Horizon temperature without space-time

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QFT on non-trivial space-times the notion of **particle** loses its universal meaning
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This rather *philosophical* observation is **directly** related to **one of the most significant insights** semiclassical gravity provides on (quantum) gravity

Puzzles of BH thermodynamics

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$\Rightarrow$ Fate of **unitarity** in BH quantum *evaporation*:

do BH **evolve pure states into mixed states**?

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Work in collaboration with J. Kowalski-Glikman
(Phys. Lett. B **788**, 82 (2019) [arXiv:1804.10550])

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ASIDE: irreps of the $ax + b$ group have been used in the literature on κ-Minkowski space in order to get a **"continuous limit"** of the model

(Agostini, J. Math. Phys. **48**, 052305 (2007); Dabrowski and Piacitelli, arXiv:1004.5091 [math-ph].)

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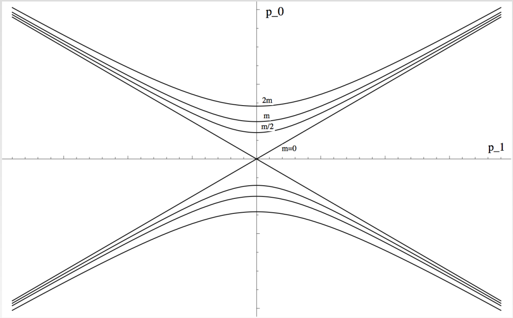
functions on **orbits of boosts** generated by N on momentum space



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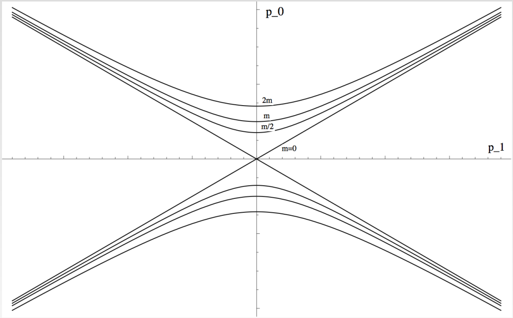

- **One-particle** Hilbert space $\equiv$ irrep labeled by m: functions on $p_0^2 - p_1^2 = m^2$, $p_0 > 0$, **one-particle states** are denoted by $|p\rangle$

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(Vilenkin and Klimyk, "Representation of Lie Groups and Special Functions" 1991)

There are just two of them labelled by the **eigenvalues of the translation generator** P

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$$\begin{aligned} T(\tau)|\omega\rangle_+ \quad &= e^{-i\tau P}\,|\omega\rangle_+ = e^{-i\tau\omega}\,|\omega\rangle_+ \\ D(\lambda)|\omega\rangle_+ \quad &= e^{-i\lambda R}\,|\omega\rangle_+ = |e^{-\lambda}\,\omega\rangle_+\,. \end{aligned}$$

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and inner product $\langle\psi|\psi'\rangle = \int_0^\infty \frac{d\omega}{\omega}\,\bar{\psi}(\omega)\psi'(\omega) = \int_0^\infty \frac{d\omega}{\omega}\,\langle\psi|\omega\rangle_{++}\langle\omega|\psi'\rangle$

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We call the kets $|\omega\rangle$: "P-particle" states

A field on the real line

Functions on the real line can be written in terms of a combination of the two irreducible representations $|\omega\rangle_+$ and $|\omega\rangle_-$ (Moses and Quesada, J. Math. Phys. **15**, 748 (1974))

$$\psi(t) \equiv \langle t|\psi\rangle = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{d\omega}{\omega}\, e^{i\omega t}\, {}_+\langle\omega|\psi\rangle + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \frac{d\omega}{|\omega|}\, e^{i\omega t}\, {}_-\langle\omega|\psi\rangle$$

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The action $T(\tau)\,\psi(t) = \psi(t+\tau)$, $D(\lambda)\,\psi(t) = \psi(e^\lambda t)$ gives

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Wave-functions associated to P-particles are given by "positive frequency" plane waves

$$\langle t|\omega\rangle_+ = \frac{1}{\sqrt{2\pi}}\, e^{i\omega t}\,, \quad \omega \in \mathbb{R}^+$$

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The coefficients $a(\omega)$ and $a^*(\omega)$ become **annihilation and creation operators** and the P-**vacuum state** $|0\rangle_P$ is defined by

$$a(\omega)\,|0\rangle_P = 0$$

Diagonalizing the R-operator: the Mellin transform

Eigenfunctions of R can be obtained via the **Mellin transform** of ${}_+\langle\omega|\psi\rangle$ (Vilenkin and Klimyk, 1991)

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Key point: the coefficient $\langle\Omega|\psi\rangle$ defines an annihilation operator $b(\Omega)$ which **shares the same vacuum** with the $a(\omega)$

$$b(\Omega)|0\rangle_P = \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{d\omega}{\omega}\,\omega^{-i\Omega}\,a(\omega)|0\rangle_P = 0$$

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from the action of the $ax + b$ group

$$T(\tau)\,\psi(t) = \psi(t + \tau)\,, \quad D(\lambda)\,\psi(t) = \psi(e^{\lambda} t) \implies \boxed{t = e^{\nu}}$$

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The generators of the $ax + b$ algebra will be now given by

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iii in order to get dimensions right one should introduce a parameter $a = [time]^{-1}$ in $t = e^{a\nu}$, we set $a = 1$ but this parameter will be **very important** at the end

The R-mode expansion

Plane waves oscillating with frequency Ω w.r.t. to R, or R-modes, are given by

$$\langle \nu | \Omega \rangle_R = \frac{1}{\sqrt{2\pi}}\, e^{i\Omega\nu} = \frac{1}{\sqrt{2\pi}}\, t^{-i\Omega} = {}_{+}\langle t | \Omega \rangle_R\,, \quad t \in \mathbb{R}^+\,.$$

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$$\psi_+(t) \equiv \theta(t)\,\psi(t) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{d\Omega}{\Omega}\left(t^{i\Omega}\, c_+(\Omega) + t^{-i\Omega}\, c_+^*(\Omega)\right)\,.$$

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Is this the same state as $|0\rangle_R$?

Linking the two representations: the Bogolubov map

Let us go back to the **expansion of the field** in terms on $ax + b$ irreps **diagonal in** P

$$\psi(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{d\omega}{\omega} \left(e^{i\omega t}\, {}_{+}\langle \omega | \psi \rangle + e^{-i\omega t}\, {}_{-}\langle -\omega | \psi \rangle \right) .$$

and substitute the **inverse Mellin transform**

$$|\omega\rangle_{+} = \int_{-\infty}^{\infty} d\Omega \, \langle \Omega | \omega \rangle_{+} \, |\Omega\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\Omega \, \omega^{-i\Omega} \, |\Omega\rangle \, ,$$

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restricting to positive t and comparing with the Ω expansion we have

$$c_+(\Omega) = \tfrac{\Omega}{\sqrt{2\pi}}\, \Gamma(-i\Omega) \left(e^{-\pi\Omega/2}\, b^*(\Omega) + e^{\pi\Omega/2}\, b(-\Omega) \right)$$

where $b(\Omega) = \langle \Omega|\psi\rangle$

The thermal spectrum

Now we can **answer our question**...in terms of operators on Fock space

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calculate the expectation value of the R-particle number operator in $|0\rangle_P$

$${}_P\langle 0|c_+^\dagger(\Omega)c_+(\Omega')|0\rangle_P = \frac{\Omega\,\delta(\Omega-\Omega')}{e^{2\pi\Omega}-1}$$

<u>**i.e. the P-vacuum contains a thermal distribution of R-particles at $T = 1/2\pi$.**</u>

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I described the simplest system exhibiting **thermal effects**
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OBJECTIVE

Contribute to **deeper understanding** of these phenomena providing a **minimal setting** in which such thermal behaviour emerges: only group theoretic ingredients concerning the **symmetries of space** and **their role as quantum observables**

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WHAT'S NEXT?

- The $ax + b$ group plays a key role in a variety of contexts: **non-commutative geometry, affine quantization, quantum cosmology**
- **Could the effect I described in this talk be relevant for these applications?**