Multitrace matrix models and fuzzy field theories

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Fuzzy spaces

- Fuzzy spaces = noncommutative compact spaces

$$[x_i, x_j] = \theta_{ij}$$

- Algebra of functions $\leftrightarrow$ finite dimensional matrix algebra

Examples

- **Fuzzy sphere**

  Hoppe '82; Madore '92; Grosse, Klimčík, Prešnajder '90s

$$[x_i, x_j] = i\theta\epsilon_{ijk}x_k \qquad x_1^2 + x_2^2 + x_3^2 = R^2$$

$$x_i = \frac{2R}{\sqrt{N^2-1}}L_i \qquad \theta = \frac{2R}{\sqrt{N^2-1}}$$

  - L_i are $SU(2)$ generators in N dimensional matrix representation.
  - Limit $N \to \infty$ recovers commutative sphere.

Examples

- $\mathbb{C}\mathbf{P}^n$ **fuzzy spaces**

  Alexanian, Balachandran, Immirzi, Ydri '02; Karabali, Nair, Randjbar-Daemi '04; Grosse, Steinacker '05

$$[x_a, x_b] = i\theta f_{abc} x_c \qquad x_a x_a = R^2$$

$$x_a = \frac{R}{\sqrt{L(L+n)}} T_a \qquad \theta = \frac{2R}{\sqrt{L(L+n)}}$$

$$N = \frac{(L+n)!}{n!L!}$$

Scalar field theory

$\boxed{\text{Scalar field theory on a fuzzy space} \leftrightarrow \text{random matrix model}}$

$$Z = \int [dM] e^{-S[M]}$$

$$S[M] = \mathrm{Tr}\left(\frac{1}{2} m^2 M^2 + g M^4 + \frac{1}{2} M \mathcal{K} M\right)$$

Kinetic term:

$$M \mathcal{K} M = M [L_i, [L_i, M]]$$

Analytical treatment

- We rewrite the integration parameters:

$$M = U^\dagger \Lambda U$$
$$\Lambda = (\lambda_1, \lambda_2, \dots \lambda_N) \qquad U \in U(n)$$
$$dM = \left[\prod_{i<j} (\lambda_i - \lambda_j)^2\right] d\Lambda dU$$

$$Z = \int \left(\prod_{i=1}^{N} d\lambda_i\right) e^{-N^2\left[\frac{1}{2} r \frac{1}{N}\sum \lambda_i^2 + g\frac{1}{N}\sum \lambda_i^4 - \frac{2}{N^2}\sum_{i<j}\log|\lambda_i - \lambda_j|\right]}$$
$$\boxed{\int dU e^{-\frac{1}{2}\mathrm{Tr}[U^\dagger \Lambda U \mathcal{K}(U^\dagger \Lambda U)]}}$$

Effective action

- To study the model analytically we introduce the effective action:

$$\boxed{e^{-N^2 S_{eff}[\Lambda]} = \int dU e^{-\frac{1}{2}\mathrm{Tr}[U^\dagger \Lambda U \kappa (U^\dagger \Lambda U)]}}$$

$$Z = \int d\Lambda e^{-N^2 \tilde{S}[\Lambda]}$$

$$\tilde{S}[\Lambda] = \frac{1}{2} r \frac{1}{N} \sum \lambda_i^2 + g \frac{1}{N} \sum \lambda_i^4 - \frac{2}{N^2} \sum_{i<j} \log |\lambda_i - \lambda_j| + S_{eff}[\Lambda]$$

- The key problem is therefore to determine $S_{eff}[\lambda]$.

Perturbative treatment

High temperature expansion:

$$\boxed{e^{-N^2 S_{eff}[\Lambda]} = \int dU e^{-\frac{1}{2}\epsilon \mathrm{Tr}[U^\dagger \Lambda U \mathcal{K}(U^\dagger \Lambda U)]}}$$

O'Connor, Sämann '07; Sämann '10; Sämann '15

In large N limit:

$$S_{eff} = \frac{1}{2}\left[\epsilon\frac{1}{2}t_2 - \epsilon^2\frac{1}{24}t_2^2 + \epsilon^4\frac{1}{2880}t_2^4\right] - \epsilon^3\frac{1}{432}t_3^2 - \epsilon^4\frac{1}{3456}(t_4 - 2t_2^2)^2$$

$$t_n = \mathrm{Tr}\left(M - \frac{1}{N}(\mathrm{Tr}M)\,\mathbb{I}_{\mathbb{N}}\right)^n$$

Natural choice of parameters due the symmetry of the kinetic term $M \to M + \alpha\mathbb{I}_{\mathbb{N}}$.

Non-perturbative treatment

- Non-perturbative treatment is based on the fact that free theory is analytically solvable.

  Steinacker '05

- The free theory results are reproduced by the effective action:

  Polychronakos '13

$$S_{eff} = \frac{1}{2} F(t_2) + \mathcal{R}$$

$$F(t_2) = \log \left(\frac{t_2}{1 - e^{-t_2}} \right)$$

Analytical treatment of the multitrace matrix models

- We can use the saddle point approximation.
- In large N limit, the integral Z is dominated by the most probable eigenvalue configuration.

$$Z = \int d\Lambda e^{-N^2 \tilde{S}[\Lambda]}$$

$$\boxed{\frac{\partial \tilde{S}}{\partial \lambda_i} = 0}$$

Solving the aproximatives models

- The basic matrix models:

$$\boxed{\begin{aligned} S &= \mathrm{Tr}\left(\frac{1}{2}rM^2 + gM^4\right) \\ S &= \mathrm{Tr}\left(aM + \frac{1}{2}rM^2 + gM^4\right) \end{aligned}}$$

Brezin, Itzykson, Parisi, Zuber '78

Tekel '18

- What we need to solve now:

$$\boxed{S = \mathrm{Tr}\left(\frac{1}{2}rM^2 + gM^4\right) + \mathcal{F}(t_2, t_3, t_4)}$$

Itzykson, Zuber '80; Das, Dhar, Sengupta, Wadia '90

Saddle point equation

- **For the symmetric solutions**:

$$\boxed{t_3 = 0, \quad t_{2n} = c_{2n} \quad c_n = \mathrm{Tr} M^n}$$

$$\frac{\partial S}{\partial \lambda_i} = r\lambda_i + 4g\lambda_i^3 + \frac{\partial \mathcal{F}}{\partial c_2}\lambda_i + 4\frac{\partial \mathcal{F}}{\partial c_4}\lambda_i^3 - \frac{2}{N}\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j} = 0$$

$$r_{eff}\lambda_i + 4g_{eff}\lambda_i^3 - \frac{2}{N}\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j} = 0$$

$$\boxed{\begin{aligned} r_{eff} &= r + \frac{\partial \mathcal{F}(c_2, 0, c_4)}{\partial c_2} \\ g_{eff} &= g + \frac{\partial \mathcal{F}(c_2, 0, c_4)}{\partial c_4} \\ c_{2n} &= \int \lambda^{2n}\rho(\lambda, r_{eff}, g_{eff}) \end{aligned}}$$

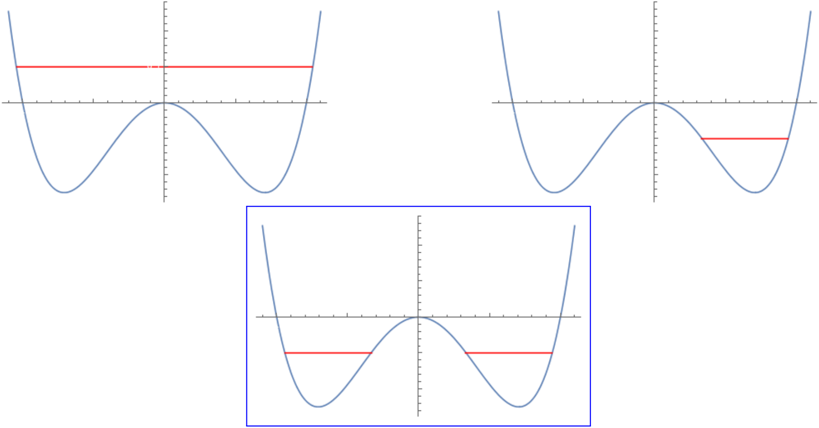
Saddle point equation

- The same procedure can be done for **the asymmetric solution** with the related effective model:

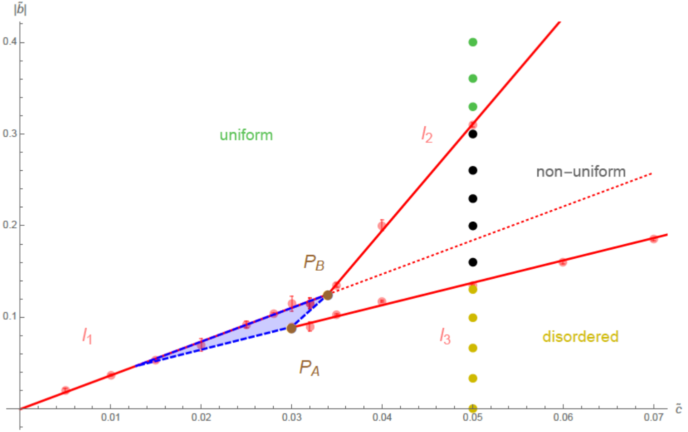
$$S_e = \mathrm{Tr}\left(a_{eff} M + \frac{1}{2} r_{eff} M^2 + g_{eff} M^4 \right)$$

and we get the equations for $\boxed{a_{eff}(c_n),\, r_{eff}(c_n),\, g_{eff}(c_n)}$.

Types of solutions



Fuzzy sphere- numerical simulations



Kováčik, O'Connor '18

Non-perturbative approximation of fuzzy sphere model

$$S = \mathrm{Tr}\left(\frac{1}{2}rM^2 + gM^4\right) + \frac{1}{2}\log\left(\frac{t_2}{1 - e^{-t_2}}\right)$$

- The equations obtained from saddle point can be solved numerically.

  Tekel '15; Tekel '18

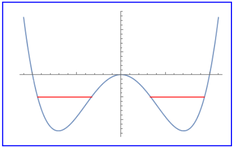
- The equations for the non-uniform and uniform order phases allow also for a nice perturbative solution in the large $|r|$, small g parameters.

  MŠ, J. Tekel '19

- The perturbative solution for the disorder phase is a little bit trickier.

  MŠ, J. Tekel '19

Non-uniform order phase



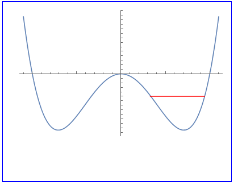
$$\mathrm{supp}\rho = \left(-\sqrt{D+\delta}, -\sqrt{D-\delta}\right) \cup \left(\sqrt{D-\delta}, \sqrt{D+\delta}\right)$$

$$\rho(\lambda) = \ldots$$

$$\boxed{4Dg + r + F'[D] = 0, \qquad \delta = \frac{1}{\sqrt{g}}}$$

$$D = -\frac{r}{4g} + \frac{1}{r} + \frac{4g}{r^3} + \frac{32g^2}{r^5} + \frac{320g^3}{r^7} + \ldots$$

Uniform order phase



$$\mathrm{supp}\rho = (D - \sqrt{\delta}, D + \sqrt{\delta})$$
$$\rho(\lambda) = \ldots$$

$$D^2 = \frac{(4+3\delta^2 g)(8-3\delta^2 g - \delta r)}{\delta(80g+36\delta^2 g^2)}$$
$$0 = 4\frac{4+15\delta^2 g + 2\delta r}{\delta(4+9\delta^2 g)} - $$
$$- F'\left[\frac{\delta(64+160\delta^2 g + 144\delta^4 g^2 + 81\delta^6 g^3 + 36\delta^3 g r + 27\delta^5 g^2 r)}{64(4+9\delta^2 g)}\right]$$

$$\delta = -\frac{2}{r} - \frac{1}{2r^2} - \frac{4+720g}{24r^3} - \frac{2+864g}{1920r^4} + \ldots$$

Phase transition

- The solution with the minimal free energy will be realized:

$$\boxed{Fe = \frac{1}{N^2} \log Z}$$

$$Fe_{nonuni} = -\frac{r^2}{16g} + \frac{1}{2}\log(-r) + \frac{3}{8} - \frac{1}{4}\log(4g) - \frac{g}{r^2} + \dots$$

$$Fe_{uni} = -\frac{r^2}{16g} + \frac{1}{2}\log(-r) + \frac{3}{4} + \frac{1}{2}\log(2) - \frac{1}{8r} + \dots$$

- The phase transition:

$$\boxed{\begin{aligned} & Fe_{nonuni} - Fe_{uni} = 0 \\ & g(r) = \frac{1}{16e^{3/2}} + \frac{1}{32e^{3/2}r} + \frac{9+5e^{3/2}}{384e^3r^2} + \frac{141+16e^{3/2}}{3072e^3r^3} + \dots \end{aligned}}$$

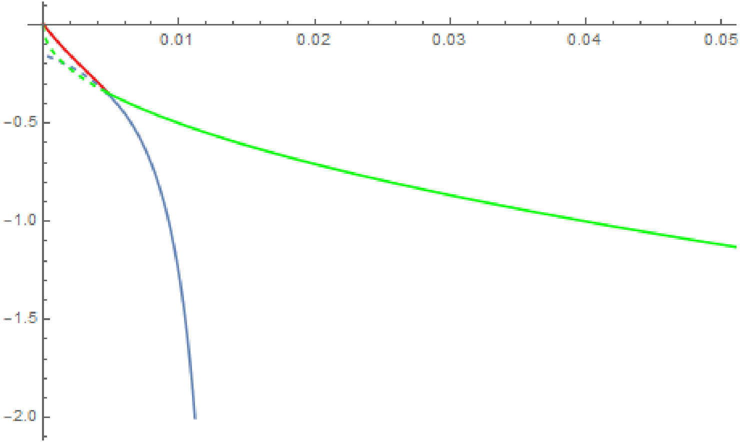
- In order to obtain reasonable line even for small $|m^2|$ we use Padé aproximantion.

The transition between disorder phase and non-uniform order phase

- This transition line can be obtained analytically from the existence boundary of the phases:

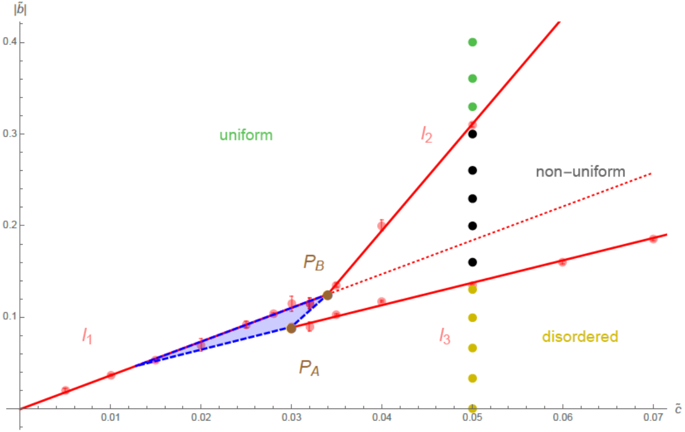
$$\boxed{\begin{gathered} r_{eff} = -4\sqrt{g} \\ r = -5\sqrt{g} - \frac{1}{1 - e^{1/\sqrt{g}}} \end{gathered}}$$

Phase diagram



Tekel ’17; MŠ, J.Tekel ’19

Numerical simulations



Kováčik, O'Connor '18

Perturbative approximation of fuzzy sphere model

$$S = \mathrm{Tr}\left(\frac{1}{2}rM^2 + gM^4\right) + \frac{1}{2}\left[\frac{1}{2}t_2 - \frac{1}{24}t_2^2 + \frac{1}{2880}t_2^4\right] - \\ - \frac{1}{432}t_3^2 - \frac{1}{3456}(t_4 - 2t_2^2)^2$$

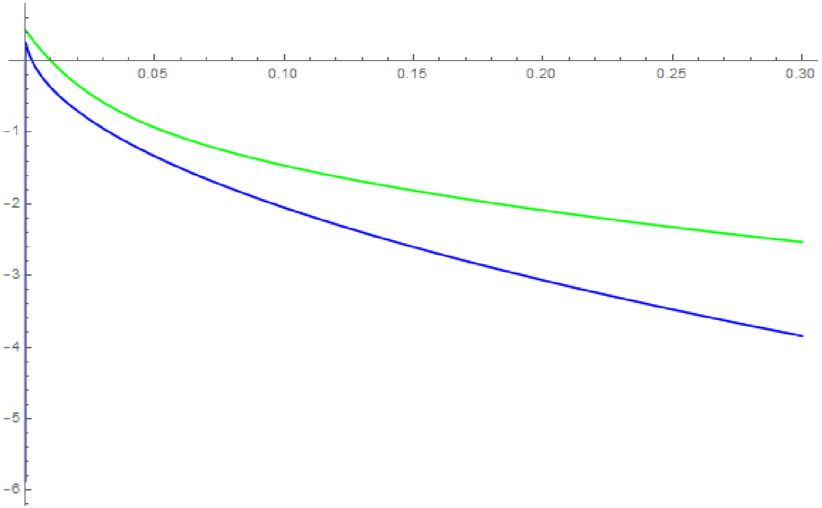
- The model was studied numerically.

  Tekel '15

- This approximation does not reproduce the behaviour of the full model for small g parameter. We need large t_n expansion of the kinetic term to obtain the disorder and non-uniform solutions correctly.

- However, the model has merit for the large g parameter.

Perturbative approximation



Tekel '15

The addition of the higher moment terms

$$S_{eff} = \frac{1}{2}\left[\frac{1}{2}t_2 - \frac{1}{24}t_2^2 + \frac{1}{2880}t_2^4\right] - \frac{1}{432}t_3^2 - \frac{1}{3456}(t_4 - 2t_2^2)^2$$

$$S_{eff} = \frac{1}{2}F(t_2) + F_3(t_3) + F_4(t_4 - 2t_2^2)$$

Summary

- Scalar field theories on fuzzy spaces can be described as matrix models.
- The kinetic term in the action leads to the multitrace terms that are known only approximatively.
- Two main approximations are considered:
  - Non-perturbative approximation gives correct behaviour around the triple point, but not further from the origin.
  - Perturbative approximation does not reproduce the theory behaviour around the triple point but further from the origin is more successful.

Outlook

- How to include the higher moments?
- Can we get rid of the non-uniform order phase?

$$\mathcal{K} = a(r,g)C_2 + b(r,g)C_2^2$$

Gubser, Sondhi '01; Dolan, O'Connor,Presnajder '02

Thank you for your attention.