_+ \oplus C_-$ with C_+, C_- spanned by $\{e_i\}$, $\{b_i\}$ respectively, defines a positive definite metric $\mathcal{H}$ on $\mathfrak{d}$ via:

$$\mathcal{H} = (\ ,\ )_{C_+} - (\ ,\ )_{C_-}$$

satisfying

$$\mathcal{H}^T \eta \mathcal{H} = \eta$$

namely $\mathcal{H}$ is a pseudo-orthogonal $O(3,3)$ metric.

- A whole family of models exists, described by the Hamiltonians labelled by the parameter τ in terms of $I_J = (I_j, K^j)$:

$$H_\tau = \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}\sigma \, I_L (\mathcal{H}_\tau^{-1})^{LM} I_M$$

with $\mathcal{H}_\tau^{-1}$ being the *Riemannian generalized metric*

$$\mathcal{H}_\tau^{-1} = \begin{pmatrix} h^{ij}(\tau) & i\tau\epsilon^{ip3}\delta_{pj} \\ i\tau\delta_{ip}\epsilon^{jp3} & \delta_{ij} \end{pmatrix}$$

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- They are related (and indeed equivalent) to the standard $SU(2)$ chiral model by the $O(3,3)$ transformation $K^i(\sigma) = J^i(\sigma) - i\tau\epsilon^{li3} I_l(\sigma)$, symmetry of the dynamics because it maps solutions into solutions .

Born Geometry

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- $T^*SU(2)$ is a *Born manifold*: a phase space equipped with a structure $(\eta, \kappa, \mathcal{H}_0)$ with $\kappa \in \mathrm{End}(\mathfrak{su}(2) \ltimes \mathbb{R}^3)$ such that $\kappa^2 = \mathbb{1}$ with $\mathfrak{su}(2)$ eigenspace of κ associated with the eigenvalue $+1$ and $\mathbb{R}^3$ eigenspace with the eigenvalue -1. The structures $< \cdot \,,\, \cdot >$ and κ satisfy a compatibility condition
$$< \kappa(\xi), \psi >= - < \kappa(\psi), \xi >, \qquad \forall \xi, \psi \in \mathfrak{su}(2) \ltimes \mathbb{R}^3,$$
which defines a two-form ω on $\mathfrak{su}(2) \ltimes \mathbb{R}^3$.

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  which defines a two-form ω on $\mathfrak{su}(2) \ltimes \mathbb{R}^3$.
- Defining relations:

$$\eta^{-1}\mathcal{H}_0 = \mathcal{H}_0^{-1}\eta \qquad \omega^{-1}\mathcal{H}_0 = -\mathcal{H}_0^{-1}\omega.$$

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- The family of equivalent Hamiltonian descriptions of the $SU(2)$ PCM can be understood in terms of a one-parameter family of Born geometries for $T^*SU(2)$, corresponding, for each choice of the parameter τ, to a specific splitting of the phase space, with the value $\tau = 0$ the canonical splitting.

Poisson-Lie structure

- The PCM, in the Hamiltonian formulation given by any H_τ is a Poisson-Lie sigma-model.

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- The PCM, in the Hamiltonian formulation given by any H_τ is a Poisson-Lie sigma-model.
- Hamiltonian vector fields $X_{K^i} \cdot := \{\cdot, K^i\}$ associated to the coordinates functions K^i for $T^*SU(2)$, close a non-Abelian algebra according to the following:

$$[X_{K^i}, X_{K^j}] = X_{\{K^i, K^j\}} = i\tau f_k^{ij} X_{K^k}$$

  because of the non-trivial Poisson bracket $\{K^i(\sigma), K^j(\sigma')\} = i\tau f^{ij}{}_k K^k(\sigma')\delta(\sigma - \sigma') \rightarrow$ the constant structures of the dual Lie algebra $\mathfrak{sb}(2, \mathbb{C})$ appear.
- A dual formulation of this property can be given in terms of the Hamiltonian vector fields associated with the currents I_i that close the Lie algebra $\mathfrak{su}(2)$.

Poisson-Lie dual models

- Two-parameter generalization in the Poisson algebra generated by I_i and K^i by adding another imaginary parameter α making the role of the subalgebras $\mathfrak{su}(2)(\mathbb{R})$ and $\mathfrak{sb}(2,\mathbb{C})(\mathbb{R})$ symmetric:

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\begin{aligned}
\{I_i(\sigma), I_j(\sigma')\} &= i\alpha\, \epsilon_{ij}{}^k I_k(\sigma)\delta(\sigma-\sigma') \\
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- In the limit $i\tau \to 0$, it reproduces the semi-direct sum $\mathfrak{su}(2)(\mathbb{R}) \ltimes \mathfrak{a}$, while the limit $i\alpha \to 0$ yields $\mathfrak{sb}(2,\mathbb{C})(\mathbb{R}) \ltimes \mathfrak{a}$.

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- In the limit $i\tau \to 0$, it reproduces the semi-direct sum $\mathfrak{su}(2)(\mathbb{R}) \ltimes \mathfrak{a}$, while the limit $i\alpha \to 0$ yields $\mathfrak{sb}(2,\mathbb{C})(\mathbb{R}) \ltimes \mathfrak{a}$.
- For all non zero values of the two parameters, the algebra is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$, and, by suitably rescaling the fields, one gets a two-parameter family of models, all equivalent to the PCM.

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$$\begin{aligned}\{I_i(\sigma), I_j(\sigma')\} &= i\alpha\, \epsilon_{ij}{}^k I_k(\sigma)\delta(\sigma-\sigma') \\ \{K^i(\sigma), K^j(\sigma')\} &= i\tau f^{ij}{}_k K^k(\sigma')\delta(\sigma-\sigma')\end{aligned}$$

$$\{I_i(\sigma), K^j(\sigma')\} = \left(i\alpha K^k(\sigma')\epsilon_{ki}{}^j + i\tau f^{jk}{}_i I_k(\sigma')\right)\delta(\sigma-\sigma') - \delta_i^j \delta'(\sigma-\sigma')$$

- In the limit $i\tau \to 0$, it reproduces the semi-direct sum $\mathfrak{su}(2)(\mathbb{R}) \ltimes \mathfrak{a}$, while the limit $i\alpha \to 0$ yields $\mathfrak{sb}(2,\mathbb{C})(\mathbb{R}) \ltimes \mathfrak{a}$.
- For all non zero values of the two parameters, the algebra is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$, and, by suitably rescaling the fields, one gets a two-parameter family of models, all equivalent to the PCM.

- The dynamics derived from this algebra is equivalent to the dynamics following from the original underfomed algebra on $T^*SU(2)$ and the undeformed Hamiltonian, if the new Hamiltonian is considered:

$$H_{\tau\alpha} = \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}\sigma\, I_L(\mathcal{H}^{-1}_{\tau,\alpha})^{LM} I_M .$$

with

$$\mathcal{H}_{\tau,\alpha}{}^{-1} = \begin{pmatrix} \frac{h^{ij}(\tau)}{(i\alpha)^2} & i\tau\epsilon^{ip3}\delta_{pj} \\ i\tau\delta_{ip}\epsilon^{jp3} & (i\alpha)^2\delta_{ij} \end{pmatrix}$$

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- Since the role of I and K is now symmetric, one can perform an $O(3,3)$ transformation which exchanges the momenta I_i with the fields K^i, thus obtaining a new two-parameter family of models, *dual* to the PCM.

- The $O(3,3)$ transformation

$$\tilde{K}(\sigma) = I(\sigma), \quad \tilde{I}(\sigma) = K(\sigma)$$

when applied to $H_{\tau\alpha}$ and to the corresponding Poisson algebra leads to a new family of models having target space configuration the group manifold of $SB(2,C)$ spanned by the fields $\tilde{K}_i$ and momenta $\tilde{I}^i$. These are the *Dual Principal Chiral Models*.

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- The Dual Principal Chiral Models are Poisson-Lie sigma models.

Principal Chiral Model $SB(2, C)$ - Lagrangian approach

- Lagrangian for the PCM $SB(2, C)$ involving $\tilde{g} : (t, \sigma) \to SB(2, C)$, one-forms valued in the Lie algebra $\mathfrak{sb}(2, \mathbb{C})$ and $\mathcal{T}r$ non-degenerate product only invariant under $SB(2, C)$ action:

$$\tilde{S} = \frac{1}{2} \int_{\mathbb{R}^{1,1}} \mathcal{T}r \left[\phi^*(\tilde{g}^{-1} d\tilde{g}) \wedge \phi^*(\tilde{g}^{-1} d\tilde{g}) \right]$$

with $\quad \tilde{\phi}^*(\tilde{g}^{-1} d\tilde{g}) = (\tilde{g}^{-1}\partial_t \tilde{g})_i \tilde{e}^i \, dt + (\tilde{g}^{-1}\partial_\sigma \tilde{g})_i \tilde{e}^i \, \mathrm{d}\sigma$

$$\tilde{A}_i = (\tilde{g}^{-1}\partial_t \tilde{g})_i \quad , \quad \tilde{J}_i = (\tilde{g}^{-1}\partial_\sigma \tilde{g})_i$$

The Lagrangian becomes then:

$$\tilde{L} = \frac{1}{2} \int_{\mathbb{R}} d\sigma (\tilde{A}_i h^{ij} \tilde{A}_j - \tilde{J}_i h^{ij} \tilde{J}_k)$$

- At fixed t, all elements $\tilde{g}$ constant at the infinity form the infinite-dimensional Lie group $SB(2, \mathbb{C})(\mathbb{R}) \equiv \mathrm{Map}(\mathbb{R}, SB(2, \mathbb{C}))$, given by smooth maps $\tilde{g} : \sigma \in \mathbb{R} \to \tilde{g}(\sigma) \in SB(2, \mathbb{C})$ which are constant at infinity.

Principal Chiral Model $SB(2, C)$ - Hamiltonian approach

- By introducing left momenta $\tilde{I}^i = \tilde{A}_j h^{ij}$ and inverting for the generalized velocities, one obtains the Hamiltonian:

$$\tilde{H} = \frac{1}{2}\int_{\mathbb{R}} d\sigma \tilde{I}_I (\tilde{\mathcal{K}}_0^{-1})^{IJ} \tilde{I}_J$$

with

$$\tilde{\mathcal{K}}_0 = \begin{pmatrix} h^{ij} & 0 \\ 0 & h_{ij} \end{pmatrix}$$

and $\tilde{I}_J = (\tilde{I}^j, \tilde{J}_j)$.

- Equal-time Poisson brackets

$$\begin{aligned} \{\tilde{I}^i(\sigma), \tilde{I}^j(\sigma')\} =& f^{ij}{}_k \tilde{I}^k(\sigma)\delta(\sigma-\sigma'), \\ \{\tilde{I}^i(\sigma), \tilde{J}_j(\sigma')\} =& \tilde{J}_k(\sigma) f^{ki}{}_j \delta(\sigma-\sigma') - \delta^i_j \delta'(\sigma-\sigma'), \\ \{\tilde{J}_i(\sigma), \tilde{J}_j(\sigma')\} =& 0 \end{aligned}$$

- Dual Born geometry.

- Group valued field $\Phi : \mathbb{R}^{1,1} \to \gamma \in SL(2, \mathbb{C})$ and the left-invariant Maurer-Cartan one-form $\gamma^{-1}\mathrm{d}\gamma \in \mathfrak{sl}(2, C)$:

$$\Phi^*(\gamma^{-1}\mathrm{d}\gamma) = \gamma^{-1}\partial_t\gamma\mathrm{d}t + \gamma^{-1}\partial_\sigma\gamma\mathrm{d}\sigma$$

The Doubled Principal Chiral Model

- Group valued field $\Phi : \mathbb{R}^{1,1} \to \gamma \in SL(2, \mathbb{C})$ and the left-invariant Maurer-Cartan one-form $\gamma^{-1}\mathrm{d}\gamma \in \mathfrak{sl}(2, C)$:

$$\Phi^*(\gamma^{-1}\mathrm{d}\gamma) = \gamma^{-1}\partial_t\gamma\mathrm{d}t + \gamma^{-1}\partial_\sigma\gamma\mathrm{d}\sigma$$

- By using the Lie algebra basis $e_I = (e_i, \tilde{e}^i)$ one has:

$$\gamma^{-1}\partial_t\gamma = \dot{\mathbf{Q}}^I e_I, \ ; \ \gamma^{-1}\partial_\sigma\gamma = \mathbf{Q}'^I e_I.$$

- $\dot{\mathbf{Q}}^I, \mathbf{Q}'^I$, left generalized coordinates, respectively given by:

$$\dot{\mathbf{Q}}^I = \mathrm{T}r\left(\gamma^{-1}\partial_t\gamma e_I\right), \quad \mathbf{Q}'^I = \mathrm{T}r\left(\gamma^{-1}\partial_\sigma\gamma e_I\right)$$

with $\mathrm{T}r$ the Cartan-Killing metric of $\mathfrak{sl}(2, \mathbb{C})$.

Lagrangian

- The Lagrangian density can be rewritten in terms of the left generalized coordinates $\dot{\mathbf{Q}}^I$ as follows:

$$\mathbf{L} = \frac{1}{2}(k\ \eta + \mathcal{H})_{IJ}\left(\dot{\mathbf{Q}}^I\dot{\mathbf{Q}}^J - \mathbf{Q}'^I\mathbf{Q}'^J\right)$$

with

$$(k\eta + \mathcal{H})_{IJ} = \begin{pmatrix} \delta_{ij} & k\delta_i^j + \epsilon_i^{j3} \\ k\delta_j^i - \epsilon^i_{j3} & (\delta^{ij} + \epsilon^i_{k3}\epsilon^j_{l3}\delta^{kl}) \end{pmatrix}$$

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- η (Lorentzian) and $\mathcal{H}$ (Riemannian) are the left-invariant metrics on $SL(2,\mathbb{C})$ induced, respectively, by the pairings $2\mathrm{ImTr}()$ and $2\mathrm{ReTr}()$ on $\mathfrak{sl}(2,\mathbb{C})$. They are two of the structures defining a Born geometry on $SL(2,\mathbb{C})$.
- The degrees of freedom are doubled. Performing a gauging of its global symmetries both the Lagrangian models, with $SU(2)$ and $SB(2,C)$ target configuration spaces, can be retrieved.

Hamiltonian Formalism

- Canonical momentum:

$$\mathbf{I}_I = (I_i, \tilde{I}^i) = \frac{\delta \mathbf{L}}{\delta \dot{\mathbf{Q}}^I} = (k\ \eta + \mathcal{H})_{IJ} \dot{\mathbf{Q}}^J .$$

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- The Legendre transform gives:

$$\mathbf{H} = \frac{1}{2} \int_{\mathbb{R}} \mathrm{d}\sigma \ \left([(k\,\eta + \mathcal{H})^{-1}]^{IJ} \mathbf{I}_I \mathbf{I}_J + (k\,\eta + \mathcal{H})_{IJ} \mathbf{J}^I \mathbf{J}^J \right).$$

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- Poisson brackets:

$$\begin{aligned}
\{\mathbf{I}_I(\sigma'), \mathbf{I}_J(\sigma'')\} &= C_{IJ}{}^K \mathbf{I}_K \delta(\sigma' - \sigma'') \\
\{\mathbf{I}_I(\sigma'), \mathbf{J}^J(\sigma'')\} &= C_{KI}{}^J \mathbf{J}^K \delta(\sigma' - \sigma'') - \delta_I^J \delta'(\sigma' - \sigma'') \\
\{\mathbf{J}^I(\sigma'), \mathbf{J}^J(\sigma'')\} &= 0
\end{aligned}$$

with $\mathbf{Q}'^I \rightarrow \mathbf{J}^I$

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- $O(3,3)$ transformations allow to find a parametric family of T-dual PCM models, with target configuration space the group $SB(2, \mathbb{C}$, the Poisson-Lie dual of $SU(2)$ in the Iwasawa decomposition of the Drinfel'd double $SL(2, \mathbb{C})$. They exhibit Poisson-Lie symmetries.

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- A further double PCM with the group manifold of $SL(2,\mathbb{C})$ has been constructed.

Outlook

- Extension with a Wess-Zumino term: this could provide a deeper insight on the geometric structures of String Theory on AdS_3.

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- Extension to String Theory on Group Manifolds.
- To apply this scheme of construction to the world-sheet action. In this case, a manifestly $O(d,d)$-invariant action may be written, considering that the configuration space is a differentiable manifold. It would be interesting to A doubled world-sheet string action, as discussed for Principal Chiral Models, could be written and then perform the low energy limit. This limit result should reproduce all the results so far obtained in Double Field Theory.

Thank you for your attention.