Thoughts on a Hamiltonian formulation of sigma models, doubling and possible susy

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Kerkyra, September , 2019

Lagrangian

$2d$ $(1,1)$ superspace

$$\int d^2\xi d\theta^+ d\theta^- \mathcal{L} = \int d^2\xi d\theta^+ d\theta^- (D_+\phi^i E_{ij}(\phi) D_-\phi^j)$$

$$E_{ij} = (E_{(ij)} + E_{[ij]}) := G_{ij} + B_{ij}$$

$$\nabla_+^{(+)} D_-\phi^i = 0 \ ,$$

Hamiltonian

$$S_{\pm i} := \frac{\partial \mathcal{L}}{\partial D_{\mp}\phi^i} \ .$$

$$S_{+i} = -D_+\phi^j E_{ji}$$

$$S_{-i} = E_{ij} D_-\phi^j .$$

$$\mathcal{H} = [S_{(+i}D_{-)}\phi^i + \mathcal{L}]|$$

$$\mathcal{H} = S_{+i}E^{ij}S_{-j}$$

$$E^{ij}E_{jk} = \delta^i_k$$

Canonical Equations

$$D_+\phi^i = \frac{\partial \mathcal{H}}{\partial S_-}$$

$$D_-\phi^i = \frac{\partial \mathcal{H}}{\partial S_+}$$

$$D_{(+}S_{-)i} = \frac{\partial \mathcal{H}}{\partial \phi^i} \; .$$

This system is an equivalent formulation of the motion.

A Phase Space Action

First order Lagrangian

$$\mathcal{L}_1 = S_{(+i}D_{-)}\phi^i - S_{+i}E^{ij}(\phi)S_{-j} \ ,$$

Can be obtained by gauge fixing a model on $\mathbb{T} \oplus \mathbb{T}^*$

$$\mathcal{D}_{(+}\tilde{\phi}_i D_{-)}\phi^i - \mathcal{D}_+\tilde{\phi}_i E^{ij}(\phi)\mathcal{D}_-\tilde{\phi}_j \ ,$$

$$\mathcal{D}_\pm\tilde{\phi}_i := D_\pm\tilde{\phi}_i + S_{\pm i} \ ,$$

$$\delta S_{\pm i} = D_\pm \Lambda_i \ .$$

The topological term is necessary for equivalence.

The Doubled Model

A $(1,1)$ Lagrangian on $\mathbb{T} \oplus \mathbb{T}^*$ subject to constraints. In a polarisation:

$$D_{(+}\tilde{\phi}_i D_{-)}\phi^i - D_+\tilde{\phi}_i E^{ij}(\phi) D_-\tilde{\phi}_j \ ,$$

$$D_+\tilde{\phi}_i = -D_+\phi^j E_{ji}$$
$$D_-\tilde{\phi}_i = E_{ij} D_-\phi^j .$$

Inserting the constraints returns the original action on $\mathbb{T}$. Equivalently this action is obtained via the gauging procedure.

Freedom

$$D_{(+}\tilde{\phi}_i D_{-)}\phi^i - D_+\tilde{\phi}_i E^{ij}(\phi) D_-\tilde{\phi}_j + \mu D_+\phi^i E_{ij}(\phi) D_-\phi^j \ .$$

Additional Supersymmetry

- Can we find additional susy for the Doubled Lagrangian $\mathcal{L}(\tilde{\phi}, \phi)$? c.f. S.Driezen, A. Sevrin,and D.C. Thompson 2016

- Can we find additional susy for the first order Lagrangian $\mathcal{L}(\phi, S_{\pm})$? U.L. 2004, U.L., R.Minasian, A.Tomasiello and M.Zabzine 2005

- Can we find additional susy for the gauged Lagrangian $\mathcal{L}(\tilde{\phi}, \phi, S_{\pm})$?

General

A general $(1,1)$ model

$$\tfrac{1}{2}D_+\phi^A E_{AB} D_-\phi^B$$

has additional susy if

$$\delta\phi^A = \epsilon^+ J^A_{(+)B} D_+\phi^B + \epsilon^- J^A_{(-)B} D_-\phi^B$$

$$J^2_{(\pm)} = -\mathbb{1}\ , \quad J^t G J = G$$

$$\nabla^{(\pm)} J_{(\pm)} = 0\ .$$

But the constraints?

Special

One (complex) chiral superfield, Kähler potential $K(\phi, \bar{\phi})$

$$\mathcal{L} = D_+\bar{\phi}K,_{\phi\bar{\phi}}D_-\phi + c.c.$$

Second susy $\phi := (\phi, \bar{\phi})^t$

$$\delta\phi := \delta\begin{pmatrix}\phi\\ \bar{\phi}\end{pmatrix} = \epsilon^+\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix}D_+\begin{pmatrix}\phi\\ \bar{\phi}\end{pmatrix} =: \epsilon^+ JD_+\phi$$

$$\tilde{\phi} = (\tilde{\phi}, \bar{\tilde{\phi}})^t$$

$$2D_{(+}\tilde{\phi}^t D_{-)}\phi - D_+\tilde{\phi}^t K^{\bar{\phi}\phi} D_-\tilde{\phi} + 2\mu D_+\phi K,_{\phi\bar{\phi}} D_-\bar{\phi}. \ ,$$

$$\boxed{\begin{aligned} D_+\bar{\tilde{\phi}} &= -D_+\phi K,_{\phi\bar{\phi}} \\ D_-\tilde{\phi} &= K,_{\phi\bar{\phi}} D_-\bar{\phi} \end{aligned}}$$

Not invariant under the unique second susy:

$$\delta\phi = \epsilon^+ J D_+\phi$$

$$\delta\tilde{\phi} = \pm\epsilon^+ J D_+\tilde{\phi}$$

If in addition

$$\delta \bar{S}_+ = -i\epsilon^+ D_+ \bar{S}_+ + 2i\epsilon^+ K,_{\phi\bar{\phi}\bar{\phi}} D_+\phi D_+\bar{\phi}$$

$$\delta S_- = -i\epsilon^+ D_+ S_- - 2i\epsilon^+ (lnK,_{\phi\bar{\phi}})_\phi S_- D_+\phi$$

then

$$2\mathcal{D}_{(+}\tilde{\phi}^t D_{-)}\phi - \mathcal{D}_+\tilde{\phi}^t K^{\bar{\phi}\phi}\mathcal{D}_-\tilde{\phi} + 2\mu D_+\phi K,_{\phi\bar{\phi}} D_-\bar{\phi}.$$

is (off-shell) supersymmetric. So implementation via gauging is supersymmetric.

Generalised

A nontrivial b transformation:

$$\begin{aligned}\mathcal{G} &= (e^{-b})^t \mathbb{G} e^{-b} \\ &= (e^{-b})^t \begin{pmatrix} \mu g & 0 \\ 0 & -g^{-1} \end{pmatrix} e^{-b} \\ &= \begin{pmatrix} \mu g + b g^{-1} b & -b g^{-1} \\ g^{-1} b & -g^{-1} \end{pmatrix} .\end{aligned}$$

The b transform of the whole Lagrangian is then

$$\begin{aligned}&(D_+\phi, D_+\tilde{\phi})^t \mathbb{G} (D_-\phi, D_-\tilde{\phi}) \\ &\to (D_+\phi, -bD_+\phi + D_+\tilde{\phi})^t \mathcal{G} (D_-\phi, -bD_-\phi + D_-\tilde{\phi})\end{aligned}$$

On the constraints

$$D_+\tilde{\phi} = -D_+\phi g$$
$$D_-\tilde{\phi} = gD_-\phi \ ,$$

and the Lagrangian becomes

$$\begin{aligned}&(D_+\phi, D_+\tilde{\phi})^t \mathbb{G}(D_-\phi, D_-\tilde{\phi})\\ &\to (D_+\phi, -eD_+\phi)^t \mathcal{G}(D_-\phi, +e^t D_-\phi) \ ,\end{aligned}$$

i.e., the metric $\mathbb{G}$, gets replaced by $\mathcal{G}$, and g gets replaced by e and e^t in the constraints.
Evaluating the corresponding action for $\mu = -1$ yields

$$D_+\phi(g+b)D_-\phi = D_+\phi e D_-\phi$$

More Generalised

$$\mathcal{G} = \begin{pmatrix} g - bg^{-1}b & bg^{-1} \\ -g^{-1}b & g^{-1} \end{pmatrix} .$$

$$\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$\mathcal{S} = \eta^{-1}\mathcal{G} = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix}$$

$$\mathcal{S}^2 = \mathbb{1}$$

$$\mathcal{P}_{(\pm)} := \tfrac{1}{2}(1 \pm \mathcal{S})$$

$$\Omega = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$\begin{aligned} \mathcal{I} &:= -\Omega^{-1}\mathcal{G}\,, \quad \mathcal{K} := \Omega^{-1}\eta \\ -\mathcal{I}^2 &= \mathcal{S}^2 = \mathcal{K}^2 = \mathbb{1} \\ \mathcal{I}\mathcal{S}\mathcal{K} &= -\mathbb{1}\,, \quad \{\mathcal{I},\mathcal{S}\} = \{\mathcal{S},\mathcal{K}\} = \{\mathcal{K},\mathcal{I}\} = 0 \end{aligned}$$

$$\Phi^M = \begin{pmatrix} \phi^\mu \\ \tilde{\phi}_{\tilde{\mu}} \end{pmatrix}$$

$$S_2 = \tfrac{1}{4}\int d^2xd^2\theta\,(D_+\Phi)^t\mathcal{K}\mathcal{G}D_-\Phi\ ,$$

$$\mathcal{P}_{(\pm)}D_\pm\Phi = 0\ , \quad \Rightarrow D_+\tilde{\phi}_{\tilde{\mu}} = -D_+\phi^\mu e_{\mu\tilde{\mu}}\ , \quad D_-\tilde{\phi}_{\tilde{\mu}} = e_{\tilde{\mu}\mu}D_-\phi^\mu$$

$$S_2 \to \tfrac{1}{2}\int d^2xd^2\theta\, D_+\phi\, e\, D_-\phi$$

or

$$S_1 \to \tfrac{1}{2}\int d^2xd^2\theta\, D_+\phi\, e\, D_-\phi - D_{(+}\tilde{\phi}D_{-)}\phi$$

$$\mathcal{J}^2 = -\mathbb{1}\ , \quad \mathcal{J}^t\mathcal{G}\mathcal{J} = \mathcal{G}\ , \quad \mathcal{J}^t\eta\mathcal{J} = \eta$$

$$[\mathcal{J}, \mathcal{S}] = 0\ , \quad [\mathcal{J}, \Omega] = 0\ ,$$

$$\Rightarrow \mathcal{J} = e^b\mathcal{J}_0 e^{-b}$$

$$\mathcal{J}_0 = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix} .$$

$$\delta\Phi = \epsilon\mathcal{J}D\Phi$$