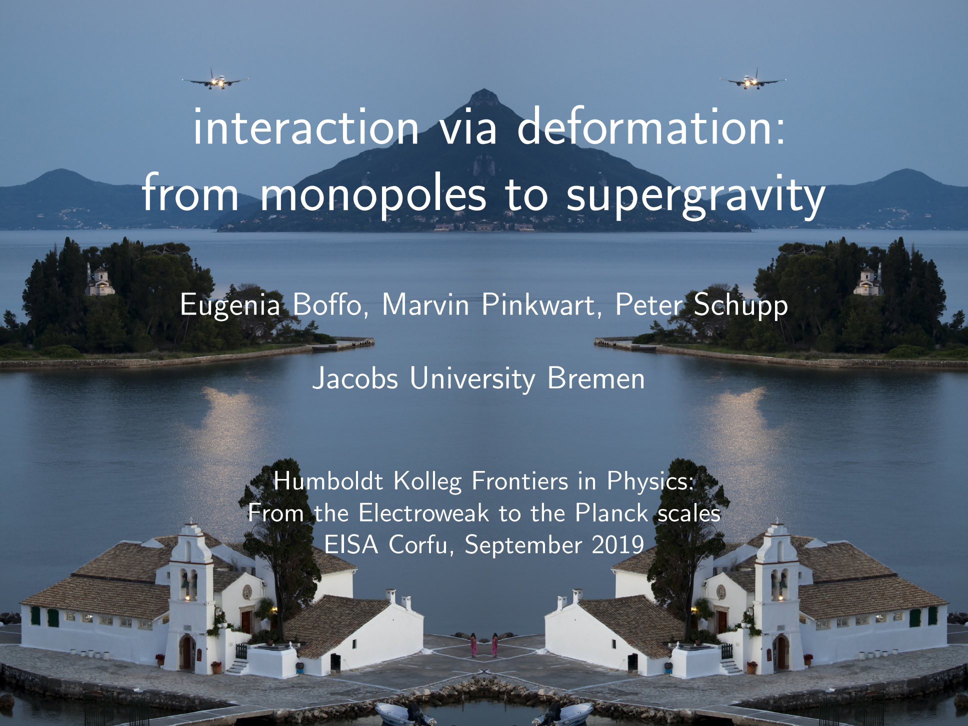


interaction via deformation: from monopoles to supergravity

Eugenia Boffo, Marvin Pinkwart, Peter Schupp

Jacobs University Bremen

Humboldt Kolleg Frontiers in Physics:
From the Electroweak to the Planck scales
EISA Corfu, September 2019

Outline

- Interaction via deformation, monopoles
- Graded (curved) spacetime mechanics
- Graded/generalized geometry and gravity
- Deformation, non-associativity, gravitipoles

Interaction via deformation

example: relativistic particle in einbein formalism

$$S = \int d\tau \left(\frac{1}{2e} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu - \frac{1}{2} e m^2 + A_\mu(x)\dot{x}^\mu \right) \rightsquigarrow p_\mu = \frac{1}{e} g_{\mu\nu}\dot{x}^\nu + A_\mu$$

$$S_H = \int p_\mu dx^\mu - \frac{1}{2} e \left((p_\mu - A_\mu)^2 + m^2 \right) d\tau \qquad \leftarrow p_\mu\text{: canonical momentum}$$

$$\boxed{S_H = \int (p_\mu + A_\mu)\, dx^\mu - \frac{1}{2} e \left(p_\mu^2 + m^2 \right) d\tau} \qquad \leftarrow p_\mu\text{: physical momentum}$$

$$\omega' = d\big(p_\mu + A_\mu\big) \wedge dx^\mu \quad \rightsquigarrow$$

$$\boxed{\{p_\mu, p_\nu\}' = F_{\mu\nu},\ \{x^\mu, p_\nu\}' = \delta^\mu_\nu,\ \{x^\mu, x^\nu\}' = 0}$$

$$\boxed{\{p_\lambda, \{p_\mu, p_\nu\}'\}' + \text{cycl.} = (dF)_{\lambda\mu\nu} = (*j_m)_{\lambda\mu\nu}} \qquad \leftarrow \text{magnetic 4-current}$$

magnetic sources $\Leftrightarrow$ non-associativity

Interaction via deformation

Quantization

- canonical? depends. . . (✓)
- deformation quantization ✓
- path integral ✓

Deformed CCR:

$$[p_\mu, p_\nu] = i\hbar F_{\mu\nu}, \quad [x^\mu, p_\nu] = i\hbar\delta^\mu_\nu, \quad [x^\mu, x^\nu] = 0, \quad [\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}$$

Let $\mathbf{p} = \gamma^\mu p_\mu$ and $H = \frac{1}{2}\mathbf{p}^2 \rightsquigarrow$ correct coupling of fields to spin

$$H = \frac{1}{8}\big([\gamma^\mu, \gamma^\nu]_+[p_\mu, p_\nu]_+ + [\gamma^\mu, \gamma^\nu][p_\mu, p_\nu]\big) = \frac{1}{2}p^2 - \frac{i\hbar}{2}S^{\mu\nu}F_{\mu\nu}$$

Lorentz-Heisenberg equations of motion (ignoring spin)

$$\dot{p}_\mu = \frac{i}{\hbar}[H, p_\mu] = \frac{1}{2}(F_{\mu\nu}\dot{x}^\nu + \dot{x}^\nu F_{\mu\nu}) \quad \text{with} \quad \dot{x}^\nu = \frac{i}{\hbar}[H, x^\nu] = p^\nu$$

this formalism allows $dF \neq 0$: magnetic sources, non-associativity

Interaction via deformation: monopoles

local non-associativity: $\frac{1}{3}[p_\lambda,[p_\mu,p_\nu]]\,dx^\lambda dx^\mu dx^\nu = \hbar^2 dF = \hbar^2 *j_m$

$j_m \neq 0 \Leftrightarrow$ no operator representation of the p_μ!

spacetime translations are still generated by p_μ, but magnetic flux Φ_m leads to path-dependence with phase $e^{i\phi}$; where $\phi = iq_e\Phi_m/\hbar$

globally:

$$\Phi_m = \int_S F = \int_{\partial S} A \qquad \leftrightarrow \text{ non-commutativity}$$

$$\Phi_m = \int_{\partial V} F = \int_V dF = \int_V *j_m = q_m \qquad \leftrightarrow \text{ non-associativity}$$

global associativity requires $\phi \in 2\pi\mathbb{Z} \;\Rightarrow \boxed{\frac{q_e q_m}{2\pi\hbar} \in \mathbb{Z}}$ Dirac quantization

non-relativistic version: Jackiw 1985, 2002

Graded spacetime mechanics

Graded Poisson algebra on $T^*[2] \oplus T[1]M$

$$\{\underset{1}{\theta^\mu}, \underset{1}{\theta^\nu}\} = 2\underset{0}{g^{\mu\nu}}(x) \qquad \{\underset{2}{p_\mu}, \underset{0}{x^\nu}\} = \underset{0}{\delta^\nu_\mu} \qquad \{p_\mu, f(x)\} = \partial_\mu f(x)$$

associativity/Jacobi identity $\Leftrightarrow$ metric connection

$$\{\underset{2}{p_\mu}, \underset{1}{\theta^\alpha}\} = \nabla_\mu \theta^\alpha = \Gamma^\alpha_{\mu\beta} \underset{1}{\theta^\beta}$$

$$\{p_\mu, \{\theta^\alpha, \theta^\beta\}\} = 2\partial_\mu g^{\alpha\beta} = \{\{p_\mu, \theta^\alpha\}, \theta^\beta\} + \{\theta^\alpha, \{p_\mu, \theta^\beta\}\}$$

and curvature

$$\{\{p_\mu, p_\nu\}, \theta^\alpha\} = [\nabla_\mu, \nabla_\nu]\theta^\alpha = \theta^\beta R_\beta{}^\alpha{}_{\mu\nu}$$

$$\Rightarrow \quad \{\underset{2}{p_\mu}, \underset{2}{p_\nu}\} = \tfrac{1}{4} \underset{1}{\theta^\beta} \underset{1}{\theta^\alpha} R_{\beta\alpha\mu\nu}$$

Graded spacetime mechanics

symmetries = canonical transformations

- generator of degree 2 (degree-preserving):

$$v^\alpha(x)p_\alpha + \tfrac{1}{2}\Omega_{\alpha\beta}(x)\theta^\alpha\theta^\beta \quad \rightsquigarrow \quad \text{local Poincare algebra}$$

- generators of degree 1:

$$V = V_\alpha(x)\theta^\alpha \quad \rightsquigarrow \quad \{V, W\} = 2g(V, W) \quad \text{Clifford algebra}$$

- generators of degree 3:

$$\Theta = \theta^\alpha p_\alpha \quad (+\tfrac{1}{6}C_{\alpha\beta\gamma}\theta^\alpha\theta^\beta\theta^\gamma)$$

- generators of degree 4:

$$H = \tfrac{1}{2}g^{\mu\nu}(x)p_\mu p_\nu + \tfrac{1}{2}\Gamma^\beta_{\mu\nu}(x)\theta^\mu\theta^\nu p_\beta + \tfrac{1}{16}R_{\alpha\beta\mu\nu}(x)\theta^\alpha\theta^\beta\theta^\mu\theta^\nu$$

$$\rightsquigarrow \quad \text{SUSY algebra} \quad \tfrac{1}{4}\{\Theta, \Theta\} = H$$

Graded spacetime mechanics

Graded Poisson algebra on $T^*[2]M \oplus T[1]M$: $\{p_\mu, x^\nu\} = \delta^\nu_\mu$

$$\{\theta^\mu, \theta^\nu\} = 2g^{\mu\nu}(x) \qquad \{p_\mu, \theta^\alpha\} = \Gamma^\alpha_{\mu\beta}\theta^\beta \qquad \{p_\mu, p_\nu\} = \tfrac{1}{2}\theta^\beta\theta^\alpha R_{\beta\alpha\mu\nu}$$

Equations of motion with Hamiltonian (Dirac op.) $\Theta = \theta^\mu p_\mu$

$$\frac{dA}{d\tau} = \tfrac{1}{2}\{\Theta, \{\Theta, A\}\} = \tfrac{1}{2}\{\{\Theta, \Theta\}, A\} - \tfrac{1}{2}\{\Theta, \{\Theta, A\}\} =: \{H, A\}$$

and derived Hamiltonian

$$H = \tfrac{1}{4}\{\Theta, \Theta\} = \tfrac{1}{2}g^{\mu\nu}p_\mu p_\nu + \tfrac{1}{2}\theta^\mu\theta^\nu\Gamma^\beta_{\mu\nu}p_\beta + \tfrac{1}{16}\theta^\alpha\theta^\beta\theta^\mu\theta^\nu R_{\alpha\beta\mu\nu}$$

For a torsion-less connection, only the first term is non-zero.

Derived anchor map applied to $V = V_\alpha(x)\theta^\alpha$:

$$h(V)f = \{\{V, \Theta\}, f\} = V_\alpha(x)g^{\alpha\beta}\partial_\beta f$$

Graded spacetime mechanics

Equations of motion (cont'd)

$$\frac{dx^\mu}{d\tau} = \tfrac{1}{2}\{\Theta, \{\Theta, x^\mu\}\} = \{\tfrac{1}{2} g^{\alpha\beta} p_\alpha p_\beta, x^\mu\} = g^{\mu\nu} p_\nu$$

$$\frac{dp_\nu}{d\tau} = \{\tfrac{1}{2} g^{\alpha\beta} p_\alpha p_\beta, p_\nu\} = \tfrac{1}{2}(\partial_\mu g^{\alpha\beta}) p_\alpha p_\beta = {}^{g}\Gamma_\mu{}^{\alpha\beta} p_\alpha p_\beta$$

with *any* metric-compatible connection ${}^{g}\Gamma$; pick a WB connection...

Geodesic equation:

$$\frac{d^2 x^\mu}{d\tau^2} = \{\tfrac{1}{2} g^{\alpha\beta} p_\alpha p_\beta, g^{\mu\nu} p_\nu\} = -\frac{dx^\alpha}{d\tau} {}^{LC}\Gamma_{\alpha\beta}{}^{\mu} \frac{dx^\beta}{d\tau}$$

Nice warmup example and useful in its own right. For the grander scheme with B-field, dilaton, stringy symmetries, we need to double up ...

Graded geometry

Graded Poisson manifold $T^*[2]T[1]M$

- degree 0: x^i "coordinates"
- degree 1: $\xi^\alpha = (\theta^i, \chi_i)$
- degree 2: p_i "momenta"

symplectic 2-form

$$\omega = dp_i \wedge dx^i + \frac{1}{2} G_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta = dp_i \wedge dx^i + d\chi_i \wedge d\theta^i$$

even (degree -2) Poisson bracket on functions $f(x, \xi, p)$

$$\{x^i, x^j\} = 0, \quad \{p_i, x^j\} = \delta_i^j, \quad \{\xi^\alpha, \xi^\beta\} = G^{\alpha\beta}$$

metric $G^{\alpha\beta}$: natural pairing of TM, T^*M:

$$\{\chi_i, \theta^j\} = \delta_i^j \ , \quad \{\chi_i, \chi_j\} = 0 \ , \quad \{\theta^i, \theta^j\} = 0$$

Graded geometry

degree-preserving canonical transformations

- infinitesimal, generators of degree 2:

$$v^\alpha(x)p_\alpha + \tfrac{1}{2}M^{\alpha\beta}(x)\xi_\alpha\xi_\beta \quad \rightsquigarrow \quad \text{diffeos and } o(d,d)$$

- finite, idempotent ("coordinate flip"): $(\tilde{\chi},\tilde{\theta}) = \tau(\chi,\theta)$ with $\tau^2 = \text{id}$

  $\rightsquigarrow$ generating function F of type 1 with $F(\theta,\tilde{\theta}) = -F(\tilde{\theta},\theta)$:

$$F = \theta\cdot g\cdot\tilde{\theta} - \tfrac{1}{2}\,\theta\cdot B\cdot\theta + \tfrac{1}{2}\,\tilde{\theta}\cdot B\cdot\tilde{\theta}$$

$$\chi = \frac{\partial F}{\partial\theta} = \tilde{\theta}\cdot g + \theta\cdot B\ , \qquad \tilde{\chi} = -\frac{\partial F}{\partial\tilde{\theta}} = \theta\cdot g + \tilde{\theta}\cdot B$$

$$\Rightarrow \quad \tau(\chi,\theta) = (\chi,\theta)\cdot\begin{pmatrix} g^{-1}B & g^{-1} \\ g - Bg^{-1}B & -Bg^{-1}\end{pmatrix}$$

  $\rightsquigarrow$ generalized metric

Generalized geometry as a derived structure

Cartan's magic identy:

$$\mathcal{L}_X = [i_X, d] \equiv i_X d + d\, i_X$$

Lie bracket $[X, Y]_{\text{Lie}}$ of vector fields as a derived bracket:

$$[[i_X, d], i_Y] = [\mathcal{L}_X, i_Y] = i_{[X,Y]_{\text{Lie}}} \quad \text{with } [d, d] = d^2 = 0$$

Generalized geometry as a derived structure

degree 3 "Hamiltonian": Dirac operator

$$\Theta = \xi^\alpha h^i_\alpha(x) p_i \; + \; \underbrace{\tfrac{1}{6} C_{\alpha\beta\gamma}\xi^\alpha\xi^\beta\xi^\gamma}_{\text{twisting/flux terms}}$$

For $e = e_\alpha(x)\xi^\alpha \in \Gamma(TM \oplus T^*M)$ (degree 1, odd):

- pairing: $\langle e, e' \rangle = \{e, e'\}$
- anchor: $h(e)f = \{\{e, \Theta\}, f\}$
- bracket: $[e, e']_D = \{\{e, \Theta\}, e'\}$

Generalized geometry as a derived structure

Courant algebroid axioms from associativity and $\{\Theta,\Theta\}=0$:

$$\begin{aligned} h(\xi_1)\left\langle \xi_2,\xi_2\right\rangle &= \{\{\Theta,\xi_1\},\{\xi_2,\xi_2\}\} \\ &= 2\{\{\{\Theta,\xi_1\},\xi_2\},\xi_2\} = 2\left\langle [\xi_1,\xi_2]\,,\xi_2\right\rangle \qquad \text{(axiom 1)} \\ &= 2\{\xi_1,\{\{\Theta,\xi_2\},\xi_2\}\} = 2\left\langle \xi_1,[\xi_2,\xi_2]\right\rangle \qquad \text{(axiom 2)} \end{aligned}$$

$$\begin{aligned} [\xi_1,[\xi_2,\xi_3]] &= \{\{\Theta,\xi_1\},\{\{\Theta,\xi_2\},\xi_3\}\} \\ &= [[\xi_1,\xi_2],\xi_3] + [\xi_2,[\xi_1,\xi_3]] + \frac{1}{2}\{\{\{\{\Theta,\Theta\},\xi_1\},\xi_2\},\xi_3\}. \end{aligned}$$

$$\{\Theta,\Theta\}=0 \quad \Leftrightarrow \quad [\,,]\text{-Jacobi identity} \qquad \text{(axiom 3)}$$

Generalized differential geometry

generalized Lie-bracket (involves anchor $h : E \to TM$)

$$[[V,W]] = -[[W,V]]\,, \quad [[V,fW]] = (h(V)f)W + f\,[[V,W]]$$

generalized connection and miracoulus identity

$$\nabla_V(fW) = (h(V)f)W + f\nabla_V W\,, \qquad \nabla_{fV}W = f\nabla_V W$$

$$\langle \nabla_V W, Z\rangle = \langle [W,Z], V\rangle - \langle [[W,Z]], V\rangle$$

generalized curvature and torsion

$$R(V,W) = \nabla_V\nabla_W - \nabla_W\nabla_V - \nabla_{[[V,W]]}$$

$$T(V,W) = \nabla_V W - \nabla_W V - [[V,W]]$$

Deformation

deformation by generalized vielbein E

$$\Omega = dx^i \wedge dp_i + d\theta^i \wedge d\chi_i$$

deformation by change of coordinates in the odd (degree 1) sector
two choices:

$$\begin{pmatrix}\theta\\ \chi\end{pmatrix} \mapsto \begin{pmatrix}1 & 0\\ g+B & 1\end{pmatrix} \cdot \begin{pmatrix}\theta\\ \chi\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}1 & \Pi + G\\ -g+B & 1\end{pmatrix} \cdot \begin{pmatrix}\theta\\ \chi\end{pmatrix}$$

Boffo, PS 1903.09112 and in preparation

now crank the “machine” (deformed derived bracket, connection, project, Riemann, Ricci) $\rightsquigarrow$ (effective) gravity actions . . .

Graded/generalized geometry and gravity

generalized Koszul formula for nonsymmetric $\mathcal{G} = g + B$

$$
\begin{aligned}
2g(\nabla_Z X, Y) &= \langle Z, [X, Y]' \rangle' \\
&= X\mathcal{G}(Y, Z) - Y\mathcal{G}(X, Z) + Z\mathcal{G}(X, Y) \\
&\quad -\mathcal{G}(Y, [X, Z]_{\mathrm{Lie}}) - \mathcal{G}([X, Y]_{\mathrm{Lie}}, Z) + \mathcal{G}(X, [Y, Z]_{\mathrm{Lie}}) \\
&= 2g(\nabla_X^{LC} Y, Z) + H(X, Y, Z)
\end{aligned}
$$

$\Rightarrow$ non-symmetric Ricci tensor

$$
R_{jl} = R_{jl}^{LC} - \frac{1}{2}\nabla_i^{LC} H_{jl}{}^i - \frac{1}{4} H_{lm}{}^i H_{ij}{}^m \qquad R = \mathcal{G}_{ij} g^{ik} g^{jl} R_{kl}
$$

$\Rightarrow$ gravity action (closed string effective action) after partial integration:

$$
S_{\mathcal{G}} = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g} \left(R^{LC} - \frac{1}{12} H_{ijk} H^{ijk} \right)
$$

Khoo, Vysoky, Jurco, Boffo; PS

Graded/generalized geometry and gravity

This formulation consistently combines all approaches of Einstein: Non-symmetric metric, Weitzenböck and Levi-Civita connections, without any of the usual drawbacks.

The dilaton $\phi(x)$ rescales the generalized tangent bundle. The deformation can be formulated in terms of vielbeins

$$E = e^{-\frac{\phi}{3}} \begin{pmatrix} 1 & 0 \\ g+B & 1 \end{pmatrix} \qquad E^{-1}\partial_i E = \begin{pmatrix} -\frac{1}{3}\partial_i\phi & 0 \\ \partial_i(g+B) & -\frac{1}{3}\partial_i\phi \end{pmatrix}$$

Going through the same steps as before we find in $d=10$

$$S = \frac{1}{2\kappa}\int d^{10}x\, e^{-2\phi}\sqrt{-g}\Big(R^{\mathrm{LC}} - \tfrac{1}{12}H^2 + 4(\nabla\phi)^2\Big)$$

Boffo, PS

More deformation

Link to gauge theory:
deformation of symplectic form $\Omega' \rightsquigarrow$ gauge field A

Moser's lemma

Let $\Omega_t = \Omega + tF$, with Ω_t symplectic for $t \in [0,1]$.

$d\Omega_t = 0 \;\Rightarrow\; dF = 0 \;\Rightarrow\;$ locally $F = dA$

$\Omega' \equiv \Omega_1$ and Ω are related by a change of phase space coordinates generated by the flow of a vector field V_t defined up to gauge transformations by the gauge field $i_{V_t}\Omega_s = A$, i.e. $V_t = \theta_s(A, -)$.

Proof: $\mathcal{L}_{V_t}\Omega_t = i_{V_t} d\Omega_t + d\, i_{V_t}\Omega_t = 0 + dA = \frac{d}{dt}\Omega_t$. Moser 1965

Quantum and Poisson versions of the lemma: Jurco, PS, Wess 2000-2002

More deformation

our initial example:

deformation by a gauge field A

$$\Omega' = dx^i \wedge dp_i + \tfrac{1}{2} F_{ij}(x) dx^i \wedge dx^j,\ dF = 0, \text{ locally } F = dA$$

$$\Omega_t = \Omega + t\, dA\,, \quad A = A_i(x) dx^i$$

$$V_t = A_i(x) \frac{\partial}{\partial p_i}\,, \quad \mathcal{L}_{V_t} \rightsquigarrow \rho_{[A]}(p) = p + A$$

$$\{p_i, x^j\}_t = \delta_i^j$$

$$\{p_i, p_j\}_t = t\, F_{ij}(x)$$

gauge transformation $\delta A = d\lambda \leftrightarrow \delta\rho_{[A]}$: canonical transformation

non-abelian versions: $A_i^\alpha(x)\ell_\alpha dx^i$ and $A_{ia}^b(x)\theta^a\chi_b dx^i$

$\rightsquigarrow$ Abelian and non-abelian gauge theory

More deformation

deformation by a spin connection ω

$$\Omega = dx^i \wedge dp_i + \tfrac{1}{2}\eta_{ab} d\theta^a \wedge d\theta^b \qquad \theta^a = e^a_i \theta^i \,, \quad g_{ij} = e^a_i e^b_j \eta_{ab}$$

$$\Omega_t = \Omega + t\, d\omega \,, \quad \omega = \omega_i(x,\theta) dx^i = \tfrac{1}{2}\omega_{iab}(x)\theta^a\theta^b dx^i$$

$$V_t = \omega_i \partial_{p_i} \,, \quad \mathcal{L}_{V_t} \rightsquigarrow \rho_{[\omega]}(p) = p + \omega$$

$$\{p_i, x^j\}_t = \delta^j_i \qquad \{\theta^a, \theta^b\}_t = \eta^{ab}$$

$$\{p_i, \theta^a\} = t\,\eta^{ab}\omega_{ibc}(x)\,\theta^c \qquad \omega_{ibc} = -\omega_{icb}$$

$$\{p_i, p_j\}_t = t\, R_{ij} \qquad R = d\omega + t\omega \wedge \omega$$

gauge transformation $\delta\omega = d\lambda \leftrightarrow \delta\rho_{[\omega]}$: canonical transformation

$\rightsquigarrow$ Einstein-Cartan gravity

More deformation

deformation by a general connection Γ

$$\Omega = dx^i \wedge dp_i + d\theta^i \wedge d\chi_i$$

$$\Omega_t = \Omega + t\, d\Gamma\,, \quad \Gamma = \Gamma_i dx^i = \Gamma_{ij}{}^k(x)\theta^j \chi_k dx^i$$

$$V_t = \Gamma_i \partial_{p_i}\,, \quad \mathcal{L}_{V_t} \rightsquigarrow \rho_{[\Gamma]}(p) = p + \Gamma$$

$$\{p_i, x^j\}_t = \delta_i^j \qquad \{\chi_i, \theta^j\}_t = \delta_i^j$$

$$\{p_i, \theta^j\} = t\,\Gamma^j_{ik}\theta^k \qquad \{p_i, \chi_j\} = -t\,\Gamma^k_{ij}\chi_k$$

$$\{p_i, p_j\}_t = t\, R_k{}^l{}_{ij}\theta^k \chi_l \qquad R_k{}^l{}_{ij} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{ik}\Gamma^l_{jm} - \Gamma^m_{jk}\Gamma^l_{im}$$

gauge transformation $\delta\Gamma = d\Lambda \leftrightarrow \delta\rho_{[\Gamma]}$: canonical transformation

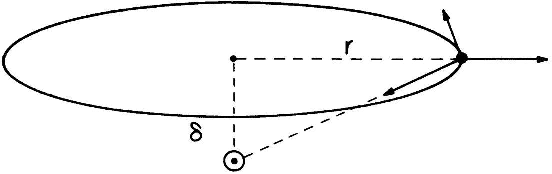
$\rightsquigarrow$ General relativity and alternative gravity theories

Non-associativity

The Jacobi identity playes a pivotal role; its violation has drastic effects:

- $\{p_\mu, \theta^\alpha, \theta^\beta\} \neq 0 \ \Rightarrow$ non-metricity of connection ∇
- $\{p_\alpha, p_\beta, p_\gamma\} \neq 0 \ \Rightarrow$ gravito-magnetic sources, mass quantization

Shifted orbit in the presence of a gravitipol:



Conclusion

- deformation: combines best aspects of Lagrange and Hamilton
- generalized geometry provides a perfect setting for the formulation of theories of gravity
- our approach is based on deformed graded geometry and is algebraic in nature: almost everything follows from associativity as unifying principle (which can be generalized)
- non-associativity $\Rightarrow$ non-metricity, gravitipoles, mass quantization

Thanks for listening!