Web construction for ABCDEFG and affine type quivers

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based on arXiv:1907.02382 JHEP09(2019)025
with T. Kimura (Keio U. -> Université de Bourgogne)

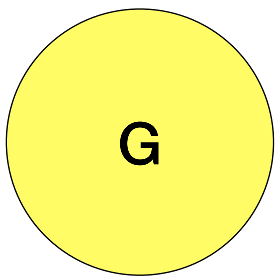
and arXiv:1709.01954 JHEP12(2017)015
with J.-E. Bourgine (KIAS), M. Fukuda (U. Tokyo), Y. Matsuo (U. Tokyo)

Plan of Talk

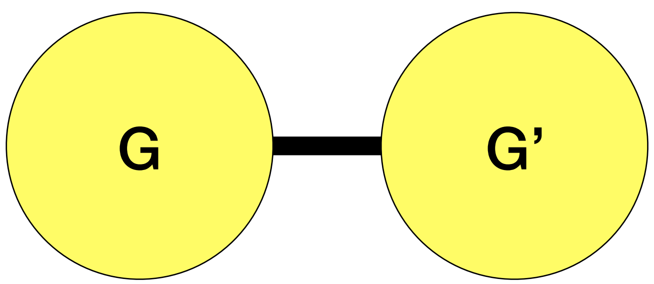
- Introduction, Motivation and review
- criteria: partition function and qq-character
- brane construction for D-type quiver
- web construction for BCFG-type quivers
- E-type construction and affine quivers

Quiver Structure of 5d N=1 Gauge Theories

field contents: vector multiplets and hypermultiplets.



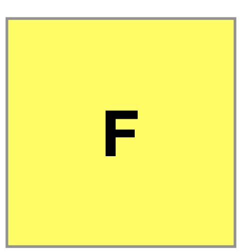
gauge node: a vector multiplet with gauge group G.



line connecting two gauge nodes:

bifundamental hypermultiplets

i.e. hypermultplets transforming in fundamental rep. of G and anti-fundamental rep. of G’

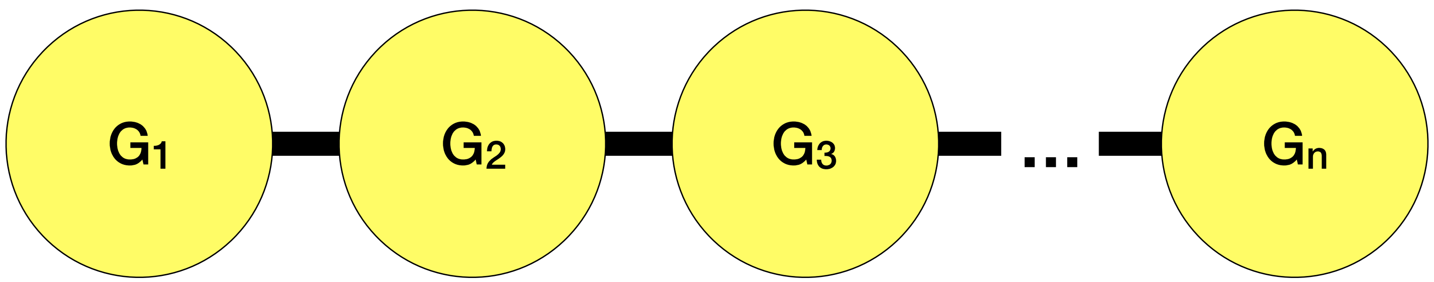


flavor symmetry of matter is usually represented by a box.

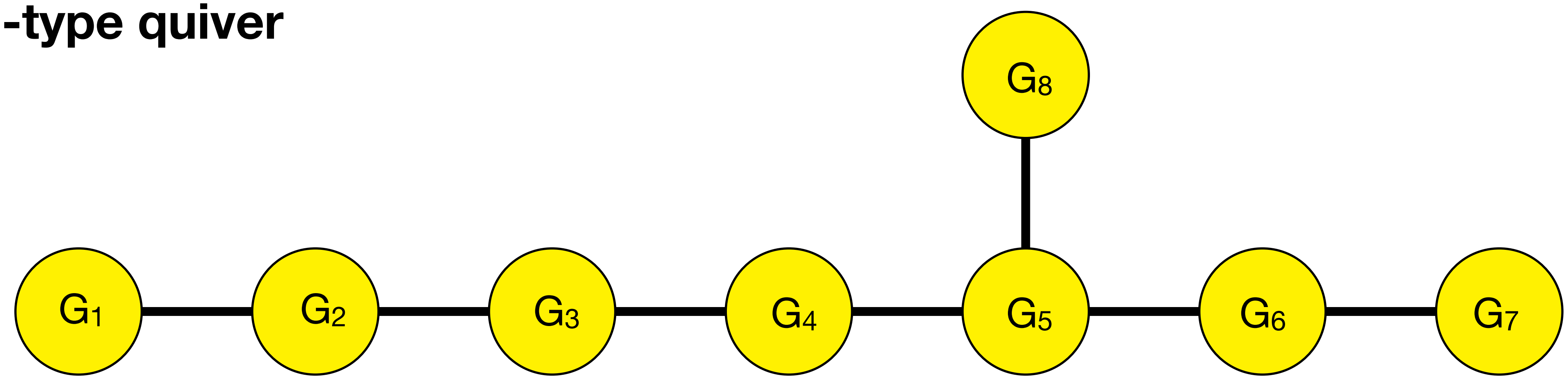
Quiver diagram → Dynkin diagram

~ quiver Lie algebra

A-type quiver



E-type quiver



Web of 5-branes

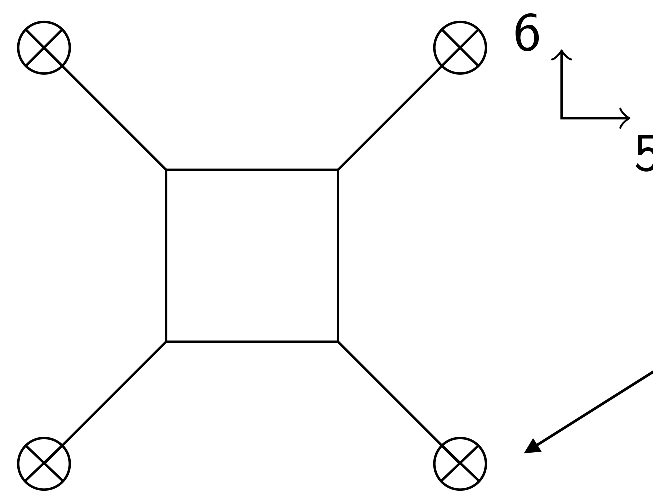
[Aharony, Hanany, Kol, 1997]

	0	1	2	3	4	5	6	7	8	9
D5	—	—	—	—	—	—	●	●	●	●
NS5	—	—	—	—	—	●	—	●	●	●
7-brane	—	—	—	—	—	●	●	—	—	—

all non-trivial information contained in this 2d plane

We draw a web diagram on this plane. (balance of tension ⇒ various kind of (p,q) 5-branes stretching along the vector (p,q))

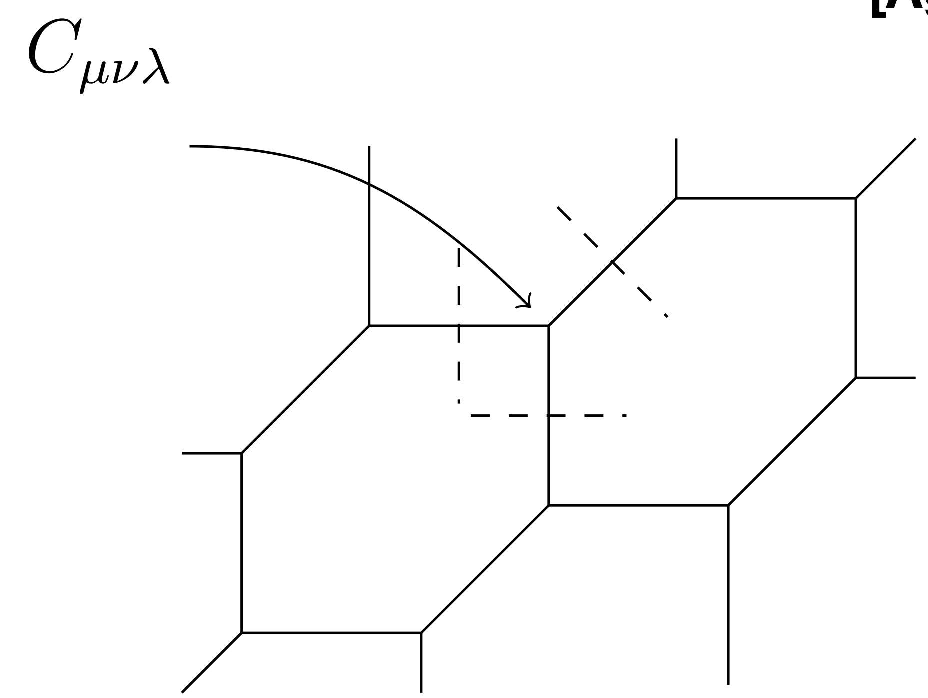
axio-dilaton charge



7-brane

The Nekrasov (instanton) partition function can be computed in a Feynman diagrammatic way with the topological vertex.

[Aganagic-Klemm-Marino-Vafa, 2003]



It can be expressed in terms of (skew) Schur functions.

$$C_{\mu,\nu,\lambda}(t,q) = q^{\frac{||\mu^t||^2}{2}} t^{-\frac{||\mu||^2}{2}} P_\lambda(t^{-\rho}, q, t) \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|+|\mu|-|\nu|}{2}} s_{\mu^t/\eta}(q^{-\lambda} t^{-\rho}) s_{\nu/\eta}(t^{-\lambda^t} q^{-\rho}).$$

λ: preferred direction

$$q_1 = e^{R\epsilon_1} = t, \quad q_2 = e^{R\epsilon_2} = q^{-1}.$$

[Awata, Kanno, 2005] [Iqbal, Kozcaz, Vafa, 2007]

It is known that the skew Schur function can be written in the form of matrix element of certain vertex operator.

$$s_{\lambda/\mu}(\vec{x}) = \langle\mu|\, V_+(\vec{x})\,|\lambda\rangle = \langle\lambda|\, V_-(\vec{x})\,|\mu\rangle\,,$$

where

$$V_\pm(\vec{x}) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\sum_i x_i^n J_{\pm n}\right),$$

$$|\lambda\rangle = (-1)^{\beta_1+\beta_2+\cdots+\beta_s+\frac{s}{2}}\psi^*_{-\beta_1}\psi^*_{-\beta_2}\ldots\psi^*_{-\beta_s}\psi_{-\alpha_s}\psi_{-\alpha_{(s-1)}}\ldots\psi_{-\alpha_1}\,|\mathrm{vac}\rangle\,,$$

$$\{\psi_n,\psi_m\} = \{\psi^*_n,\psi^*_m\} = 0,\quad \{\psi_n,\psi^*_m\} = \delta_{n+m,0},\quad J_n := \sum_{j\in\mathbb{Z}+1/2}\psi_{-j}\psi^*_{j+n},$$

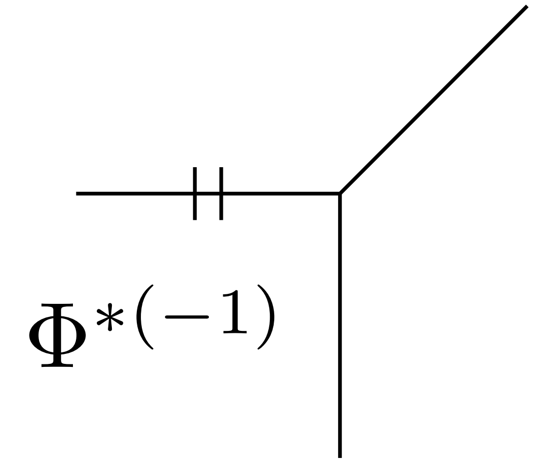
$$[J_n,\psi_k] = \psi_{n+k},\quad [J_n,\psi^*_k] = -\psi^*_{n+k},\quad [J_n,J_m] = n\delta_{n+m,0}.$$

The gluing of vertices in the unpreferred direction is just a correlator of vertex operators.

$$C_{\mu,\nu,\lambda}(t,q) \propto \sum_\eta s_{\mu^t/\eta}(t^{-\lambda}q^{-\rho+\{1/2\}})s_{\nu/\eta}(q^{-\lambda^t}t^{-\rho-\{1/2\}})$$

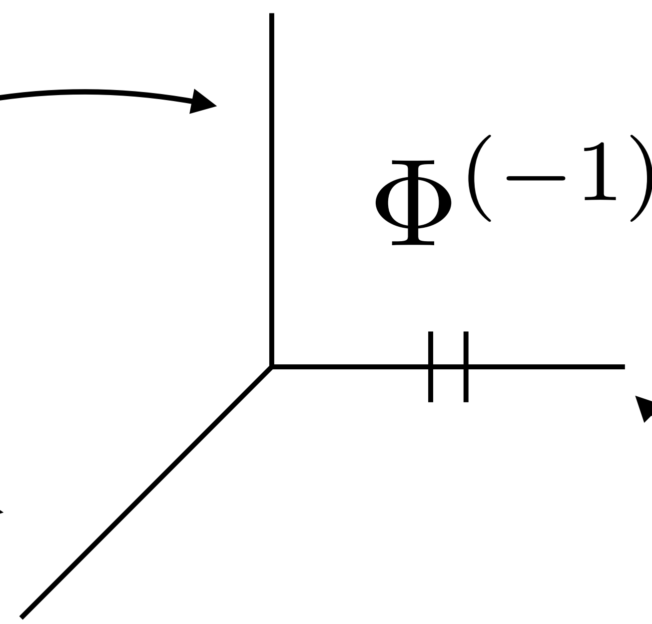
$$= \langle\mu^t|\, V_-(t^{-\lambda}q^{-\rho+\{1/2\}})V_+(q^{-\lambda^t}t^{-\rho-\{1/2\}})\,|\nu\rangle\,.$$

*Note that each brane (line) in the unpreferred direction corresponds to a Fock space.



*there are two species of vertices, classified by its leg in the preferred direction.

where vertex operators $V_{\pm}(\vec{x})$ correlates.



preferred direction
(specified by a Young diagram λ)

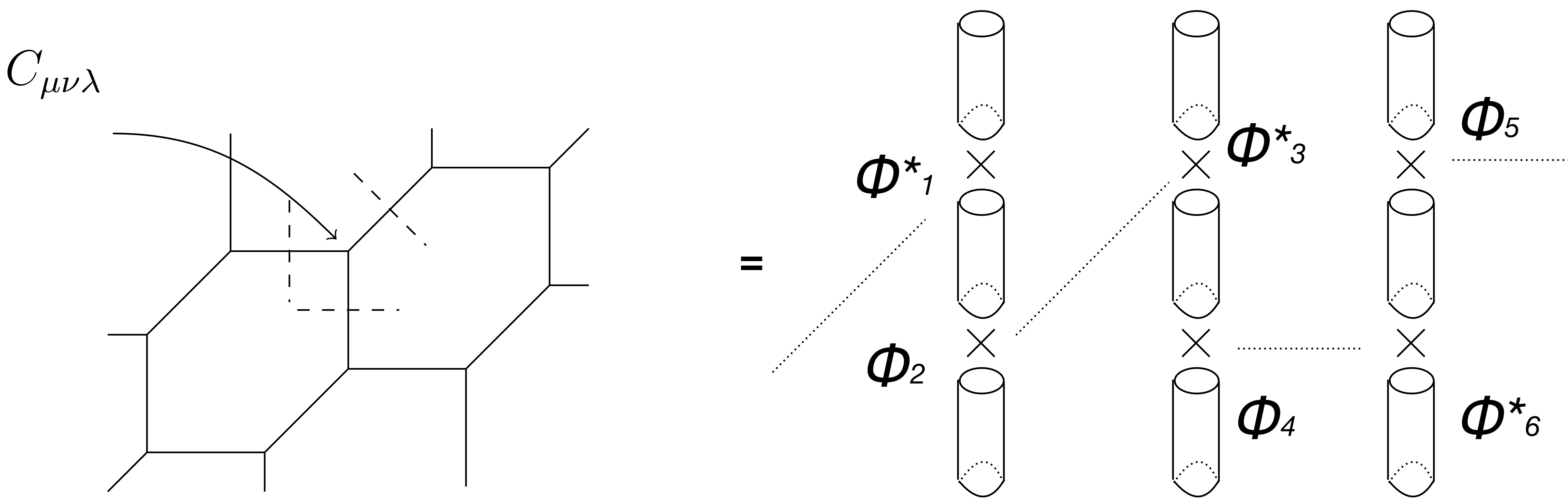
*when evaluated in the Schur basis, we obtain the IKV refined vertex.

$$\Phi^{(-1)} \left|\lambda\right\rangle = V_{-}(t^{-\lambda} q^{-\rho+\{1/2\}}) V_{+}(q^{-\lambda^t} t^{-\rho-\{1/2\}})$$

5d N=1 gauge theory on S^1 x Ω-background

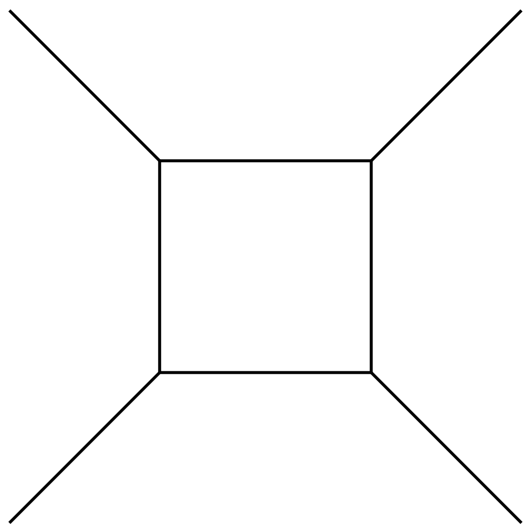
=

A web of vertex operators glued together

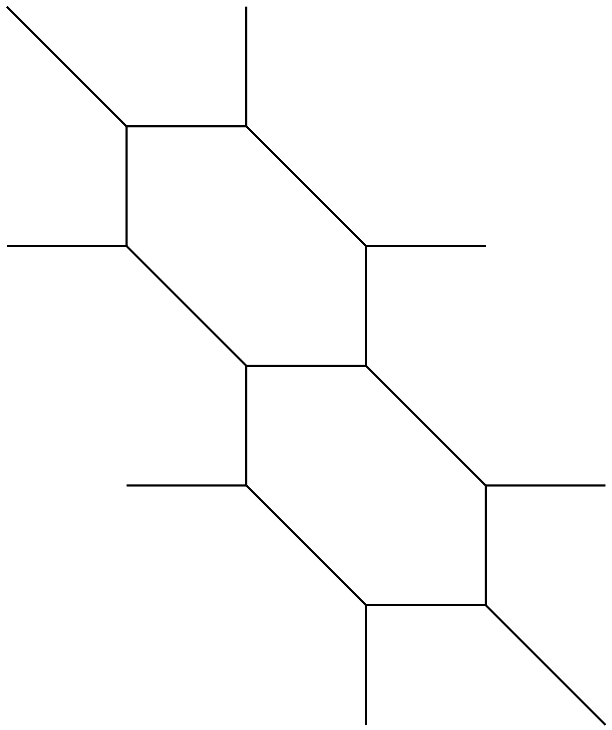


VEVs of vertex operators
(sharing labels with its neighbors)

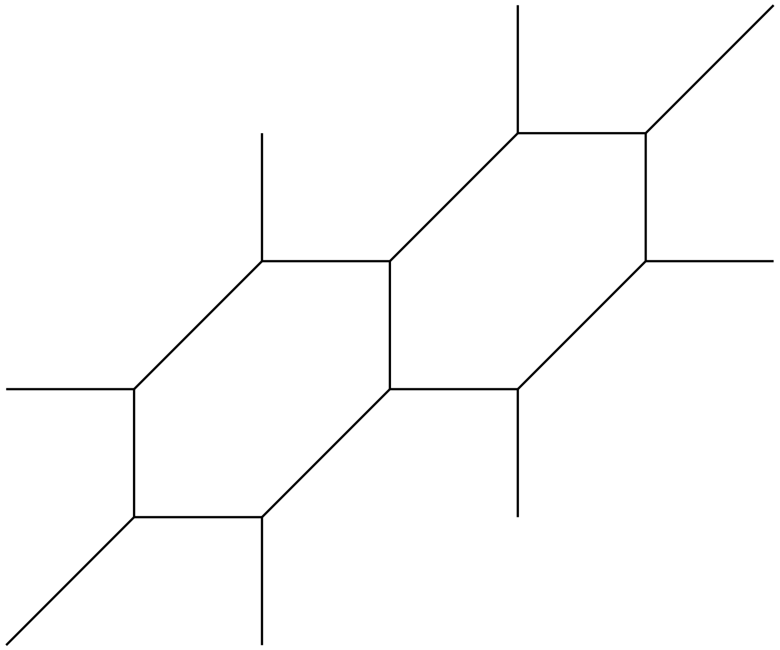
Let us examine some well-known examples:



SU(2) gauge A_1 quiver theory

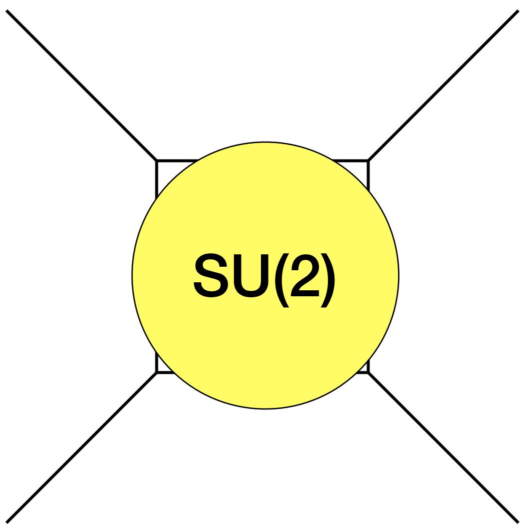


SU(3) gauge A_1 quiver theory

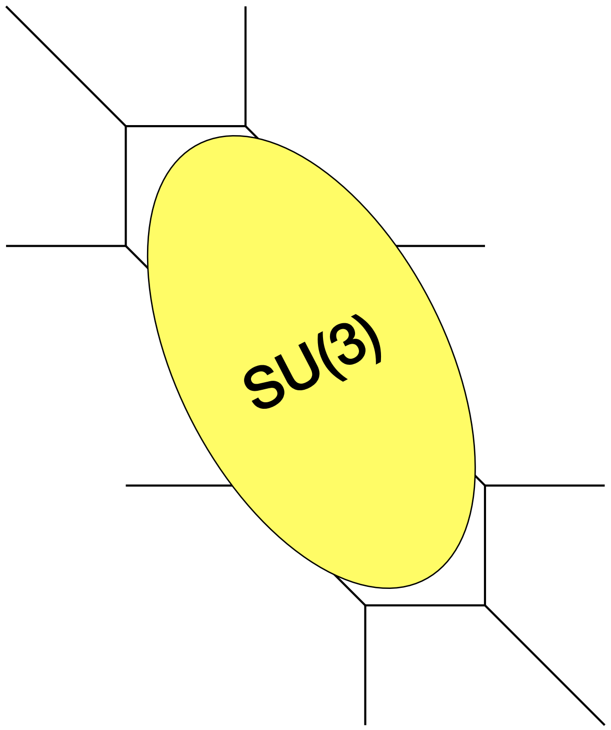


SU(2) gauge A_2 quiver theory

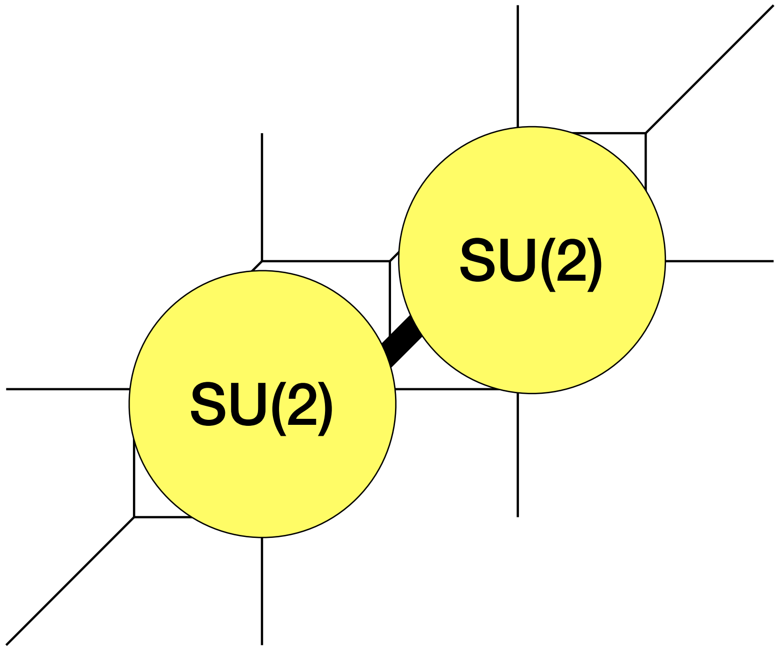
Let us examine some well-known examples:



SU(2) gauge A_1 quiver theory



SU(3) gauge A_1 quiver theory



SU(2) gauge A_2 quiver theory

All Lagrangian theories built in this way are with A-type gauge group and A-type quiver structure.

But certainly we have theories specified by other Lie algebras.

Especially, we can set the gauge group to be ABCDEFG, and the fiber-base duality exchanges the gauge group and quiver.

[Katz, Mayr, Vafa (1998)]

(In string theory, ADE-type ALE space ↔ ADE quiver.)

We have ADHM construction for ABCD-type gauge groups.

[Nekrasov, Shadchin (2004)]

However, the instanton counting for BCD-type gauge group is a mess.

[Nakamura, Okazawa, Matsuo, 2014]

We want to complete the list of ABCDEFG quivers.

Before we discuss the non-simply-laced quivers, we recall the relation with W-algebra.

AGT relation:

instanton partition function (on Ω-background) of gauge theory with gauge group G	**=**	**conformal block in W_G algebra**

[Alday, Gaiotto, Tachikawa (2009)]
[Wyllard (2009)]

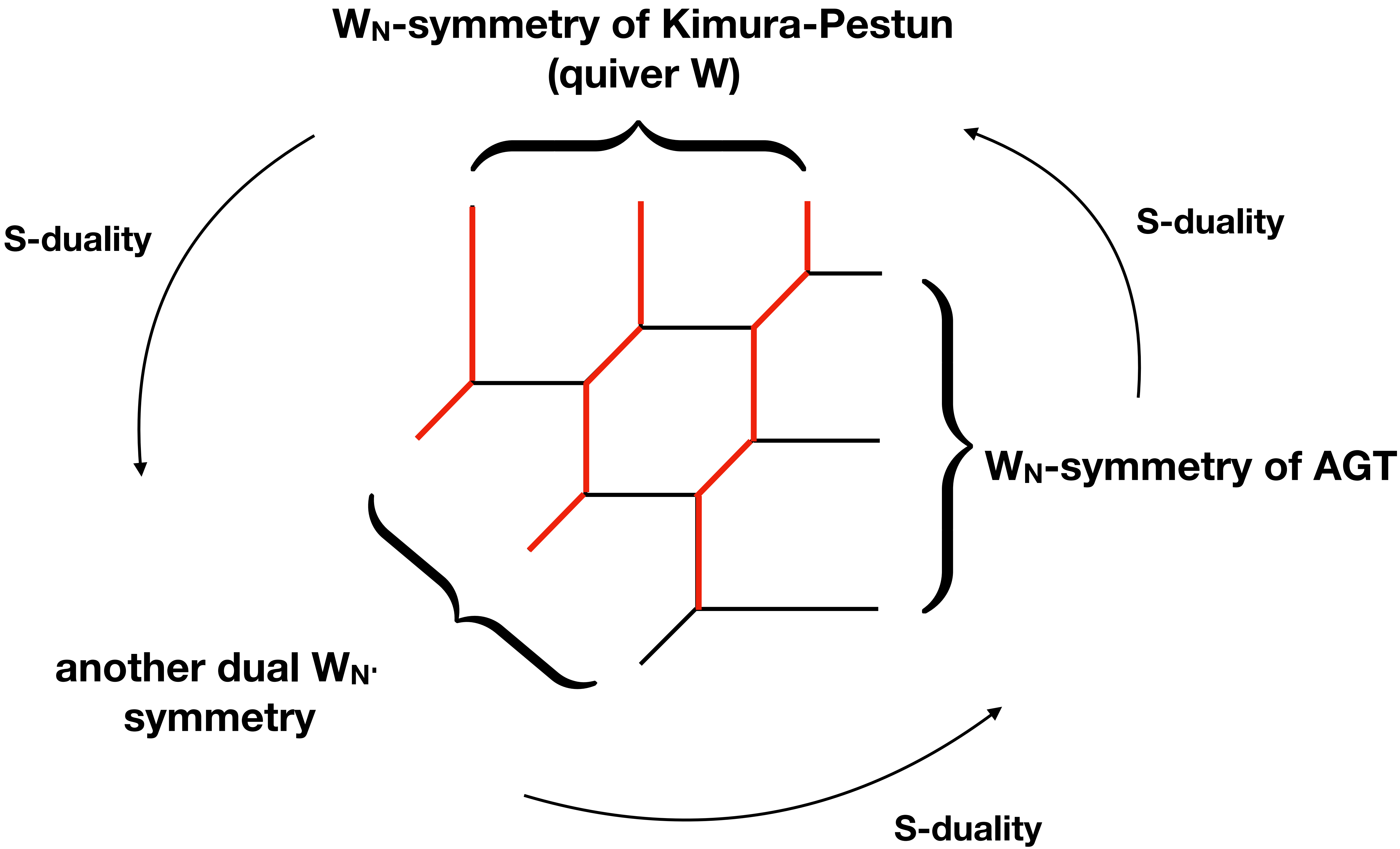
***This whole story can be uplifted to 5d.**

[Awata, Yamada (2009)]

Quiver W-algebra W_Γ

fiber-base dual version of AGT relation

[Kimura, Pestun (2015)]



W_N-symmetry of Kimura-Pestun
(quiver W)
S-duality
S-duality
W_N-symmetry of AGT
another dual $W_{N'}$
symmetry
S-duality

“Gauge Theory” with Fractional Quiver

[Kimura, Pestun, 2017]

constructed so that realizing W-algebra of fractional quiver

To each node, we need to assign an integer $d_i = (\alpha_i, \alpha_i)$

roughly speaking, we perform instanton counting with $q_1 \to q_1^{d_i}$

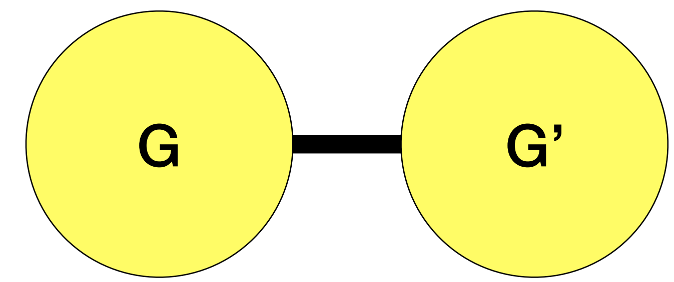
$$Z_i^{\text{vec}} = \mathbb{I}\left[\mathbf{V}_i\right] = \prod_{(x,x')\in\mathcal{X}_i^2} \left(q_1^{d_i} q_2 \frac{x}{x'}; q_2\right)_\infty \left(q_2 \frac{x}{x'}; q_2\right)_\infty^{-1}$$

where

$$x_{i,\alpha,k} = \nu_{i,\alpha} q_1^{d_i(k-1)} q_2^{\lambda_{i,\alpha,k}}$$

related to the variable in the vertex

$$V_-(t^{-\lambda} q^{-\rho+\{1/2\}}) V_+(q^{-\lambda^t} t^{-\rho-\{1/2\}})$$



Bifundamental contribution:

$$Z^{\mathrm{bf}}_{e:i\to j}[\mathcal{X}_i,\mathcal{X}_j,d_i,d_j;\mu_e]=\prod_{(x,x')\in\mathcal{X}_i\times\mathcal{X}_j}\frac{(\mu_e^{-1}q_2x/x';q_2)_\infty}{\left(\mu_e^{-1}q_1^{d_i}q_2x/x';q_2\right)_\infty}.$$

Of course it depends on two Young diagrams.

$$\frac{Z^{\mathrm{bf}}_{e:i\to j}[\mathcal{X}_{i+s},\mathcal{X}_j,d_i,d_j;\mu_e]}{Z^{\mathrm{bf}}_{e:i\to j}[\mathcal{X}_i,\mathcal{X}_j,d_i,d_j;\mu_e]}=\prod_{x'\in\mathcal{X}_j}\frac{1-\mu_e^{-1}q_1^{d_i}x_s/x'}{1-\mu_e^{-1}x_s/x'}$$

$$\frac{Z^{\mathrm{bf}}_{e:i\to j}[\mathcal{X}_i,\mathcal{X}_{j+s},d_i,d_j;\mu_e]}{Z^{\mathrm{bf}}_{e:i\to j}[\mathcal{X}_i,\mathcal{X}_j,d_i,d_j;\mu_e]}=\prod_{x'\in\mathcal{X}_i}\frac{1-\mu^{-1}q_2x'/x_s}{1-\mu^{-1}q_1^{d_i}q_2x'/x_s}$$

It behaves differently when varying two Young diagrams.

qq-character

[Nekrasov, 2015]

- operator

- roughly speaking, double-quantized Seiberg-Witten curve

  (in the classical limit $\epsilon_{1,2} \to 0$, reduces to the curve.)

- kind of character for quiver Lie algebra

  expression encodes representation data

- expectation value = partition function with Wilson lines

[Kim, 2016]

qq-character

A_1 quiver, fundamental rep. qq-character

$$\bar{\chi}_1(z) = Y(z) + Y(zq_1^{-1}q_2^{-1})^{-1},$$

where the expectation value of Y-operator is determined by Z_{k+1}/Z_k.

Nice properties:

1. qq-characters play the role of generators of quiver W-algebra W_Γ.

[Kimura, Pestun (2015)]

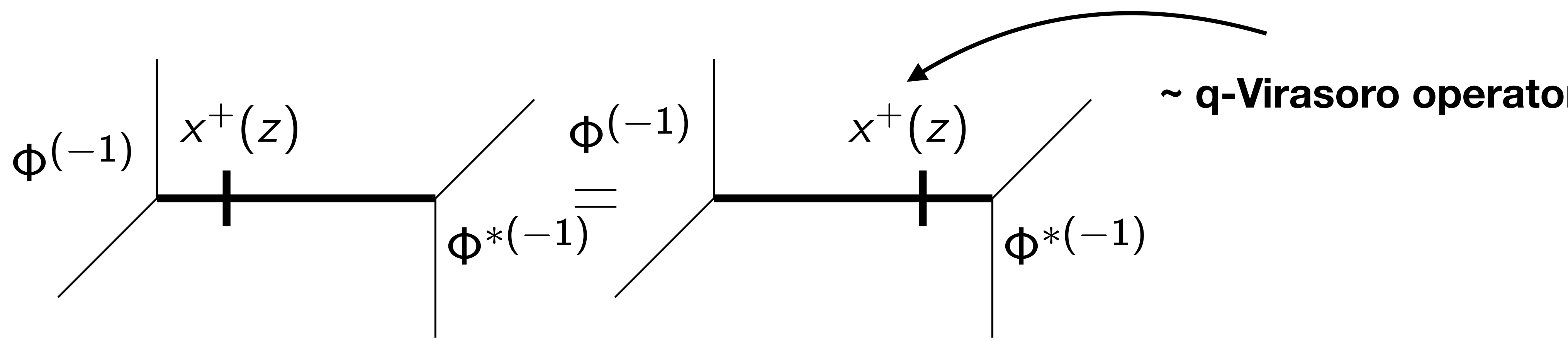
2. No poles at $z = \chi_{(i,j)} := vq_1^{i-1}q_2^{j-1}$.

poles in the expectation value of Y-operator

2. No poles at $z = \chi_{(i,j)} := v q_1^{i-1} q_2^{j-1}.$

comes from the Virasoro constraint in terms of matrix model, or equivalently from the Ward identity in terms of correlation functions in 2d CFT.

We can see it from the web construction



$$\sum_\lambda \mathfrak{q}^{|\lambda|} Z_{vect}(\lambda) \sum_{x \in R(\lambda)} \frac{1}{z - \chi_x} \operatorname*{Res}_{z \to \chi_x} \mathcal{Y}_\lambda(z q_3^{-1}) = \sum_\lambda \mathfrak{q}^{|\lambda|+1} z^{-1} Z_{vect}(\lambda) \sum_{x \in A(\lambda)} \frac{1}{z - \chi_x} \operatorname*{Res}_{z \to \chi_x} \frac{1}{\mathcal{Y}_\lambda(z)}.$$

$$\Rightarrow \quad \sum_\lambda \mathfrak{q}^{|\lambda|} Z_{vect}(\lambda) \left(\mathcal{Y}_\lambda(z q_3^{-1}) + \frac{\mathfrak{q}}{\mathcal{Y}_\lambda(z)} \right) = (\text{residue at } x \sim \infty).$$

Ward identity ↔ Weyl reflection

[Bourgine, Fukuda, Matsuo, Zhang, RZ, 2016]

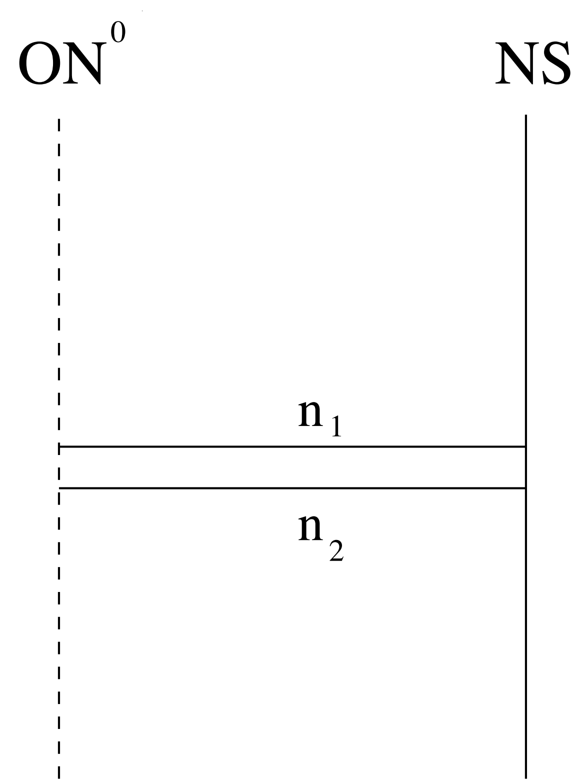
D-type quiver seems to be the easiest one to attack.

A known brane construction with orientifold.

[Kapustin, 1998]

[Hanany, Zaffaroni, 1999]

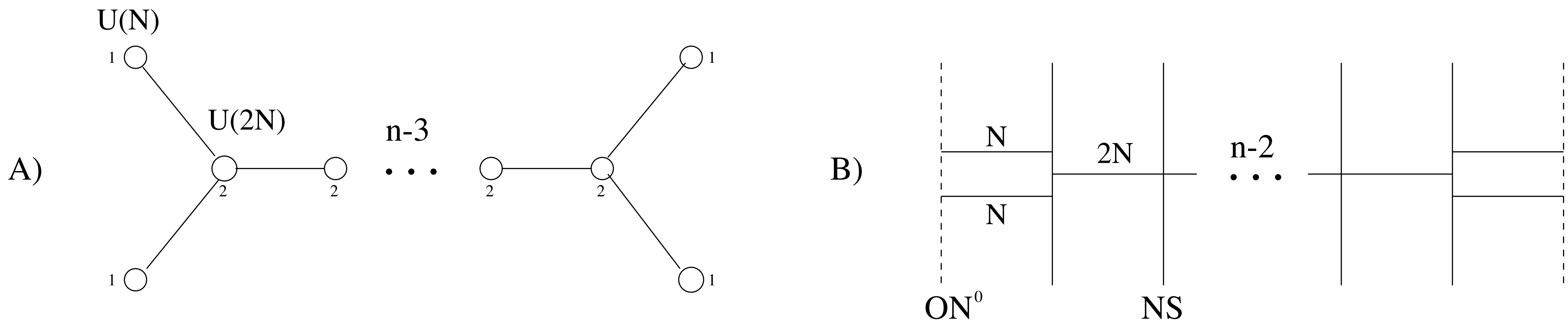
originally in 4d



$U(n_1) \times U(n_2)$

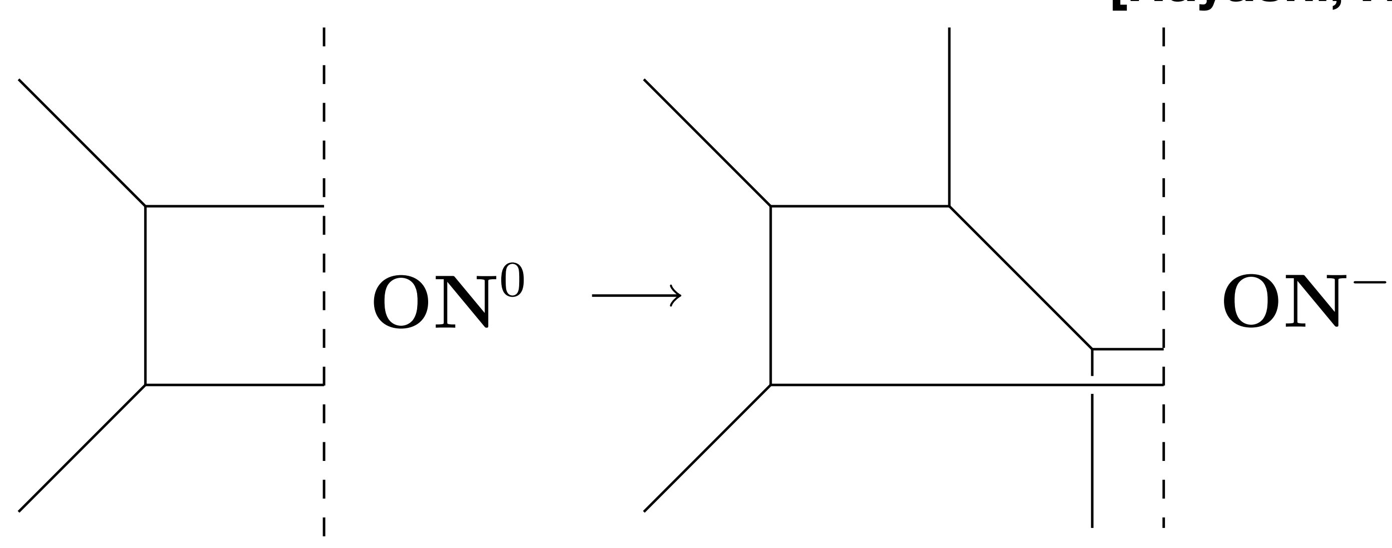
no bifundamental sectors in this configuration!

We can even reproduce the affine D-type quiver structure with ON^0 planes.



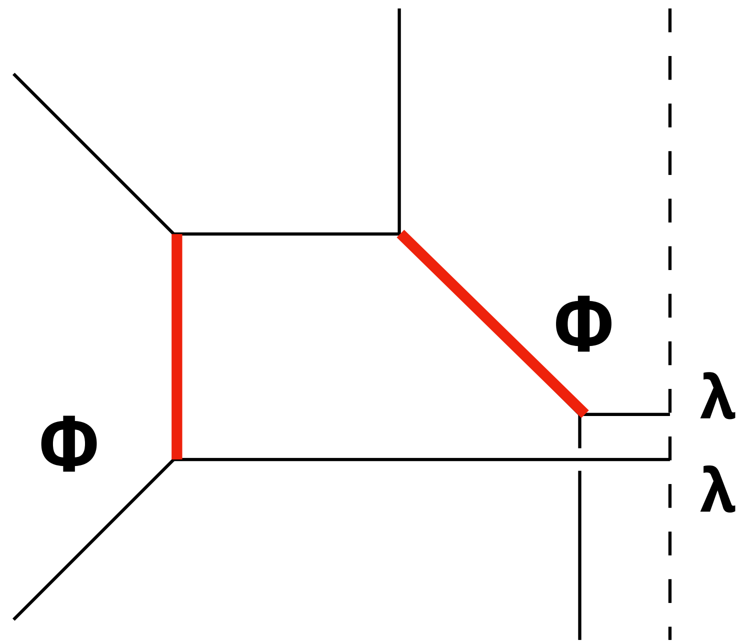
It seems to be straightforward to uplift this picture to 5d. But instead, a microscopic ("refined") brane web was proposed to be used. This microscopic picture seems to account the D structure more explicitly.

[Hayashi, Kim, Lee, Taki, Yagi, 2015]



ON^0=ON^-+NS5

In the unrefined case, the topological vertex formalism is extremely simple.



force two vertices connected by the orientifold to share the same Young diagram label.

||

reflection state $\sum_{\lambda} |v,\lambda\rangle \otimes |v,\lambda\rangle$

[Bourgine, Fukuda, Matsuo, RZ (2018)]

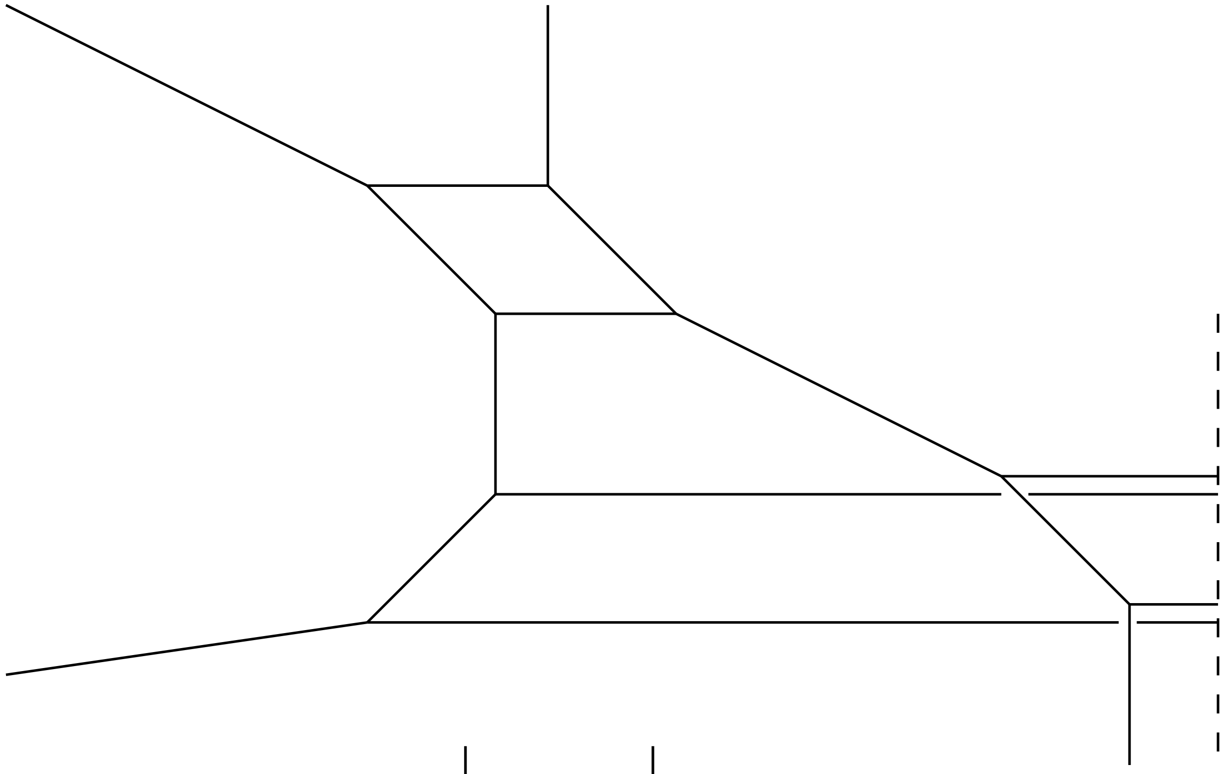
In the refined case, it is more tricky to realize this decoupling.

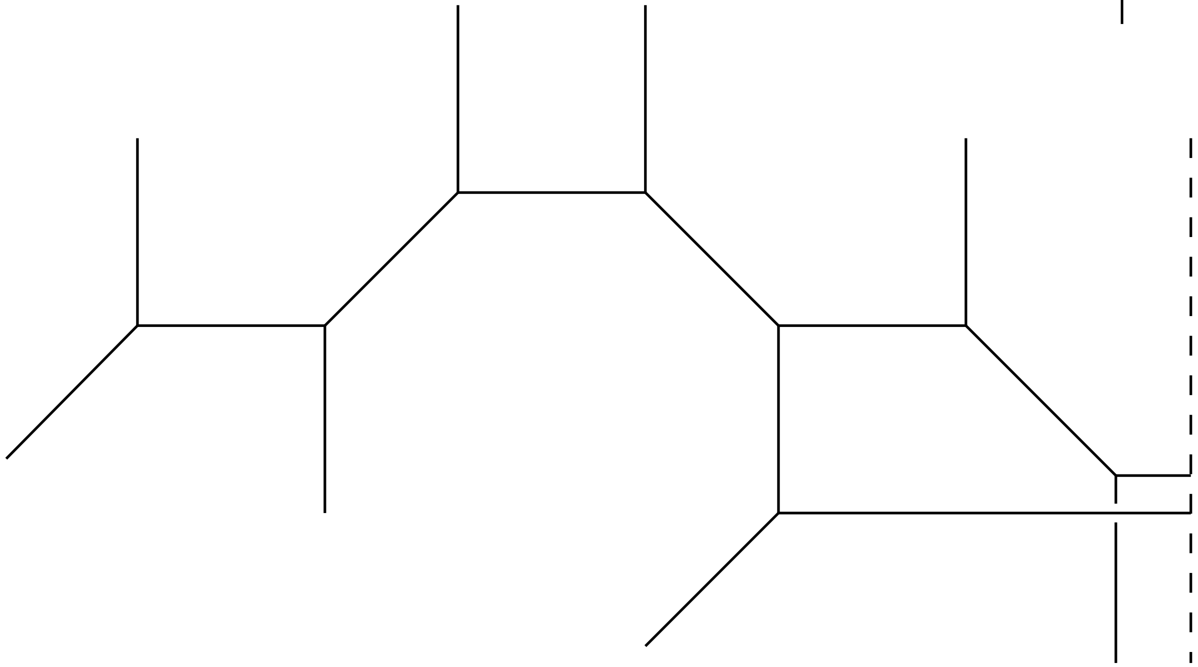
***in the unrefined limit,** $\Phi_\lambda = \overset{*}{\Phi}{}_{\lambda}^{-1}$

in the refined case, we instead use

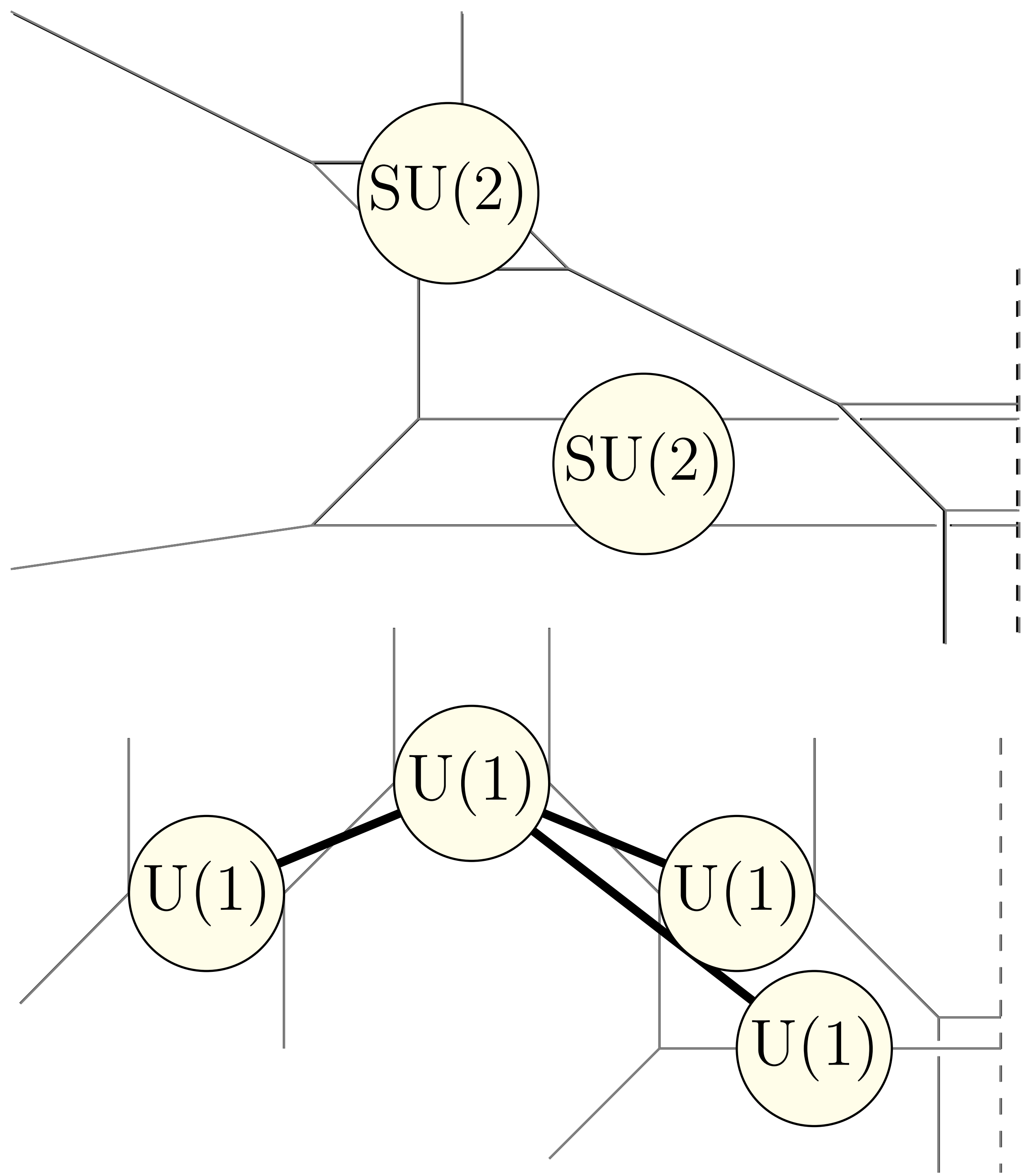
$$\tilde{\Phi}^* = \overset{*}{\Phi}{}_{\lambda}^{-1}[vq_1q_2]$$

Generalization: adding more branes

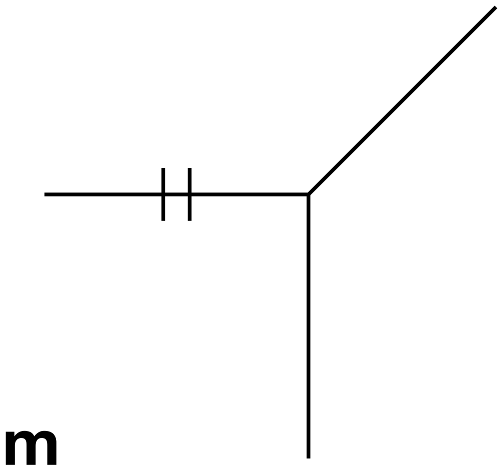




Generalization: adding more branes



How to realize the bifundamental contribution in non-simply-laced quiver?



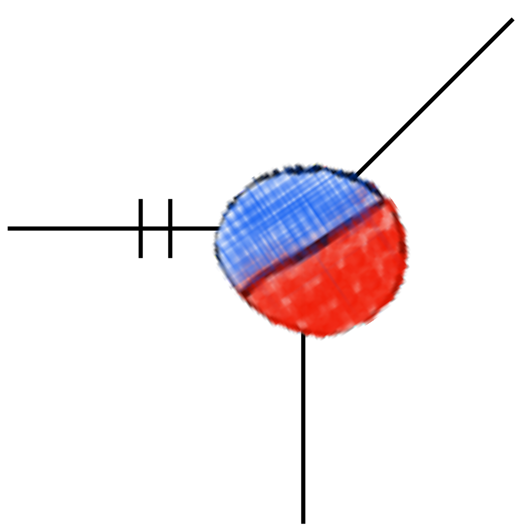
$$\gcd(d_i, d_j) = d_j$$

for example the vertex operator for empty Young diagram

$$\tilde{\Phi}_{\emptyset}^{*(d_i)}[v_j] := \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\frac{(\gamma_j^2/\gamma_i)^n v_j^n}{1-q_1^{-nd_i}}a_{-n}\right)\exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\frac{1-q_1^{-nd_i}}{1-q_1^{-nd_j}}\frac{\gamma_i^n v_j^{-n}}{1-q_2^{-n}}a_n\right),$$

The positive modes give rise to an instanton counting with q^{d_j}**,**

the negative modes give rise to an instanton counting with q^{d_i}**,**



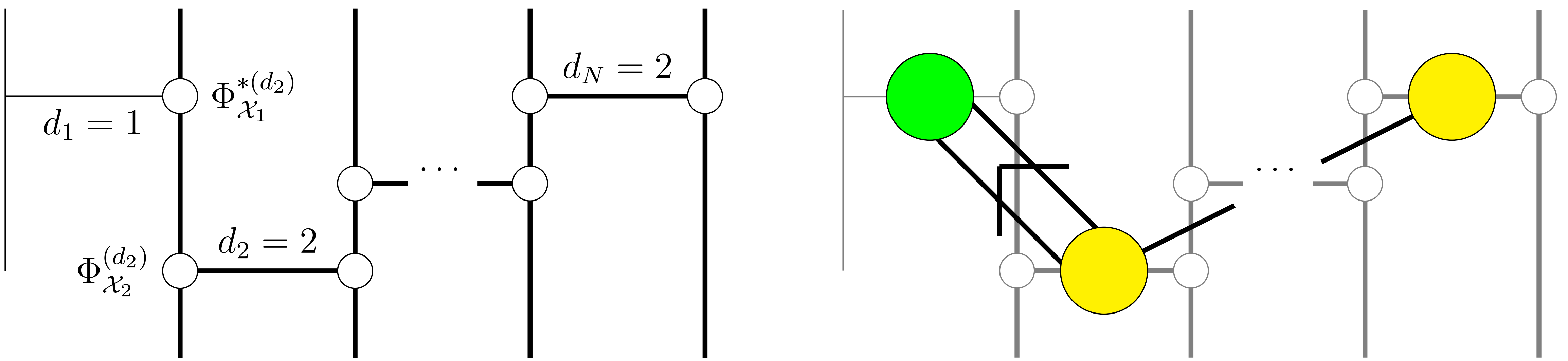
Vertex operator with mixed natures.

→ half-blood vertex

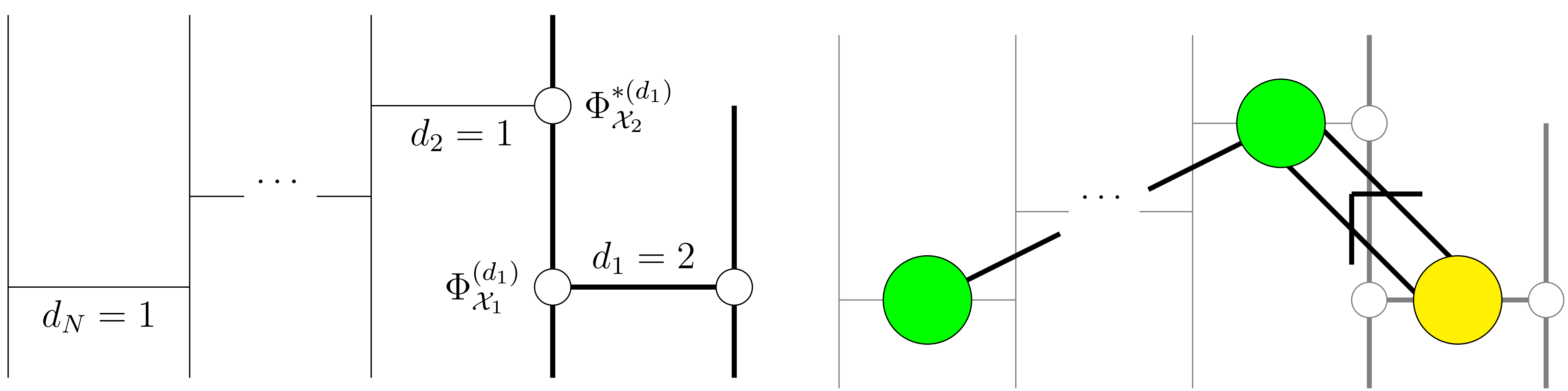
[Kimura, RZ (2019)]

Examples: BC-type construction

B-type



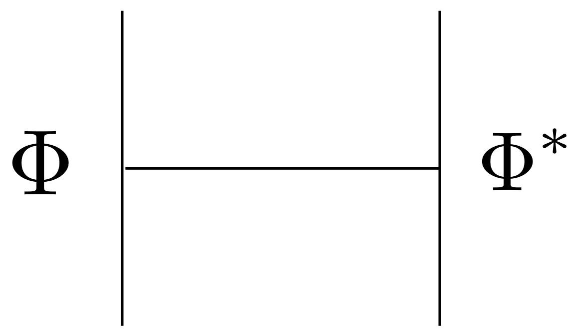
C-type



correspondence between vertices and simple roots I

A-type: n+1 Fock spaces for A_n quiver

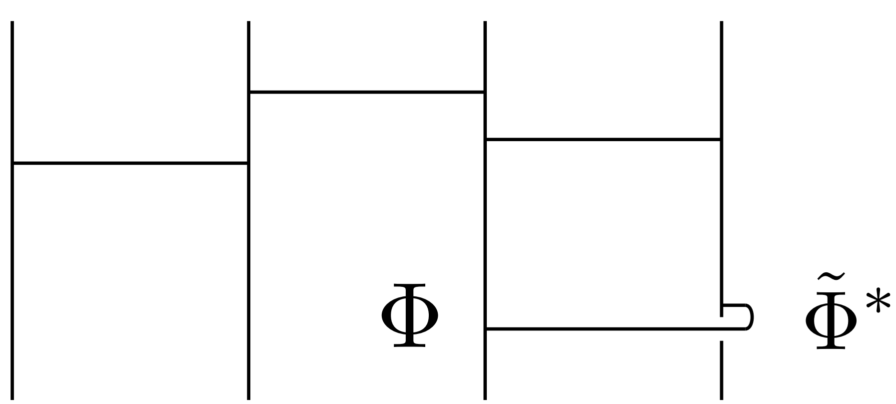
building block



$\longleftrightarrow$ $\alpha_i = e_i - e_{i+1},$

$\Phi \leftrightarrow e_i$ $\Phi^* \leftrightarrow -e_{i+1}$

D-type:



We have a special node

$\tilde{\Phi}^* \sim \Phi \leftrightarrow e_{n+1}$

recovers

$$\alpha_n = e_n + e_{n+1}.$$

E-type from simple roots

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \quad \alpha_3 = e_3 - e_4,$$

$$\alpha_4 = e_4 - e_5, \quad \alpha_5 = e_5 - e_6, \quad \alpha_6 = e_6 + e_7,$$

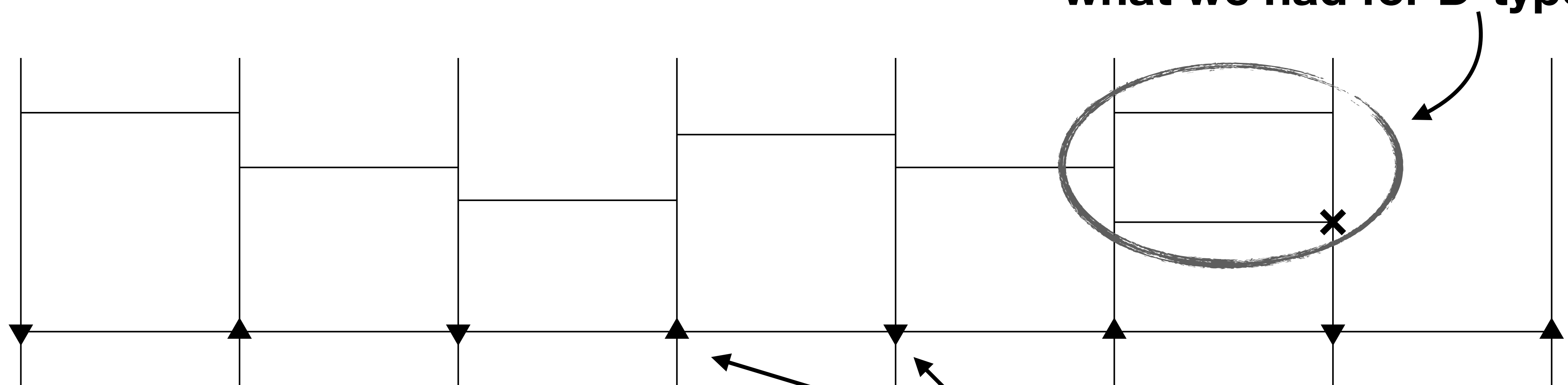
$$\alpha_7 = -\frac{1}{2}e_1 - \frac{1}{2}e_2 - \frac{1}{2}e_3 - \frac{1}{2}e_4 - \frac{1}{2}e_5 - \frac{1}{2}e_6 - \frac{1}{2}e_7 - \frac{1}{2}e_8, \quad \alpha_8 = e_6 - e_7,$$

we have to use fractional coefficients to write down the simple roots.

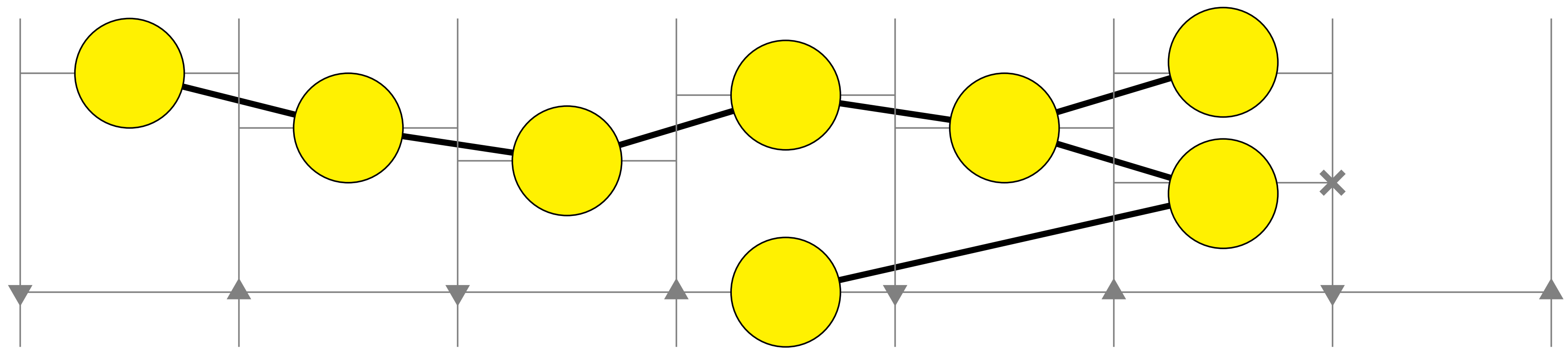
We need to introduce a new type of vertex, “square-root vertex”.

1/2-power of the usual vertices

E_8 quiver



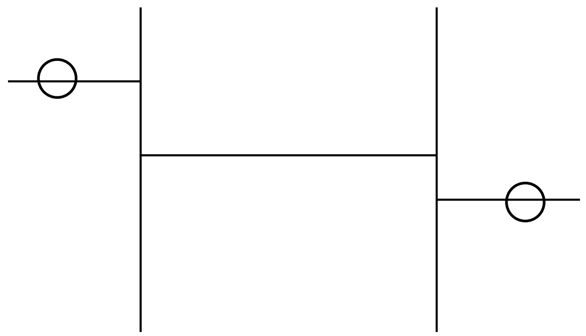
quiver structure



removing the branes corresponding to unnecessary nodes,
we obtain E_6 and E_7 quivers.

Affine quivers?

affine A-type: well-known

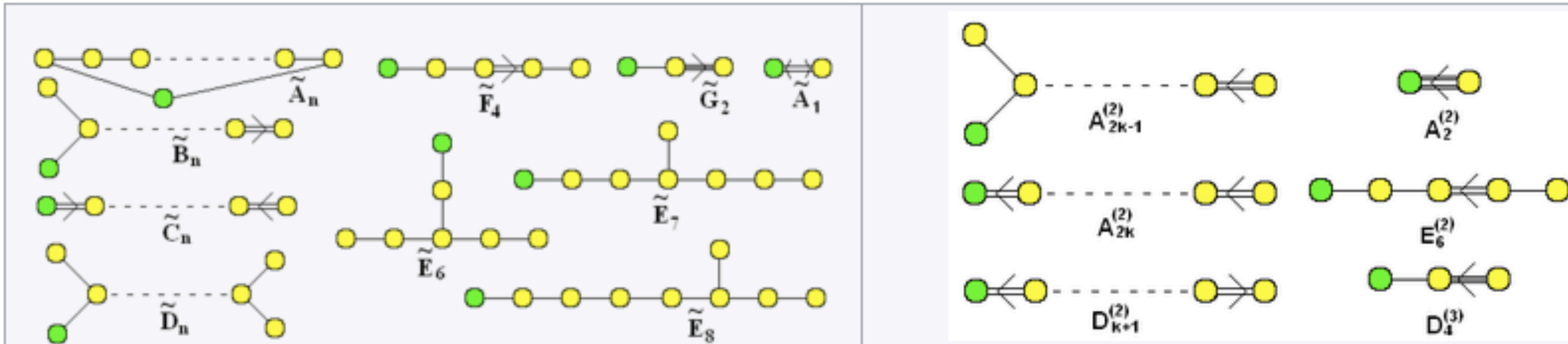


identify the branes on the two ends

affine D-type: already mentioned

affine BC-type: more or less the same as BC

Affine Dynkin diagrams



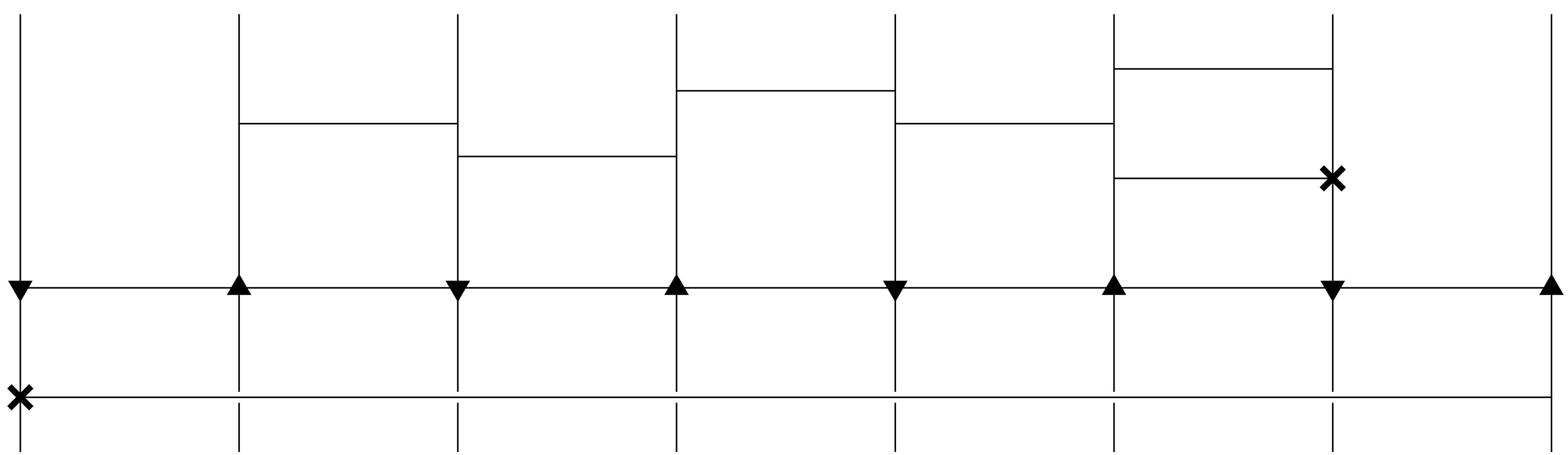
The set of extended affine Dynkin diagrams, with added nodes in green ($n \geq 3$ for B_n and $n \geq 4$ for D_n)

"Twisted" affine forms are named with (2) or (3) superscripts. (The subscript *k* always counts the number of *yellow* nodes in the graph, i.e. the total number of nodes minus 1.)

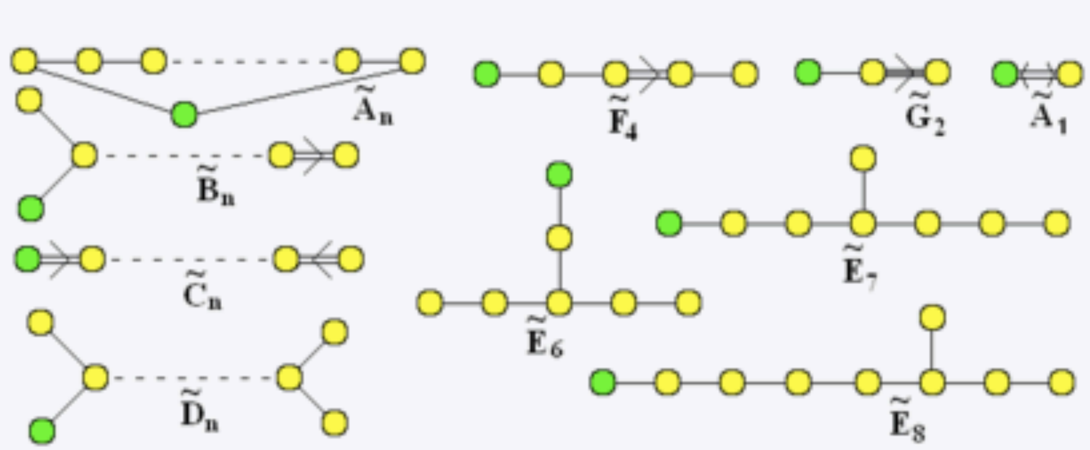
From Wikipedia

The only non-trivial ones in our approach are again the affine E-type quivers.

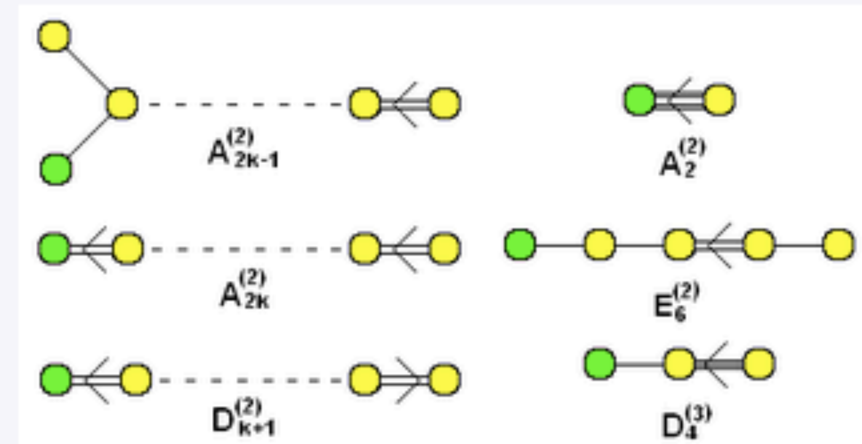
for example, $E^{(1)}{}_7$



Affine Dynkin diagrams



The set of extended affine Dynkin diagrams, with added nodes in green ($n \geq 3$ for B_n and $n \geq 4$ for D_n)



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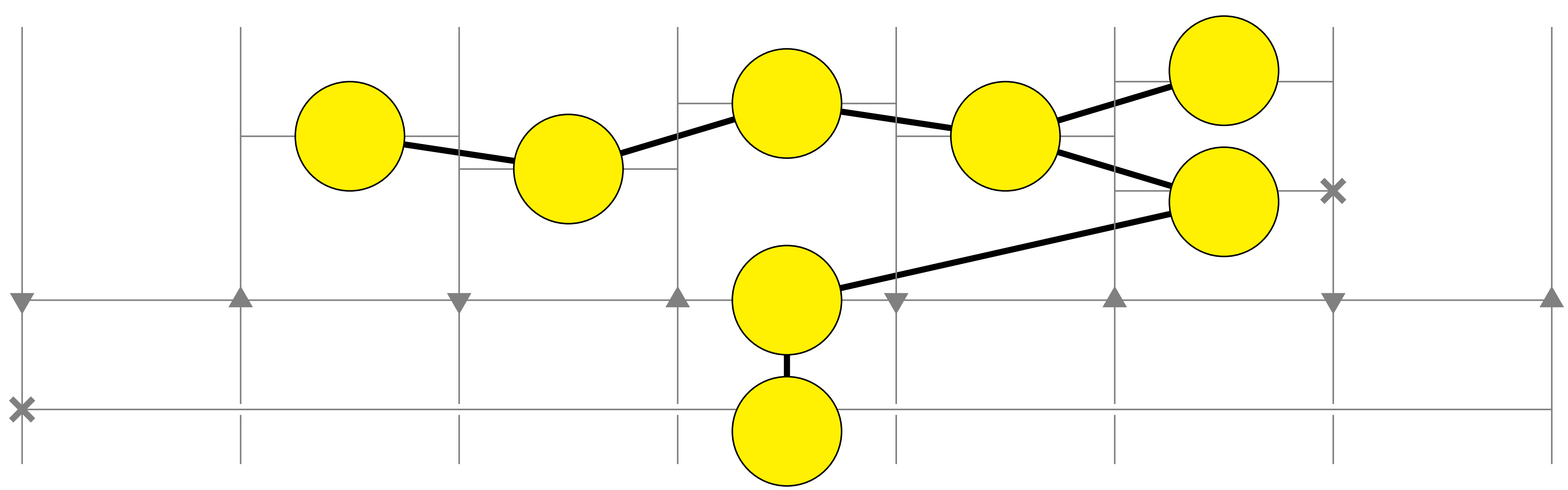


Interestingly, by using the “square-root” vertices and usual vertices, we could only reproduce all affine E-type quivers, but not to go beyond.

Conclusion

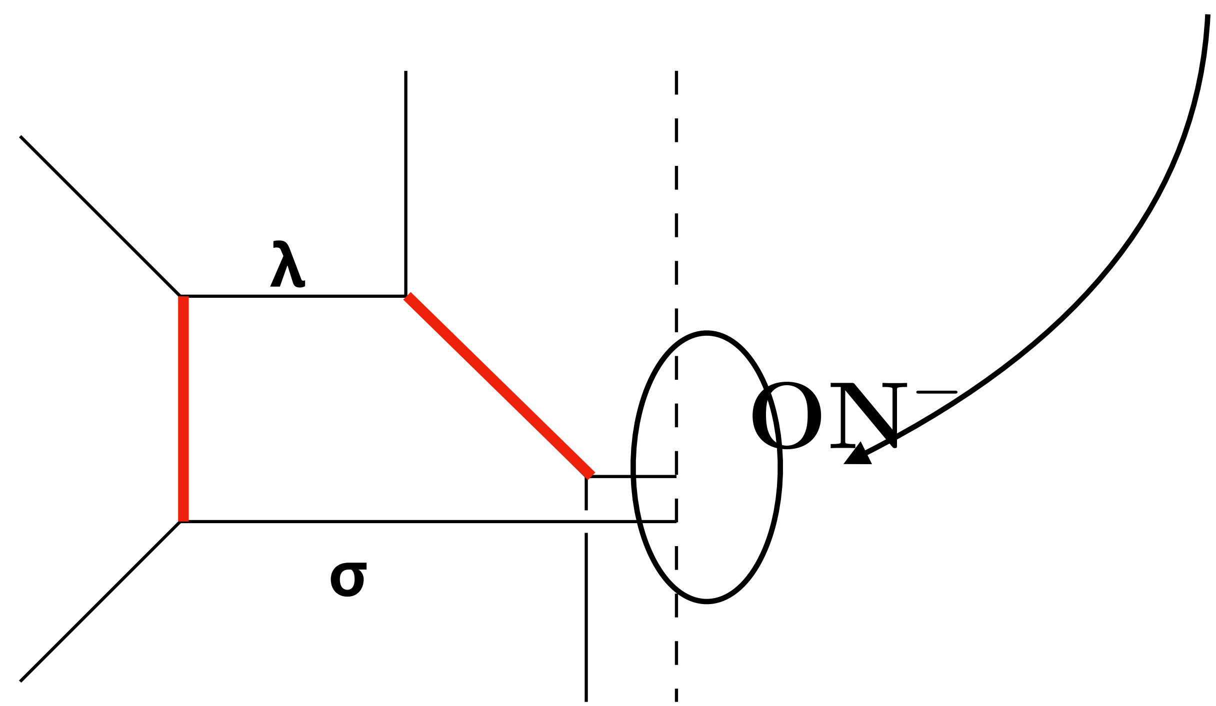
- We built the “brane” web for ABCDEFG-type and affine quivers by introducing new vertices, half-blood vertex and “square-root” vertex.

- Our construction not only reproduces the Nekrasov partition function, but also realizes qq-characters as Ward identities.

謝謝儂

Thank you very much for your attention!!

Let us assign the refined topological vertex and ignore this part first.



Left:

$$\sum_{\mu}(-Q\sqrt{q/t})^{|\mu|}s_{\mu}(t^{-\lambda^t}q^{-\rho})s_{\mu}(q^{-\sigma}t^{-\rho})$$

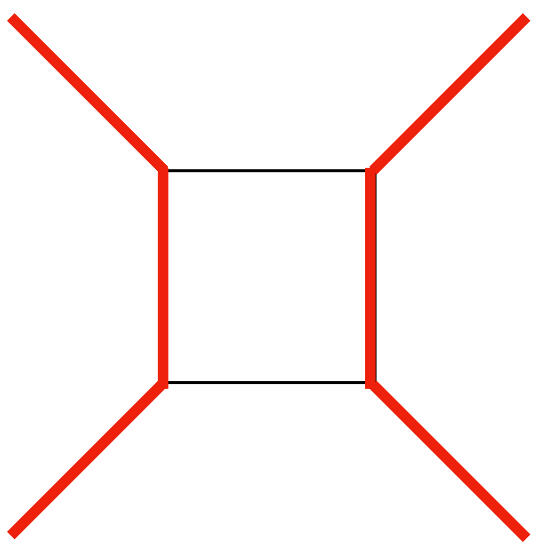
Right:

$$\sum_{\nu}(Q\sqrt{q/t})^{|\nu|}s_{\nu^t}(t^{-\lambda^t}q^{-\rho})s_{\nu}(q^{-\sigma}t^{-\rho})$$

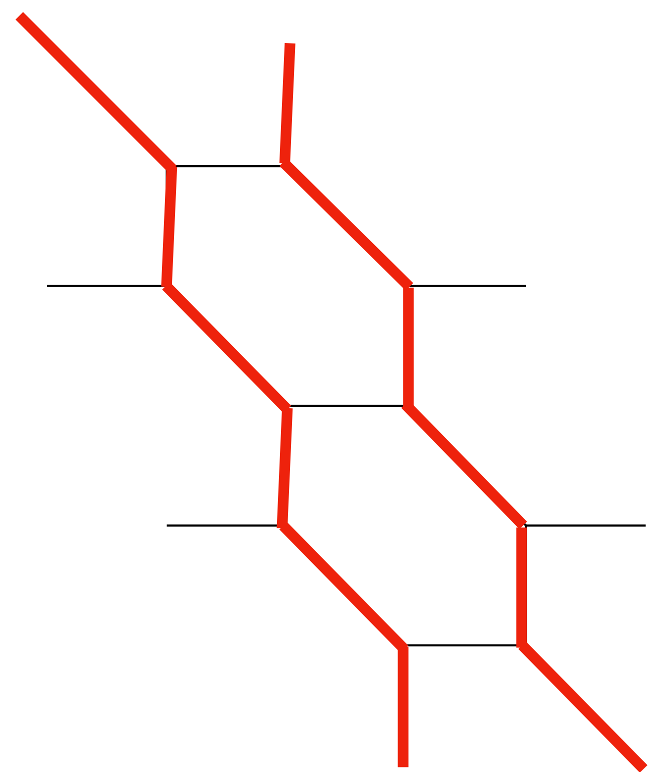
They cancel each other due to the Cauchy identity,

$$\sum_{\mu}s_{\mu}(x)s_{\mu}(y)=\prod_{i,j}(1-x_iy_j)^{-1},\quad \sum_{\nu}s_{\nu^t}(x)s_{\nu}(y)=\prod_{i,j}(1+x_iy_j).$$

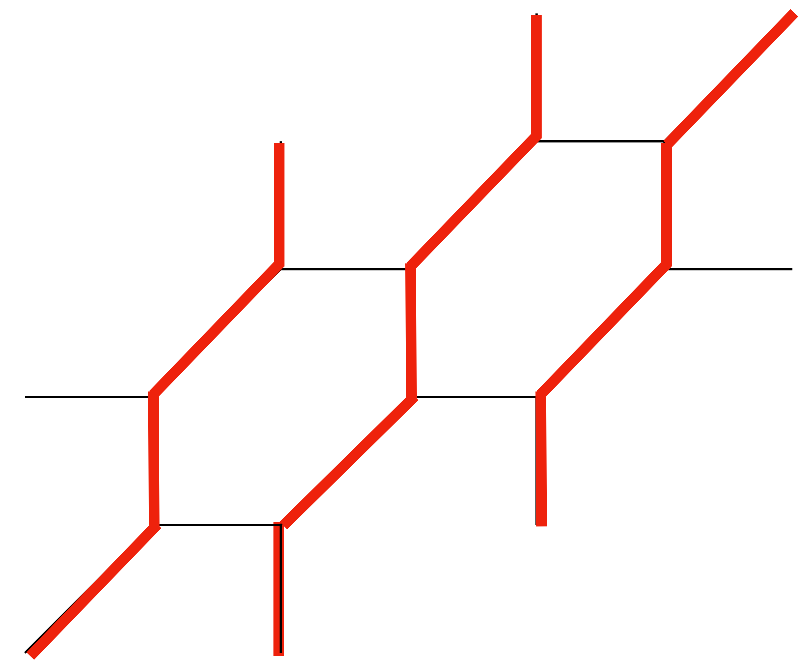
Let us examine some well-known examples:



SU(2) gauge A_1 quiver theory



SU(3) gauge A_1 quiver theory



SU(2) gauge A_2 quiver theory

Fock space structure ↔ quiver structure

One can easily confirm that the vertex operator

$$V_-(t^{-\lambda}q^{-\rho+\{1/2\}})V_+(q^{-\lambda^t}t^{-\rho-\{1/2\}})$$

matches with the vertex written down by Awata, Feigin and Shiraishi, on which they found that the so-called Ding-Iohara-Miki algebra acts in the adjoint way.

[Awata, Feigin, Shiraishi, 2011]

Ding-Iohara-Miki algebra

[Ding, Iohara, 1997] [Miki, 2007]

doubly affinized quantum group

$U_{q,t}(\widehat{\widehat{\mathfrak{gl}}}_1)$ $\longleftrightarrow$ **q-deformed** $\mathcal{W}_{1+\infty}[\mu]$

$\downarrow$ $q \to 1,\ t = q^{\beta}$ $\qquad\qquad\qquad\qquad$ $\downarrow$

$\widehat{\mathfrak{gl}}_1$ **Yangian/SH$_c$ algebra** $\longleftrightarrow$ **(A-type)** $\mathcal{W}_{1+\infty}[\mu]$

[Shiffmann, Vasserot, 2012]

[Maulik, Okounkov, 2012]

… [Prochazka, 2015]

One thing I would like to mention is

Elliptic Extension of the Whole Story

- **elliptic Ding-Iohara-Miki algebra**

[Saito, (2013)]

- **elliptic AGT/Kimura-Pestun**

[Nieri, (2015)] [Iqbal, Kozcaz, Yau, (2015)] [Kimura, Pestun, (2016)]

- **elliptic topological vertex,**
  with elliptic Ding-Iohara-Miki acting on it in the adjoint way

[RZ, (2017)] [Foda, RZ, (2018)]

elliptic Schur and Macdonald functions (might be) related

~ two copies q-Whittaker function

Some properties of Ding-Iohara-Miki (DIM)

- **Ding-Iohara-Miki algebra on the coproduct of N Fock spaces contains a U(1)x (q-deformed) W_N algebra.**

[Feigin, Hoshino, Shibahara, Shiraishi, Yanagida, 2010]

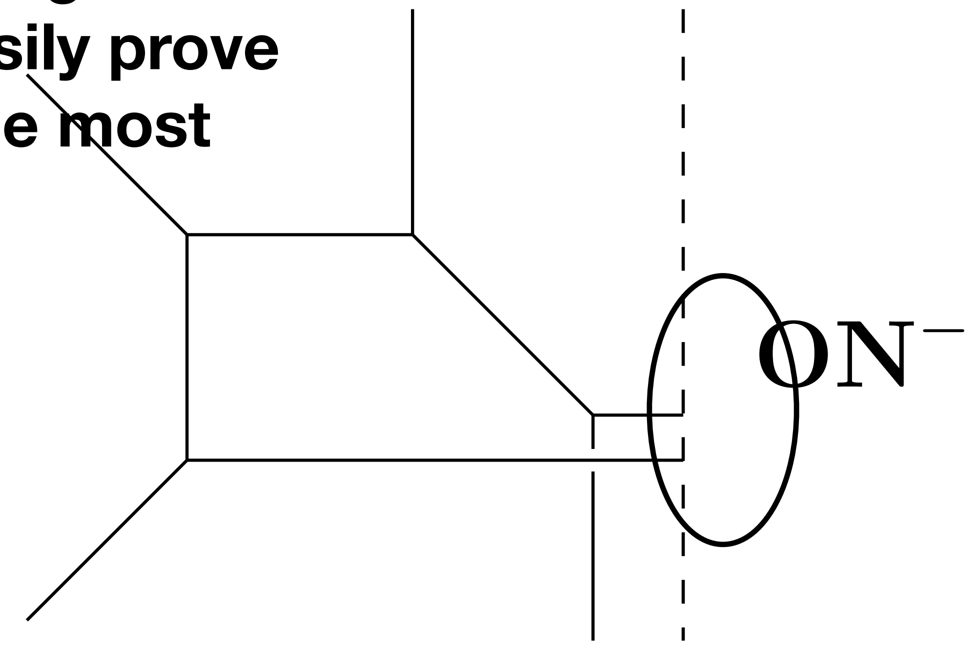
- **There is an SL(2,Z) automorphism of the algebra, among which there is an S-duality symmetry permutes three legs of the refined topological vertex.**

[Miki, 2007]
[Awata, Feigin, Shiraishi, 2011]

- **Since the algebra is a quantum group, it is equipped with a universal R-matrix, which reduces to Maulik-Okounkov's R-matrix in the 4d limit,** $q \to 1,\ t = q^{\beta}$

[Feigin, Jimbo, Miwa, Mukhin, 2015]
[Fukuda, Harada, Matsuo, RZ, 2017]

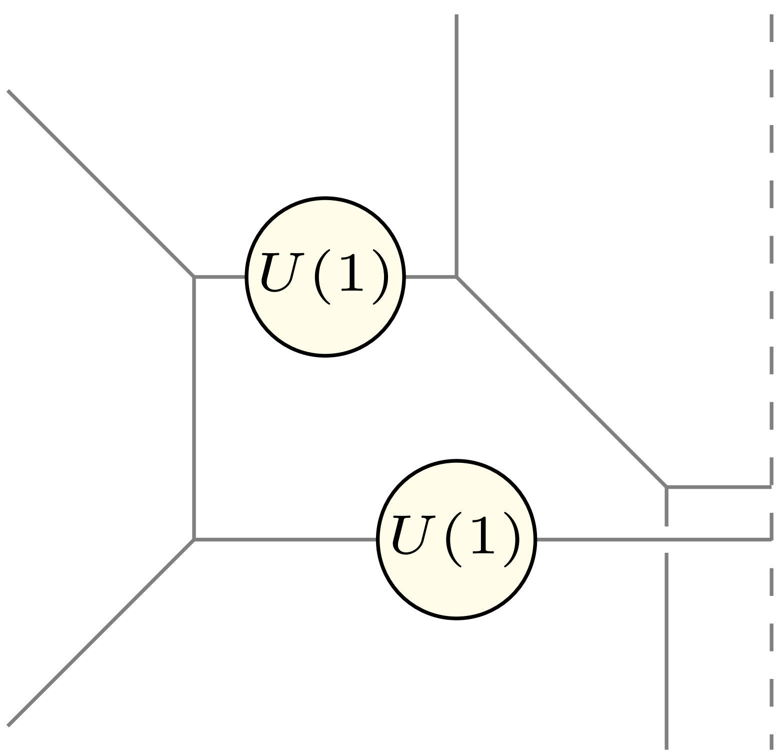
This can certainly be translated into the language of Awata-Feigin-Shiraishi vertex, and we can easily prove that this cancellation mechanism occurs in the most general case.



Now, let us consider the ignored part.

Expected structure: $D_2 \simeq A_1 \times A_1$

two decoupled U(1) instanton sectors.



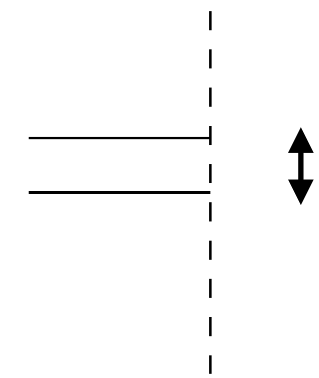
➡ need the orientifold to replace the behavior in the preferred direction of $C_{\nu^t \emptyset \sigma}(t,q)$ to $C_{\nu^t \emptyset \sigma^t}(q,t)$

In the calculation of partition function, we just need to divide a Macdonald polynomial and multiply its transposed one.

- **What is interesting is that this replacement is an automorphism of the Ding-Iohara-Miki algebra.**

The action of the orientifold can thus be represented as a reflection state:

$$\sum_{\lambda} |v, \lambda\rangle \otimes |v\gamma, \lambda\rangle \quad \longleftrightarrow$$



$\gamma = \sqrt{t/q}$

position of D5-brane

- **One intriguing observation here is that the reflection state above reduces to the boundary state in the 4d limit** $q \to 1,\ t = q^{\beta}$ **.**

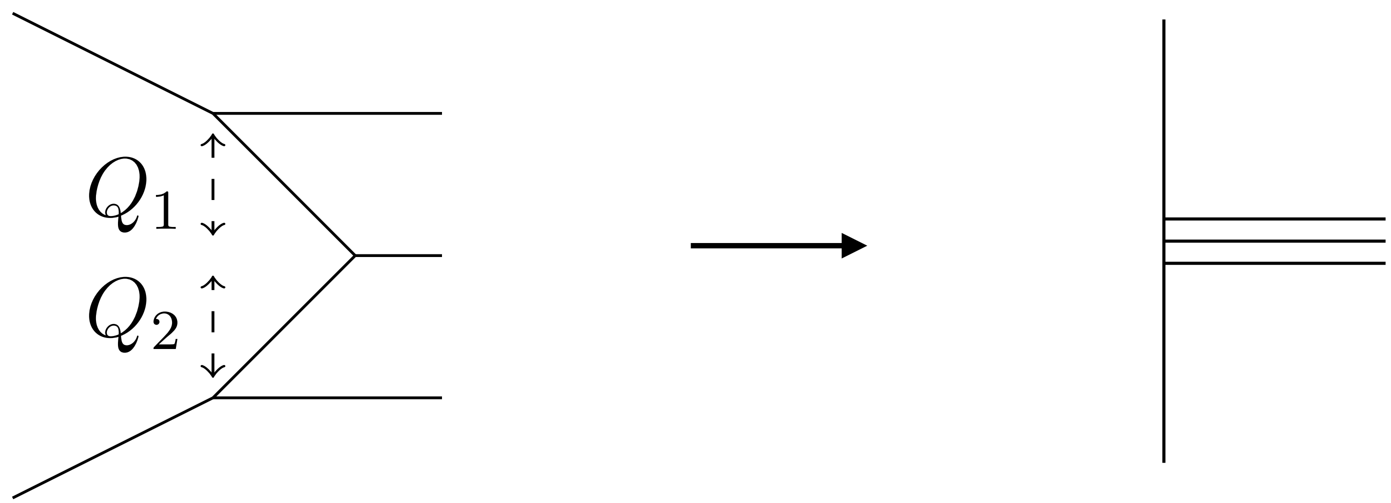
$$(L_n \otimes 1 - 1 \otimes L_{-n})|\Omega\rangle = 0.$$

“2d” picture of this construction?

***recall**

$$\Phi^{(-1)}\,|\lambda\rangle = V_-(t^{-\lambda}q^{-\rho+\{1/2\}})V_+(q^{-\lambda^t}t^{-\rho-\{1/2\}})$$

In fact, we can put the product of refined topological vertices in the unpremerred direction into normal ordering and redefine it as a new object.



Factors from the contraction in the unpremerred direction are absorbed into the preferred direction.

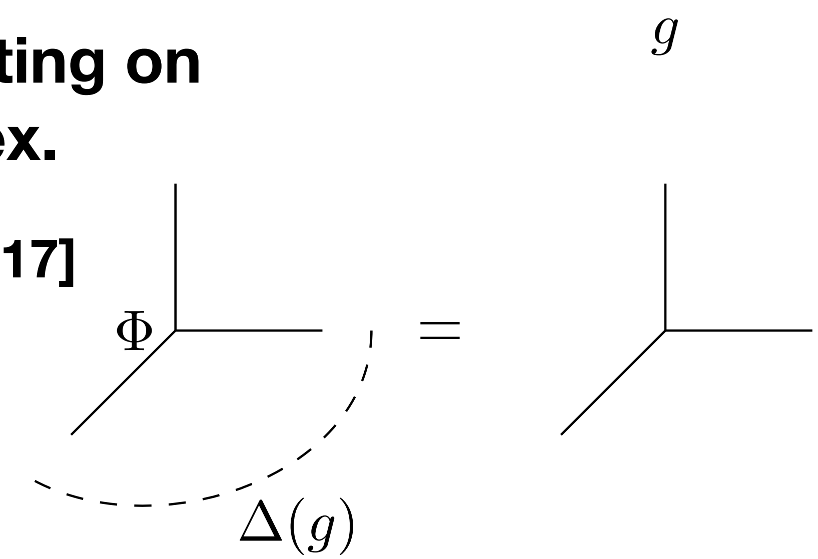
Generalized Vertex

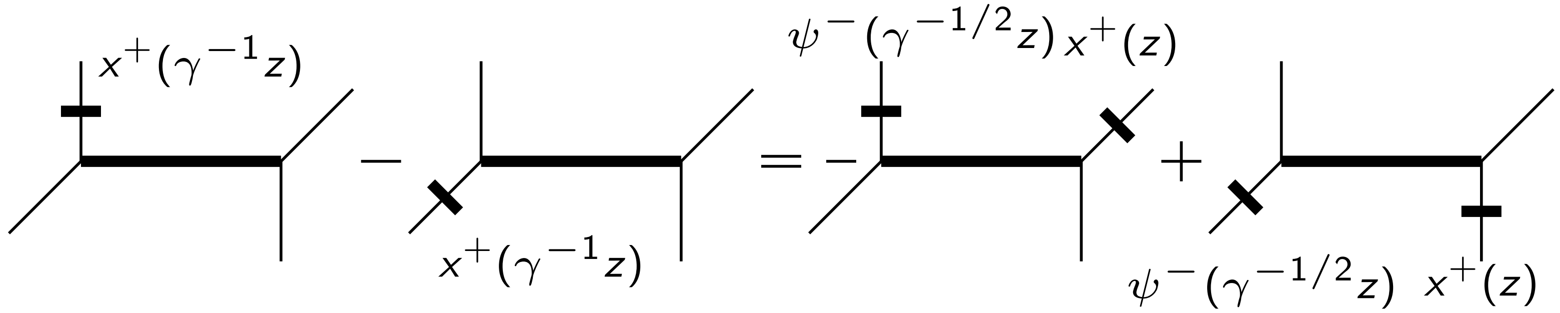
[Bourgine, Fukuda, Harada, Matsuo, RZ, 2017]

This object captures the feature of quiver, enabling us to work on web with “U(1) gauge group”.

The Ward identity can be converted to operators acting on Fock spaces by using the adjoint nature of the vertex.

[Bourgine, Fukuda, Harada, Matsuo, RZ, 2017]





With this rewriting, we reach to that the qq-character commutes with a screening-charge-like object, in analogy to [Kimura, Pestun, 2015].

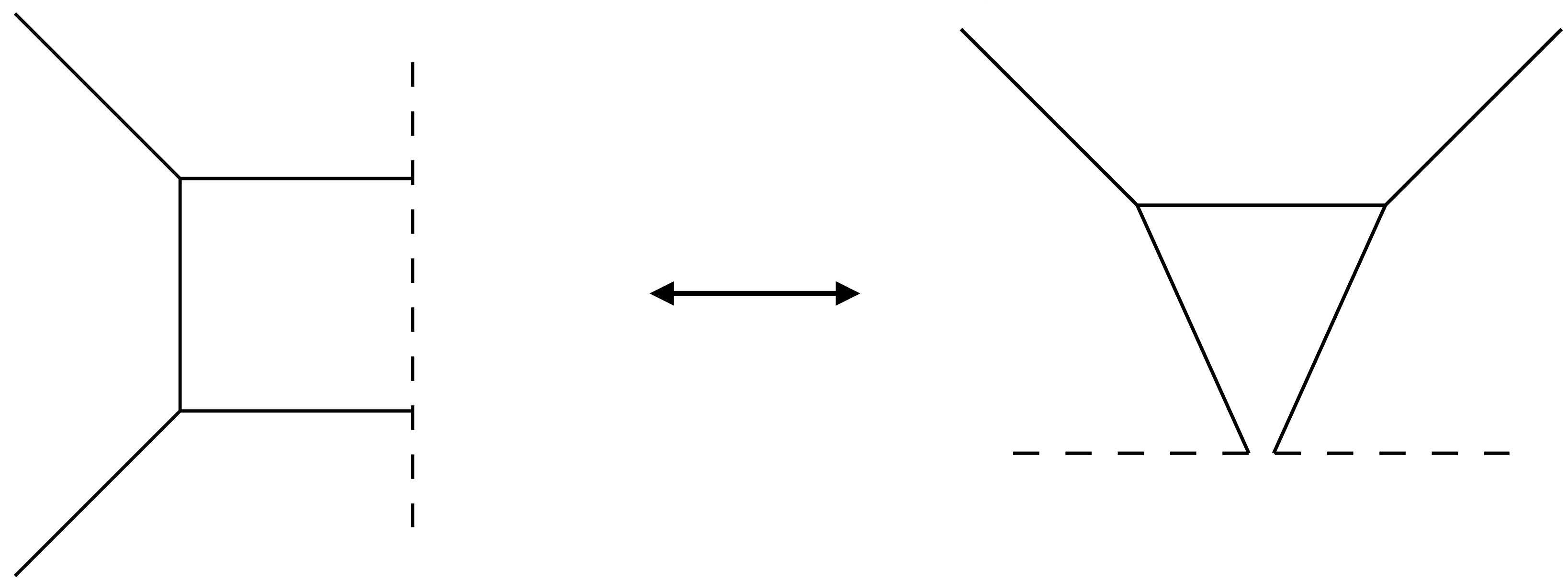
$$\left[\Delta\left(x^+(z)\right), \mathcal{T}\right] = 0.$$

Kimura-Pestun: qq-characters are generators of quiver W-algebra.

S-duality

the dual reflection state

$$\left(J_k \otimes 1 + 1 \otimes J_{-k}\right) \left|\Omega\right\rangle_s = 0.$$



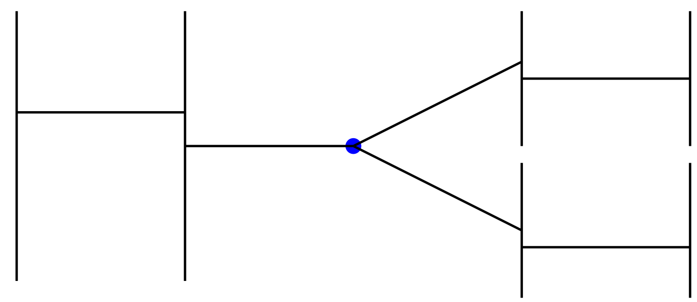
[work in progress]
see also [Kim, Yagi, 2017]

The partition function is invariant only in the unrefined limit.

Trivalent reflection operator?

with this object and its generalization, we can do ABCDEFG and affine, hyperbolic…

It was used in a recent (different) proposal for D, E instanton.



[Hayashi, Ohmori, 2017]

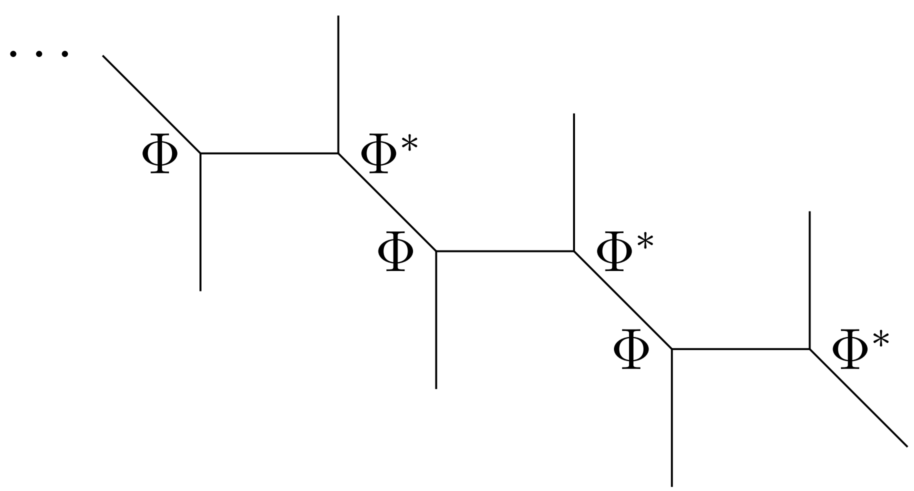
The qq-character can be checked. ✓

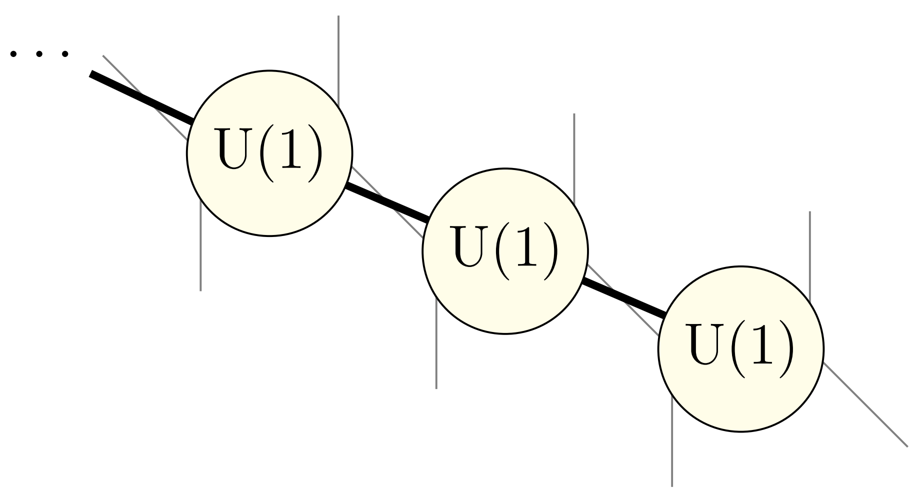
Problem: no idea how to formally define it in the algebra.

DIM algebra has charges corresponding to axio-dilaton charge.
This trivalent operator looks charged.

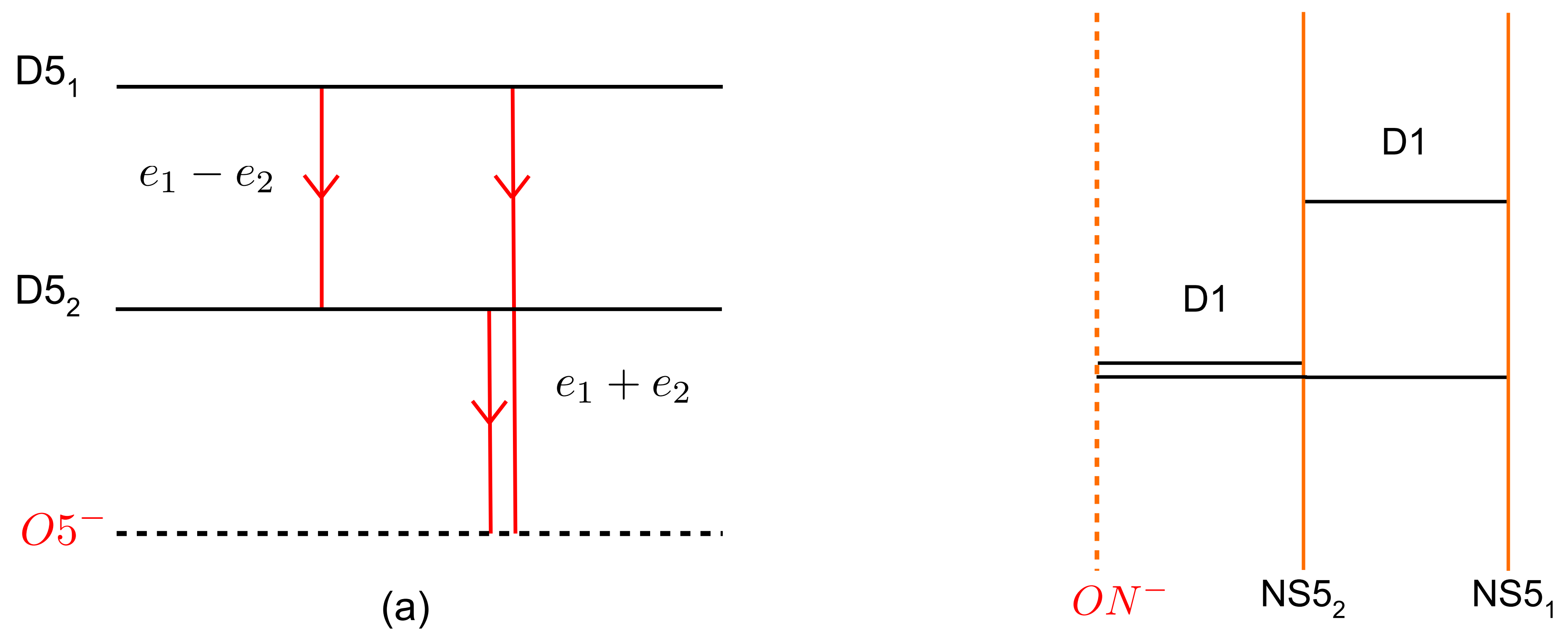
What are they (reflection objects with more than two legs) physically and mathematically?

Backup1: A_n quiver





Backup2: simplest D-type strings.



(a)

(b)

Backup3: Ding-Iohara-Miki

$$[\psi^{\pm}(z), \psi^{\pm}(w)] = 0, \quad \psi^{+}(z)\psi^{-}(w) = \frac{g(\hat{\gamma}w/z)}{g(\hat{\gamma}^{-1}w/z)}\psi^{-}(w)\psi^{+}(z)\,,$$

$$\psi^{+}(z)x^{\pm}(w) = g(\hat{\gamma}^{\mp 1/2}w/z)^{\mp 1}x^{\pm}(w)\psi^{+}(z), \quad \psi^{-}(z)x^{\pm}(w) = g(\hat{\gamma}^{\mp 1/2}z/w)^{\pm 1}x^{\pm}(w)\psi^{-}(z)\,,$$

$$x^{\pm}(z)x^{\pm}(w) = g(z/w)^{\pm 1}x^{\pm}(w)x^{\pm}(z)\,,$$

$$[x^{+}(z), x^{-}(w)] = \frac{(1-q_1)(1-q_2)}{(1-q_1q_2)}\left(\delta(\hat{\gamma}^{-1}z/w)\psi^{+}(\hat{\gamma}^{1/2}w) - \delta(\hat{\gamma}z/w)\psi^{-}(\hat{\gamma}^{-1/2}w)\right).$$

where

$$x^{\pm}(z) = \sum_{k\in\mathbb{Z}} z^{-k}x_k^{\pm}, \quad \psi^{+}(z) = \sum_{k\geq 0} z^{-k}\psi_k^{+}, \quad \psi^{-}(z) = \sum_{k\geq 0} z^{k}\psi_{-k}^{-},$$

$$\delta(z) = \textstyle\sum_{k\in\mathbb{Z}} z^k,$$

$$g(z) = \prod_{a=1,2,3}\frac{1-q_a z}{1-q_a^{-1}z}.$$

Backup4: "horizontal" representation of DIM

$$x^+(z) \mapsto u\gamma^n z^{-n}\eta(z), \quad x^-(z) \mapsto u^{-1}\gamma^{-n} z^n \xi(z), \qquad \hat{\gamma} = \gamma,$$

$$[a_m, a_n] = m\frac{1-q^{|m|}}{1-t^{|m|}}\delta_{m+n,0}.$$

Backup5: Awata-Feigin-Shiraishi vertex

$$\Phi_\lambda[u, t_i] = t_n(\lambda, u, t_i) : \Phi_\emptyset(t_i)\prod_{x\in\lambda}\eta(\chi_x) :,$$

$$\Phi^*_\lambda[u, t_i] = t^*_n(\lambda, u, t_i) : \Phi^*_\emptyset(t_i)\prod_{x\in\lambda}\xi(\chi_x) :,$$

where

$$\Phi_\emptyset(z) = \exp\left(-\sum_{n>0}\frac{1}{n}\frac{z^n}{1-q^n}a_{-n}\right)\exp\left(\sum_{n>0}\frac{1}{n}\frac{z^{-n}}{1-q^{-n}}a_n\right),$$

$$\Phi^*_\emptyset(z) = \exp\left(\sum_{n>0}\frac{1}{n}\frac{(\gamma z)^n}{1-q^n}a_{-n}\right)\exp\left(-\sum_{n>0}\frac{1}{n}\frac{\gamma^n z^{-n}}{1-q^{-n}}a_n\right),$$

$$\eta(z) = \exp\left(\sum_{n\geq 1}\frac{1-t^{-n}}{n}z^n a_{-n}\right)\exp\left(-\sum_{n\geq 1}\frac{1-t^n}{n}z^{-n}a_n\right),$$

$$\xi(z) = \exp\left(-\sum_{n\geq 1}\frac{1-t^{-n}}{n}\gamma^n z^n a_{-n}\right)\exp\left(\sum_{n\geq 1}\frac{1-t^n}{n}\gamma^n z^{-n}a_n\right).$$