Relativistic fluids, gravity and the fate of hydrodynamic frames

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Highlights

Relativistic hydrodynamics

Lorentz-invariant effective long-distance description of classical or quantum many-body states out of global thermal equilibrium [Eckart '40; Landau and Lifshitz '60]

- Fluid variables obey conservation equations
- Fluid variables are expressed as expansions in derivatives of $u^\nu(x)$, $T(x)$ and $\mu(x)$ – constitutive relations
- The coefficients of the derivative expansions describe transport properties – related to microscopic correlation functions

Key features *[Israel and Stewart '76; see book by Rezzolla and Zanotti '2013]*

- Derivative expansions are asymptotic series
- One usually treat first-order hydrodynamics
- This approximation raises stability and causality issues
- Hydrodynamic-frame invariance: u can be transformed upon simultaneous transformations of all other variables

Last feature is blurred by the derivative expansions – choosing a hydrodynamic frame is subtle and crucial

Recent revival *[Kovtun '07–'19; Romatschke, Son, Starinets, Stephanov '08; Indian group from '07]*

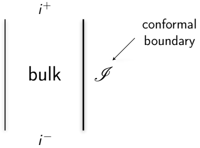
- Experimentally: subnuclear flows in heavy-ion collisions
- Theory: ideas from fluid/gravity holographic correspondence

Fluid/gravity correspondence

Macroscopic spin-off of AdS/CFT: Einstein's and relativistic Euler's equations in dimension D [Bhattacharyya, Haack, Hubeny, Loganayagam, Minwalla, ...'07]

$$\text{Einstein asymptotically locally AdS spacetime } \mathscr{E} \text{ with } \Lambda < 0$$
$$\updownarrow$$
$$\text{relativistic fluid on } \mathscr{I} = \partial\mathscr{E} \equiv \text{ conformal boundary}$$

Anti-de Sitter space: homogeneous spacetime with $\Lambda < 0$



Central question & method

- *The velocity u is presumed redundant in relativistic fluids – is it really arbitrary (putting aside any perturbation artefact)?*
- *If yes, is it also redundant in the dual Einstein spacetime?*

Naively yes ***but*** *...* *[Ciambelli, Petkou, Petropoulos, Siampos '17]*

...at least ***not*** *globally invariant* *[Campoleoni, Ciambelli, Marteau, Petropoulos, Siampos '18]*

Framework: 3-dim bulk vs. 2-dim boundary – simple and efficient

- *a systematic and exact bulk reconstruction is achievable*
- *asymptotically AdS Einstein spacetimes are known*
  - *locally AdS (e.g. Bañados – a subset is the BTZ family)*
  - *labelled with their conserved charges (mass, spin, ...)*

Approach: boundary fluid velocity $\leftrightarrow$ bulk conserved charges

Highlights

Relativistic hydrodynamics

Obey $\nabla_\mu T^{\mu\nu} = f^\nu$ *plus an equation of state plus Gibbs–Duhem with*

$$\boxed{T^{\mu\nu} = \varepsilon \frac{u^\mu u^\nu}{k^2} + p h^{\mu\nu} + \tau^{\mu\nu} + \frac{u^\mu q^\nu}{k^2} + \frac{u^\nu q^\mu}{k^2}}$$

- $\|\mathrm{u}\|^2 = -k^2, \quad h^{\mu\nu} = g^{\mu\nu} + \frac{u^\mu u^\nu}{k^2}$
- energy density $\varepsilon = \frac{1}{k^2} T_{\mu\nu} u^\mu u^\nu$ & thermodynamic pressure p
- q^μ, $\tau^{\mu\nu}$: heat current and viscous stress tensor – transverse

$$u^\mu q_\mu = 0 \quad u^\mu \tau_{\mu\nu} = 0 \quad q_\nu = -\varepsilon u_\nu - u^\mu T_{\mu\nu}.$$

expressed as u^ν- and T-derivative expansions with transport coefficients (heat conductivity, shear and bulk viscosity etc.)

The hydrodynamic-frame invariance

Landau–Lifshitz's statement for non-perfect fluids [Theoretical Physics vol. 6 §136]

First of all, however, we must discuss more closely the concept of the velocity u^μ itself. In relativistic mechanics, an energy flux necessarily involves a mass flux. Hence, when there is (e.g.) a heat flux, the definition of the velocity in terms of the mass flux density (as in non-relativistic fluid dynamics) has no direct meaning.

Translation in the formalism

Any arbitrary transformation of u^μ (local Lorentz transformation) can be compensated by an appropriate modification of T, ε, p, q^μ and $\tau^{\mu\nu}$ such that $T^{\mu\nu}$ ***and*** the entropy current S^μ remain invariant

Note: This is ***not*** Lorentz invariance (generally absent globally)

Consequences & features

Special hydrodynamic frames ("gauge conditions")

- Landau–Lifshitz: $q^\nu = 0$
- Eckart in the presence of $J^\nu = \varrho u^\nu + j^\nu$ (and μ): $j^\nu = 0$

Subtleties

- Generically T, ε, p, q^μ and $\tau^{\mu\nu}$ are transformed *order by order* in the derivative expansion for changes $\mathrm{u} \to \mathrm{u} + \delta\mathrm{u}$
- Global issues (as in gauge transformations) are ignored
- No microscopic definition for $S^\mu = su^\mu + \frac{R^\mu}{T}$ with R^μ built order by order to comply with macroscopic requirements

In 2 dimensions

The transverse direction to u is unique: $*u$

- $q_\mu = \chi * u_\mu$ (χ: heat density)
- $\tau_{\mu\nu} = \tau \frac{*u_\mu * u_\nu}{k^2}$ (τ: viscous bulk pressure or dynamic pressure)

$\longrightarrow T_\mu{}^\mu = p - \varepsilon + \tau$

Local Lorentz boosts are captured by a unique $\psi(x)$ *and act exactly*

- on the velocity $\begin{pmatrix} \mathrm{u}' \\ *\mathrm{u}' \end{pmatrix} = \begin{pmatrix} \cosh\psi(x) & \sinh\psi(x) \\ \sinh\psi(x) & \cosh\psi(x) \end{pmatrix} \begin{pmatrix} \mathrm{u} \\ *\mathrm{u} \end{pmatrix}$
- on every scalar $\varepsilon, p, \chi, \tau$ in a way that keeps $T^{\mu\nu}$ invariant

An invariant entropy current can be defined in closed form

Highlights

Reconstruct $ds^2_{bulk}\left[g_{\mu\nu}, T^{\mu\nu}\right]$ *in any D*

The u^μ*-derivative expansion – fluid/gravity* [Bhattacharyya et al '07; Haack et al '07]

- Guideline: Weyl covariance – the bulk metric must be invariant under boundary Weyl transformations
- Tool: Weyl connection $\mathrm{A} = \frac{1}{k^2}\left(\mathrm{a} - \frac{\Theta}{2}\mathrm{u}\right)$ and Weyl covariant derivative $\mathscr{D} = \nabla + w\mathrm{A}$ (a is the acceleration and $\Theta = \nabla \cdot \mathrm{u}$)
- Output: $\mathrm{d}s^2_{\mathrm{bulk}} =$ complicated expression based on the boundary data & their derivatives

The reconstruction from 2 to 3 dimensions

Is simpler than in higher dimensions

Most velocity-derivative tensors vanish (shear, vorticity etc.) – the would-be series terminates & the bulk is locally AdS_3 ($\Lambda = -3k^2$)

The general expression for bulk spacetime in $2+1$ dimensions

$$\boxed{ds^2_{bulk} = 2\frac{u}{k^2}\left(dr + rA\right) + r^2 ds^2 + \frac{8\pi G}{k^4} u \left(\varepsilon u + \chi * u\right)}$$

invariant under Weyl: $ds^2 \to {}^{ds^2}/_{\mathcal{B}^2} \Rightarrow r \to r\mathcal{B}$

The metric is Einstein provided the fluid

- has conformal state equation $p = \varepsilon$ & *anomalous* viscous bulk pressure $\tau = \frac{R}{8\pi G}$
- obeys Euler's equations

The space of solutions

The bulk metric is always locally AdS_3 – with different charges

- Solutions explicitly depend on the fluid velocity u
- Changing u is necessarily a bulk diffeomorphism

The central question is: does $u \to u'$ induce a small or a large bulk diffeomorphism?

Equivalently: can we choose a u-gauge (Eckart, LL, ...) and still scan the entire solution space?

- *If yes: hydrodynamic frame invariance is fully valid*
- *If no: hydrodynamic frame invariance has global issues*

Highlights

The flat & Weyl-flat boundary

A more restricted – still sufficient – framework: $R = 0$ *&* $dA = 0$
genuinely conformal *i.e.* $\boldsymbol{\partial} \cdot \mathsf{T} = 0$

Two extreme frames

- Dissipative fluid at rest $\mathsf{u} = -k^2 \mathrm{d}t$
- Perfect fluid $\chi = 0$ with arbitrary velocity (Landau–Lifshitz)

Note: the latter has been considered as sufficient in the literature

[Bhattacharyya, Haack, Hubeny, Loganayagam, Minwalla, Rangamani, Yarom, ...'07]

Dissipative boundary fluid at rest

Reconstructed bulk spacetime in terms of $L_\pm(x^\pm)$

$$
\begin{aligned}
ds^2_{Einstein} &= -\frac{1}{k}\left(dx^+ - dx^-\right)dr + r^2 dx^+ dx^- \\
&\quad + \frac{1}{k^2}\left(L_+ dx^+ - L_- dx^-\right)\left(dx^+ - dx^-\right)
\end{aligned}
$$

General Bañados locally AdS_3 solutions in BMS gauge [Bañados '99]
Note: Bañados zero-modes include BTZ solutions [BTZ '92; BHTZ '93]

- $L_+ + L_- = 4\pi G\varepsilon = M$ mass of the black hole
- $L_+ - L_- = -4\pi G\chi = kJ$ spin of the black hole

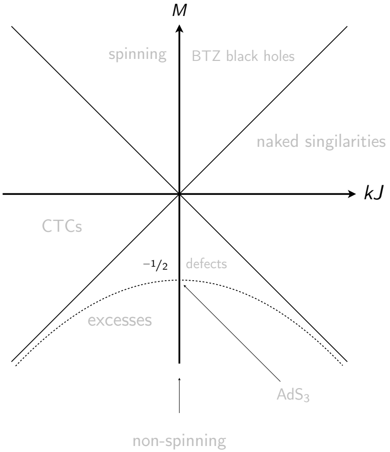


Figure: Bañados zero modes – "good" only inside the cone (fluid energy–momentum tensor has real eigenvalues)

Charges and algebra

Associated with asymptotic Killing vectors – compatible with fall-offs

Ask Glenn Barnich for details

For Bañados solutions

$$L_m^\pm = \frac{1}{8\pi kG}\int_0^{2\pi} \mathrm{d}x\, e^{imx^\pm}\left(L_\pm + \frac{1}{4}\right)$$

(here $x = \frac{x^+ + x^-}{2}$) obey Virasoro with $c = {}^3/_{2kG}$ [Brown, Henneaux '86]

$$\begin{aligned} i\left\{L_m^\pm, L_n^\pm\right\} &= (m-n)L_{m+n}^\pm + \frac{c}{12}m\left(m^2-1\right)\delta_{m+n,0} \\ \left\{L_m^\pm, L_n^\mp\right\} &= 0 \end{aligned}$$

Perfect fluid with arbitrary velocity (LL frame)

Reconstructed spacetime in terms of $\xi^{\pm}(x^{\pm})$ [Haack, Yarom; Bhattacharyya et al '08]

$$
\begin{aligned}
ds^2_{Einstein} \quad = \quad & -\frac{1}{k}\left(\sqrt{-\frac{\xi^-}{\xi^+}}dx^+ - \sqrt{-\frac{\xi^+}{\xi^-}}dx^-\right)dr \\
& + \left(\frac{M}{2k^2} - \frac{r}{2k}\sqrt{-\xi^+\xi^-}\xi^{+\prime}\right)\left(\frac{dx^+}{\xi^+}\right)^2 \\
& + \left(\frac{M}{2k^2} - \frac{r}{2k}\sqrt{-\xi^+\xi^-}\xi^{-\prime}\right)\left(\frac{dx^-}{\xi^-}\right)^2 \\
& + \left(r^2 + \frac{r}{2k}\frac{1}{\sqrt{-\xi^+\xi^-}}\left(\xi^{+\prime} + \xi^{-\prime}\right) + \frac{M}{k^2\xi^+\xi^-}\right)dx^+dx^-
\end{aligned}
$$

not BMS unless $\xi^{\pm}$ are constant – resulting in BTZ & all non-spinning zero-modes of Bañados family

Charges from asymptotic Killing vectors

$$L_m^{\pm} = \frac{1}{16\pi kG} \int_0^{2\pi} \mathrm{d}x \, \mathrm{e}^{\mathrm{i}mx^{\pm}} \left(\frac{1}{\zeta^{\pm 2}} - 1 \right)$$

obey de Witt rather than Virasoro with $\tilde{L}_m^{\pm} = L_m^{\pm} + \frac{1}{8kG}\delta_{m,0}$

$$\left\{ \tilde{L}_m^{\pm}, \tilde{L}_n^{\pm} \right\} = \mathrm{i}(m-n)\tilde{L}_{m+n}^{\pm} \qquad \left\{ \tilde{L}_m^{\pm}, \tilde{L}_n^{\mp} \right\} = 0$$

The family of locally AdS_3 spacetimes obtained holographically from 2-dim fluids in the Landau–Lifshitz frame overlap only partially Bañados solutions obtained in the frame where the fluid is at rest → hydrodynamic-frame invariance is violated (or does not survive the holographic bulk reconstruction)

Highlights

Setting the stage

Relativistic fluids

The general AdS_3 bulk reconstruction and the question

The answer from charge indentification

Summary

In 2-dim boundary 3-dim bulk

Achievements

- general $\mathrm{d}s^2_{\text{Einstein}}$ from *arbitrary* boundary metric & fluid data
- charges and algebra of the bulk sensitive to the boundary-fluid hydrodynamic frame – revealed in the class $R = \mathrm{d}A = 0$

The "same" conformal fluid viewed

- *at rest with heat current gives Bañados spacetimes*
- *as perfect with arbitrary velocity (i.e. LL) gives a different class of partially overlapping Bañados (non-spinning zero-modes)*

Doubts on the validity of fluid-frame invariance

Concretely in 3-dim fluid/gravity holography: global issues

- the derivative expansion is clearly sensitive to the choice of u
- the fluid itself might also be – how to check ?

In higher dimensions – infinite series – possibly also local issues

- in fluid/gravity original literature LL frame was assumed in the derivative expansion – possibly inaccurate
- *exact closed* resummation of the derivative expansion needs the *Eckart* frame [Ciambelli, Gath, Mukhopadhyay, Petkou, Petropoulos, Siampos '15–17]

Highlights

Fefferman–Graham vs. derivative expansion

Miscellaneous in two dimensions

The detailed charge computation

The AdS bulk reconstruction from the boundary fluid

The "initial data": first and second fundamental forms

- *boundary metric ds^2 (neither flat nor conformally flat)*
- *conserved energy–momentum tensor T*

Two options exist to reconstruct perturbatively the asymptotically AdS bulk (*pure gravity* with $\Lambda = -3k^2$):

1. Fefferman–Graham expansion: mathematically robust, locks fall-offs, not resummable, does not discriminate asympt. locally vs. globally AdS bulks, with singular $k \to 0$ limit
2. Derivative expansion (close to Eddington–Finkelstein): for fluid/gravity correspondence avoids latter caveats but requires an extra piece of bry. data – *time-like fluid congruence u* – and does not define a precise gauge

The reconstruction from 2 to 3 dimensions

Is simpler than in higher dimensions

Most velocity-derivative tensors vanish (shear, vorticity etc.) – the would-be series terminates & the bulk is locally AdS_3 ($\Lambda = -3k^2$)

The general expression for bulk spacetime in $2+1$ *dimensions*

$$\boxed{ds^2_{bulk} = 2\frac{u}{k^2}\left(dr + rA\right) + r^2 ds^2 + \frac{8\pi G}{k^4}u\left(\varepsilon u + \chi * u\right)}$$

invariant under Weyl: $ds^2 \to {}^{ds^2}/_{\mathcal{B}^2} \Rightarrow r \to r\mathcal{B}$

- $\mathrm{d}s^2 = \frac{1}{k^2}\left(-\mathrm{u}^2 + *\mathrm{u}^2\right)$: the boundary metric
- $\varepsilon, \chi, \mathrm{u}$: conformal fluid data of weight $2, 2, -1$
- $\mathrm{A} = \frac{1}{k^2}\left(\Theta^* * \mathrm{u} - \Theta\mathrm{u}\right)$: Weyl connection ($\mathrm{A} \to \mathrm{A} - \mathrm{d}\ln\mathcal{B}$)

The metric is Einstein provided the fluid

- has conformal state equation $p = \varepsilon$ & *anomalous* viscous bulk pressure $\tau = \frac{R}{8\pi G}$
- obeys Euler's equations with *external force* of geometric nature

$$\begin{cases} \left(u^\mu + *u^\mu\right) \mathscr{D}_\mu \left(\varepsilon + \chi\right) = \frac{1}{4\pi G} * u^\mu \mathscr{D}_\mu F, \\ \left(u^\mu - *u^\mu\right) \mathscr{D}_\mu \left(\varepsilon - \chi\right) = \frac{1}{4\pi G} * u^\mu \mathscr{D}_\mu F. \end{cases}$$

with $F = *\mathrm{d}A$ the Weyl curvature

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Hydrodynamic frame

The energy–momentum tensor

$$\mathsf{T} = \frac{1}{2k^2}\left((\varepsilon+\chi)\,(\mathsf{u}+*\mathsf{u})^2 + (\varepsilon-\chi)\,(\mathsf{u}-*\mathsf{u})^2\right) + \frac{1}{k^2}(p-\varepsilon+\tau)*\mathsf{u}^2$$

Hydrodynamic-frame transformation with $\psi = \psi(x)$

- on the velocity $\begin{pmatrix} \mathsf{u}' \\ *\mathsf{u}' \end{pmatrix} = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix}\begin{pmatrix} \mathsf{u} \\ *\mathsf{u} \end{pmatrix}$
- $2\chi' = (\varepsilon+\chi)\mathrm{e}^{-2\psi} - (\varepsilon-\chi)\mathrm{e}^{2\psi} - (p+\tau-\varepsilon)\sinh 2\psi$
  $2\varepsilon' = (\varepsilon+\chi)\mathrm{e}^{-2\psi} + (\varepsilon-\chi)\mathrm{e}^{2\psi} + 2(p+\tau-\varepsilon)\sinh^2\psi$
  $p'+\tau'-\varepsilon' = p+\tau-\varepsilon$ (trace) leave $T_{\mu\nu}$ invariant

The Landau–Lifshitz frame: $\chi_{LL} = 0$

$$e^{4\psi_{LL}} = \frac{\tau + p + \varepsilon + 2\chi}{\tau + p + \varepsilon - 2\chi}$$

ε_{LL} *is an eigenvalue and* u_{LL} *an eigenvector of* T

- $\varepsilon_{LL} = \sqrt{\left(\frac{p+\varepsilon+\tau}{2} + \chi\right)\left(\frac{p+\varepsilon+\tau}{2} - \chi\right)} - \frac{\tau+p-\varepsilon}{2}$
- $u_{LL} = u \cosh \psi_{LL} + *u \sinh \psi_{LL}$

$\varepsilon_{LL} + \tau$ *is the other eigenvalue associated with* $*u_{LL}$

Entropy current

Local thermodynamic equilibrium

- conformal equation of state: $p = \varepsilon$
- Stefan: $\varepsilon = \sigma T^2$
- Gibbs–Duhem: $Ts = p + \varepsilon$

$$s = 2\sqrt{\sigma\varepsilon}$$

Invariant entropy current (valid for $\tau = \tau_{LL} \neq 0$)

- $\mathsf{S}_\mathbf{0} = s_{\mathsf{LL}}\mathsf{u}_{\mathsf{LL}} = 2\sqrt{\sigma\varepsilon_{\mathsf{LL}}}\mathsf{u}_{\mathsf{LL}}$
- $\nabla \cdot \mathsf{S}_\mathbf{0} = -\sqrt{\frac{\sigma}{\varepsilon_{\mathsf{LL}}}}\left(\Theta_{\mathsf{LL}}\tau + \mathsf{u}_{\mathsf{LL}} \cdot \mathsf{f}\right)$

$S_{\mathbf{0}}$ up to second order in $\chi, \tau \ll \varepsilon$

$$\mathsf{S}_{\mathbf{0}} = s\mathsf{u} + \frac{\mathsf{q}}{T} - \frac{\chi^2}{4\varepsilon T}\mathsf{u} - \frac{\tau}{2\varepsilon T}\mathsf{q} + \cdots$$

$(\mathsf{q} = \chi * \mathsf{u})$

$\nabla \cdot S_{\mathbf{0}}$ up to first order for $\chi, \tau \ll \varepsilon$

$$\nabla \cdot \mathsf{S}_{\mathbf{0}(1)} = -\frac{1}{T}\Theta\tau_{(1)} = \frac{\zeta}{T}\Theta^2$$

$(\tau_{(1)} = -\zeta\Theta)$

General requirements – all met

1. free perfect limit: $\mathsf{S}\big|_{\chi=\tau=0} = \mathsf{S}_{(0)} = s\mathsf{u} = 2\sqrt{\sigma\varepsilon}\mathsf{u}$
2. stability: $\frac{\partial \mathsf{S}\cdot\mathsf{u}}{\partial\tau}\Big|_{\chi=\tau=0} = 0$
3. first-order (CIT) correction: $\mathsf{S}_{(1)} = \frac{\mathsf{q}}{T}$
4. second-order (EIT) corrections: $\mathsf{S}_{(2)}$ might contain $\frac{\tau^2}{\varepsilon T}\mathsf{u}$, $\frac{\chi^2}{\varepsilon T}\mathsf{u}$ and $\frac{\tau}{\varepsilon T}\mathsf{q}$
5. second law: $\nabla\cdot\mathsf{S} \geqslant 0$ implies $\zeta \geqslant 0$

Generalizable in the presence of chemical potential μ with density ϱ and conserved current J^ν

Asymptotic Killings

Dissipative fluid at rest (Bañados)

$$\zeta = \zeta^r \partial_r + \zeta^+ \partial_+ + \zeta^- \partial_-$$

with

$$\begin{aligned}
\zeta^r &= -\frac{r}{2}\left(Y^{+\prime} + Y^{-\prime}\right) + \frac{1}{2k}\left(Y^{+\prime\prime} - Y^{-\prime\prime}\right) \\
&\quad - \frac{1}{2k^2 r}\left(L_+ - L_-\right)\left(Y^{+\prime} - Y^{-\prime}\right) \\
\zeta^\pm &= Y^\pm - \frac{1}{2kr}\left(Y^{+\prime} - Y^{-\prime}\right)
\end{aligned}$$

for arbitrary chiral functions $Y^+(x^+)$ and $Y^-(x^-)$

Obey an algebra for the modified Lie bracket [Barnich '10]

$$\zeta_3 = [\zeta_1, \zeta_2]_{\mathsf{M}} = [\zeta_1, \zeta_2] - \delta_{\zeta_2}\zeta_1 + \delta_{\zeta_1}\zeta_2$$

with

$$Y_3^{\pm} = Y_1^{\pm}\partial_{\pm} Y_2^{\pm} - Y_2^{\pm}\partial_{\pm} Y_1^{\pm}$$

Perfect fluids with arbitrary velocity: $\eta\ (\epsilon^{\pm})$

$\eta = \eta^r\partial_r + \eta^+\partial_+ + \eta^-\partial_-$ with $\eta^r = -\frac{r}{2}\left(\epsilon^{+\prime} + \epsilon^{-\prime}\right)$, $\eta^{\pm} = \epsilon^{\pm}$
form an algebra for the Lie bracket

$$[\eta_1, \eta_2] = \eta_3$$

with

$$\epsilon_3^{\pm} = \epsilon_1^{\pm}\epsilon_2^{\pm\prime} - \epsilon_2^{\pm}\epsilon_1^{\pm\prime}$$

Highlights

Fefferman–Graham vs. derivative expansion

Miscellaneous in two dimensions

The detailed charge computation

Dissipative boundary fluid at rest: $\xi^{\pm} = \pm 1$

Reconstructed bulk spacetime in terms of $L_{\pm}(x^{\pm})$

$$
\begin{aligned}
ds^2_{Einstein} &= -\frac{1}{k}\left(dx^+ - dx^-\right) dr + r^2 dx^+ dx^- \\
&\quad + \frac{1}{k^2}\left(L_+ dx^+ - L_- dx^-\right)\left(dx^+ - dx^-\right)
\end{aligned}
$$

General Bañados locally AdS_3 solutions in BMS gauge [Bañados '99]
Note: Bañados zero-modes include BTZ solutions [BTZ '92; BHTZ '93]

- $L_+ + L_- = 4\pi G \varepsilon = M$ mass of the black hole
- $L_+ - L_- = -4\pi G \chi = kJ$ spin of the black hole

Determining the charges in gravitational backgrounds

Most popular: Komar charges ("surface" charges)

Killing $\zeta^\mu \to$ conserved current $K_\mu = R_{\mu\nu}\zeta^\nu = \nabla^\nu \nabla_\mu \zeta_\nu = \nabla^\nu A_{\mu\nu}$

$$Q_K = \oint_{S_\infty^{D-2}} *\mathsf{A}$$

e.g. M and J associated with ∂_t and ∂_φ in Kerr

More associated with asymptotic Killing vectors

$\mathscr{L}_\zeta g_{MN} = -\delta_\zeta g_{MN}$ compatible with fall-offs

For Bañados solutions

$\zeta = \zeta^r \partial_r + \zeta^+ \partial_+ + \zeta^- \partial_-$ in terms of $Y^\pm(x^\pm)$

$$\delta_\zeta L_\pm = -Y^\pm L'_\pm - 2L_\pm Y^{\pm\prime} + \frac{1}{2} Y^{\pm\prime\prime\prime}$$

The charge computation

$$Q_Y\left[g-\bar{g},\bar{g}\right]=\frac{1}{8\pi kG}\int_0^{2\pi}\mathrm{d}x\left(Y^+\left(L_++\tfrac{1}{4}\right)-Y^-\left(L_-+\tfrac{1}{4}\right)\right)$$

- $\bar{g}$: reference metric with $L_+=L_-={}^{-1}\!/_4$ (empty AdS_3)
- charge algebra: $\{Q_{Y_1},Q_{Y_2}\}=\delta_{\zeta_1}Q_{Y_2}=-\delta_{\zeta_2}Q_{Y_1}$

A seminal result *[Brown, Henneaux '86]*

- set $Y^\pm=e^{imx^\pm}$: get the modes (here $x=\frac{x^++x^-}{2}$)

$$Q_{e^{imx^\pm}}=\frac{1}{8\pi kG}\int_0^{2\pi}\mathrm{d}x\,e^{imx^\pm}\left(L_\pm+\frac{1}{4}\right)=L_m^\pm$$

- determine the algebra: Virasoro with $c={}^3\!/_{2kG}$

$$\begin{aligned}
i\left\{L_m^\pm,L_n^\pm\right\} &= (m-n)L_{m+n}^\pm+\frac{c}{12}m\left(m^2-1\right)\delta_{m+n,0}\\
\left\{L_m^\pm,L_n^\mp\right\} &= 0
\end{aligned}$$

Perfect fluid with arbitrary velocity (LL frame): $L_\pm = \frac{M}{2}$

Reconstructed spacetime in terms of $\xi^\pm(x^\pm)$ *[Haack, Yarom; Bhattacharyya et al '08]*

$$
\begin{aligned}
ds^2_{Einstein} \;\; = \;\; & -\frac{1}{k}\left(\sqrt{-\frac{\xi^-}{\xi^+}}dx^+ - \sqrt{-\frac{\xi^+}{\xi^-}}dx^-\right)dr \\
& + \left(\frac{M}{2k^2} - \frac{r}{2k}\sqrt{-\xi^+\xi^-}\xi^{+\prime}\right)\left(\frac{dx^+}{\xi^+}\right)^2 \\
& + \left(\frac{M}{2k^2} - \frac{r}{2k}\sqrt{-\xi^+\xi^-}\xi^{-\prime}\right)\left(\frac{dx^-}{\xi^-}\right)^2 \\
& + \left(r^2 + \frac{r}{2k}\frac{1}{\sqrt{-\xi^+\xi^-}}\left(\xi^{+\prime} + \xi^{-\prime}\right) + \frac{M}{k^2\xi^+\xi^-}\right)dx^+dx^-
\end{aligned}
$$

not BMS unless $\xi^\pm$ are constant – case captured by $\xi^\pm = \pm 1$ resulting in BTZ & all non-spinning zero-modes of Bañados family

Asymptotic Killing vectors

$\eta = \eta^r \partial_r + \eta^+ \partial_+ + \eta^- \partial_-$ with $\eta^r = -\frac{r}{2}\left(\epsilon^{+\prime} + \epsilon^{-\prime}\right)$, $\eta^\pm = \epsilon^\pm$

$$\delta_\eta \zeta^\pm = \epsilon^\pm \zeta^{\pm\prime} - \zeta^\pm \epsilon^{\pm\prime}$$

Charges and algebra

- choose empty AdS_3 as reference: $\zeta^\pm = \pm 1$ and $M = {}^{-1}/_{2}$
- set $\epsilon^\pm = e^{imx^\pm}$: get the modes

$$Q_{e^{imx^\pm}} = \frac{1}{16\pi kG} \int_0^{2\pi} dx\, e^{imx^\pm} \left(\frac{1}{\zeta^{\pm 2}} - 1 \right) = L_m^\pm$$

- determine the algebra: de Witt rather than Virasoro

$$\{\tilde{L}^{\pm}_m, \tilde{L}^{\pm}_n\} = \mathrm{i}(m-n)\tilde{L}^{\pm}_{m+n}, \quad \{\tilde{L}^{\pm}_m, \tilde{L}^{\mp}_n\} = 0$$

$$\tilde{L}^{\pm}_m = L^{\pm}_m + \frac{1}{8kG}\delta_{m,0}$$

The family of locally AdS_3 spacetimes obtained holographically from 2-dim fluids in the Landau–Lifshitz frame overlap only partially Bañados solutions (non-spinning BTZ and excess/defects geometries i.e. $L_{\pm} = {}^{M}/{}_{2}$ and $\xi_{\pm} = \pm 1$)

- *either hydrodynamic-frame invariance is **only local***
- *or its **global invariance does not survive** the holographic bulk reconstruction*