Localization and Reference Frames in κ-Minkowski Spacetime

Corfù Summer Institute: "Workshop on Quantum Geometry, Field Theory and Gravity "

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This talk is based on...

PRD

"Localization and Reference Frames in κ-Minkowski Spacetime"

(F.Lizzi, M. Manfredonia, F. Mercati, T. Poulain)

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Figure: Scan to open PDF version of the paper

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κ-Minkowski space time

Q.G. suggests classical space-time (at basis of GR and QFT) to be replaced with quantum structure.

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Non Commutative Geometry

$$\Downarrow$$

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κ-Minkowski

$$[x^0, x^i] = \mathrm{i}\lambda x^i, \quad [x^i, x^j] = 0,$$

$$i, j \in \{1, 2, 3\}$$

where $x^0 = ct$, sometimes $\lambda = 1/\kappa$, hence the name

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Invariant under **non commutative generalization**: κ-Poincaré

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Invariant under **non commutative generalization**: κ-Poincaré

κ-Poincaré, κ-Minkowski $\Rightarrow$ group manifold becomes a n.c. space

$$\Downarrow$$

Principle of Non Localizability

$$\Delta x^0 \Delta x^i \geq \frac{\lambda}{2} |\langle x^i \rangle|$$

Somehow similar to Heisenberg principle

$$\Delta p_i \Delta q^j \geq \frac{\hbar}{2}$$

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Operator Representation for κ-Minkowski

We introduce the following representation as operators on $L^2(\mathbb{R}^3)$

$$\begin{aligned} \hat{x}^i \psi(x) &= x^i \psi(x), \\ \hat{x}^0 \psi(x) &= \mathrm{i}\lambda \left(\sum_i x^i \partial_{x^i} + \frac{3}{2} \right) \psi(x) = \mathrm{i}\lambda \left(r\partial_r + \frac{3}{2} \right) \psi(x). \end{aligned}$$

which is compatible with κ-Minkowski commutators.

Caution!

Other representation are allowed. We want to consider those in future works.

The Time Eigenvalue Equation

Since $\hat{x}^0$ commutes with all spherical harmonics, we adopt polar basis (r, θ, φ) and we solved **eigenvalue equation**.

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$$\mathrm{i}\lambda\left(r\partial_r + \frac{3}{2}\right) r^\alpha = \mathrm{i}\lambda(\alpha + \frac{3}{2})r^\alpha = \ell_\alpha r^\alpha, \quad \ell_\alpha = \mathrm{i}\lambda(\alpha + \frac{3}{2})$$

We want eigenvalues to be real!

$$\{\ell_\alpha \in \mathbb{R}\} \Leftrightarrow \{\alpha = -\frac{3}{2} - \mathrm{i}\tau, \tau \in \mathbb{R}\}$$

The adimensional τ is related to time $t = \frac{\lambda}{c}\tau$

Between Time and Space

Eigenfunction of Time

$$T_\tau[r] = \frac{r^{-\frac{3}{2}-\mathrm{i}\tau}}{\lambda^{-\mathrm{i}\tau}} = r^{-\frac{3}{2}}\mathrm{e}^{-\mathrm{i}\tau\log\left(\frac{r}{\lambda}\right)}$$

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Self-adjointness of operators allow us to expand $\psi(r, \theta, \varphi)$ on functions $\{T_\tau[r]\}$

$$\psi(r, \theta, \varphi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\tau\, r^{-\frac{3}{2}} \mathrm{e}^{-\mathrm{i}\tau \log\left(\frac{r}{\lambda}\right)} \widetilde{\psi}(\tau, \theta, \varphi)$$

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Of course this is not a Fourier transform...

...it is a **Mellin** transform!

$$\psi(r,\theta,\varphi) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\tau}{\sqrt{2\pi}}\, r^{-\frac{3}{2}} \mathrm{e}^{-\mathrm{i}\tau \log\left(\frac{r}{\lambda}\right)} \widetilde{\psi}(\tau,\theta,\varphi) = \mathcal{M}^{-1}\left[\widetilde{\psi}(\tau,\theta,\varphi),\ r\right]$$

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Good News: It is still an isometry!

The above transformation preserves the norm

$$\int_0^\infty \mathrm{d}r\ r^2 |\psi(r)|^2 = \int_{-\infty}^{\infty} \mathrm{d}\tau |\widetilde{\psi}(\tau)|^2$$

Physical Interpretation

Our Conjecture

We assume the usual theory of measurements and observables to stand in this framework.

As an example, the average time measured by an observer would be

$$\langle \hat{x}^0 \rangle_\psi = 4\pi \int r^2 \mathrm{d}r \overline{\psi}(r) \mathrm{i}\lambda \left(r\partial_r + \frac{3}{2} \right) \psi(r)$$

Points at the origin of space

We recall space-time uncertainty

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NOTE

$\langle x^i \rangle$ on the right hand side suggests **spatial localized states** to be possible at spatial origin.

We have to search for states that saturate uncertainty bound.

~~Gaussian~~ $\rightarrow$ log-Gaussian

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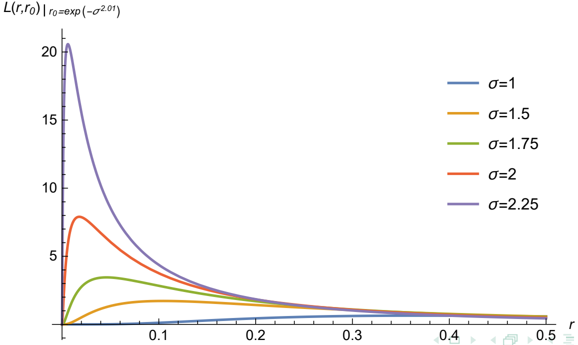
log-Gaussians play the role of *coherent states*

$$L(r, r_0) = N \mathrm{e}^{-\frac{(\log r - \log r_0)^2}{\sigma^2}}$$



The average values of r^n and $\left(\hat{x}^0\right)^n$ are

$$\langle \hat{r}^n \rangle_L = \mathrm{e}^{\frac{\sigma^2}{8} n(n+6)} r_0^n ; \quad \langle (\hat{x}^0)^n \rangle_L = \frac{1}{4\pi} \left(\frac{\lambda}{\sigma}\right)^n \begin{cases} 0 & n \text{ odd} \\ (n-1)!! & n \text{ even} \end{cases}$$

Eigenstate of the origin

There is a localized state $|o\rangle$ localized at origin $(\tau = 0, \ r = 0)$.

Multiplication by $r^{\mathrm{i}\tau'}$ shifts time

We have a one parameter family of states localized at spatial origin at different time

$$\{|o\rangle_\tau\} := \{|o\rangle_\tau \text{ is localized at } r = 0, \ \tau \neq 0\}$$

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κ-Poincaré quantum group

The noncommutative algebra of functions $\mathcal{P}_\kappa$, generated by $\Lambda^\mu{}_\nu$ and a^μ that leave κ-Minkowski commutation relations invariant under the transformation:

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1 \,.$$

$$[x'^\mu, x'^\nu] = \mathrm{i}\lambda \left(\delta^\mu{}_0\, x'^\nu - \delta^\nu{}_0\, x'^\mu\right)$$

κ-Poincaré commutation relations

$$[a^\mu, a^\nu] = \mathrm{i}\lambda \left(\delta^\mu{}_0\, a^\nu - \delta^\nu{}_0\, a^\mu\right) \,, \qquad [\Lambda^\mu{}_\nu, \Lambda^\rho{}_\sigma] = 0 \,,$$
$$[\Lambda^\mu{}_\nu, a^\rho] = \mathrm{i}\lambda \left[\left(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0\right) \Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu \delta^0{}_\sigma - \delta^0{}_\nu\right) \eta^{\mu\rho}\right] \,.$$

Hopf Algebra Structure

Coproduct $\Delta : \mathcal{P}_\kappa \to \mathcal{P}_\kappa \otimes \mathcal{P}_\kappa$:

$$\Delta(a^\mu) = a^\nu \otimes \Lambda^\mu{}_\nu + 1 \otimes a^\mu \,, \qquad \Delta(\Lambda^\mu{}_\nu) = \Lambda^\mu{}_\rho \otimes \Lambda^\rho{}_\nu \,,$$

Antipode $S : \mathcal{P}_\kappa \to \mathcal{P}_\kappa$

$$S(a^\mu) = -a^\nu (\Lambda^{-1})^\mu{}_\nu \,, \qquad S(\Lambda^\mu{}_\nu) = (\Lambda^{-1})^\mu{}_\nu \,,$$

Counit $\varepsilon : \mathcal{P}_\kappa \to \mathbb{C}$,

$$\varepsilon(a^\mu) = 0 \,, \qquad \varepsilon(\Lambda^\mu{}_\nu) = \delta^\mu{}_\nu \,,$$

Have to be homorphisms with respect to the commutation relations

A representation for κ- Poincarè

Long story short, a representation on $L^2(so(3,1))$ turns out to be non faithful. Thus we enlarged the Hilbert space with 3 more coordinates $q^i \in \mathbb{R}^3$:$L^2(SO(3,1)) \Rightarrow L^2(so(3,1) \times \mathbb{R}^3)$

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A faithful (although complicated) representation

$$a^\rho \phi = \mathrm{i}\lambda \delta^\rho{}_0 \left(\frac{3}{2}\,\phi(q,\omega) + q^i \frac{\partial \phi}{\partial q^i} \right) + \delta^\mu{}_i\, q^i\, \phi$$

$$- \mathrm{i}\lambda : \left[\left(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0 \right) \Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu \delta^0{}_\sigma - \delta^0{}_\nu \right) \eta^{\mu\rho} \right] \Lambda^\nu{}_\alpha \frac{\partial}{\partial \omega^\mu{}_\alpha} : \phi \,,$$

$$\Lambda^\mu{}_\nu \phi = \Lambda^\mu{}_\nu(\omega)\phi = (\exp \omega)^\mu{}_\nu \phi \,,$$

where $\phi = \phi(q,\omega)$ with $q \in \mathbb{R}^3$ and $\omega \in so(3,1)$.

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Observers and Reference Frames

Algebra of $\hat{x}^i$,$\hat{x}^0$ and its states represent position and time in κ-Minkowski.

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Question:How do we change the observer?

State of the observer!

Algebra of a's and Λ's represent the state of a translated and Lorentz transformed observer

Identity Transformation state

We define the state $|o\rangle_{\mathcal{P}}$ of $\mathcal{P}_\kappa$ such that

$$ {}_{\mathcal{P}}\langle o|\, f(a,\Lambda)|o\rangle_{\mathcal{P}} = \varepsilon(f) $$

The state returns the value of the function on the identity transformation.

It describes the Poincaré transformation between two coincident observers

Physical Interpretation

Consider two different observers

Alice	**Bob**
(at the origin)	(in relative motion)
$\mathbb{1} \otimes x^\nu$	$\Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes \mathbb{1}$

both coordinates are $\hat{x}^\mu, \hat{x}'^\mu \in \mathcal{P}_\kappa \otimes \mathcal{M}_\kappa$

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Bob's coordinates must carry information on the transformation between the twos!

We take direct sum of the representation of $\mathcal{P}_\kappa$ and $\mathcal{M}_\kappa$.

States of a transformed observer

$$|g\rangle \in L^2\left(SO(3,1)\times\mathbb{R}^3_q\right) \qquad |\psi\rangle \in L^2\left(\mathbb{R}^3\right)$$

(related to the observer) (related to the event)

Generic element of Hilbert space

$$|g,\psi\rangle = |g\rangle \otimes |\psi\rangle$$

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$$|g,\psi\rangle = |g\rangle \otimes |\psi\rangle$$

Expectation values measured by Bob are related to those measured by Alice through relation

$$\begin{aligned}\langle x'^{\mu}\rangle &= \langle g| \otimes \langle\psi| \left(\Lambda^{\mu}{}_{\nu} \otimes x^{\nu} + a^{\mu} \otimes 1\right) |g\rangle \otimes |\psi\rangle \\ &= \langle g|\Lambda^{\mu}{}_{\nu}|g\rangle\langle\psi|x^{\nu}|\psi\rangle + \langle g|a^{\mu}|g\rangle\end{aligned}$$

(averages transforms with averaged κ-Poincaré transformation rule)

Example: Transforming the origin state

The origin state for Alice is:

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...and for Bob:

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We want to know what Bob measures with the coordinates centred on his reference frame

$$\langle x'^{\mu}\rangle = \langle g|\Lambda^{\mu}{}_{\nu}|g\rangle\langle o|x^{\nu}|o\rangle + \langle g|a^{\mu}|g\rangle\langle o|o\rangle \, ,$$

$$\langle x'^{\mu} \rangle = \langle g|\Lambda^{\mu}{}_{\nu}|g\rangle \langle o|x^{\nu}|o\rangle + \langle g|a^{\mu}|g\rangle \langle o|o\rangle \,,$$

Since $\langle o|\, x^{\nu}\, |o\rangle = 0$ and $\langle o|o\rangle = 1$ we have:

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$$\langle x'^{\mu} \rangle = \langle g | \Lambda^{\mu}{}_{\nu} | g \rangle \langle o | x^{\nu} | o \rangle + \langle g | a^{\mu} | g \rangle \langle o | o \rangle \,,$$

Since $\langle o | \, x^{\nu} \, | o \rangle = 0$ and $\langle o | o \rangle = 1$ we have:

$$\langle x'^{\mu} \rangle = \langle g | a^{\mu} | g \rangle$$

The expectation value of coordinates x' is fully determinated by the expectation value of translation a

This is natural: the different observers are comparing positions, not directions.

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Conclusions ans Outlooks

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- We adopted observables interpretation and discussed physical meaning
- We gave representation of κ-Poincaré group as operators on Hilbert space $L^2\left(SO(3,1) \times \mathbb{R}^3_q\right)$
- We implemented all the above results to introduce transformations between different observers
- We developed a new framework based on a representation of $\mathcal{P}_\kappa \otimes M_\kappa$ as operators on $L^2\left(SO(3,1) \times \mathbb{R}^3_q \times \mathbb{R}^3\right)$ in which a state has informations about both the event and the observer who is staring at it.

Future Perspectives

- Introduce the whole phase space or at least some notion of **momentum space**

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- Introduce the whole phase space or at least some notion of **momentum space**
- Introduce **dynamics** and evolution parameter for observers and events
- Better understand the role of $\hat{x}_0$ and time t in the dynamics.


THANK YOU
FOR
YOUR
ATTENTION!
ANY QUESTIONS?