λ-deformations: exact results without loops

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- Background work: with G. Georgiou, G. Itsios, E. Sagkrioti & K. Siampos, in various combinations.
- New work here: with G. Georgiou, P. Panopoulos & E. Sagkrioti, 1906.00984 [hepth].

General motivation

- The quantum behavior of interacting field theories is encoded on how the couplings change with energy and on how the operators acquire anomalous dimensions.

- Studied in the framework of RG (since [Wilson 71]). Results into 1st order non-linear differential systems, i.e. β-function eqs.

- Traditionally computed perturbatively.

- In rare occasions it is possible to compute them exactly, i.e. re-sum perturbative expansion.
  - Smooth RG flows (UV to IR) between CFTs.
  - New fixed point theories in the IR.
  - We will learn more on the structure of QFTs. i.e. fate of degrees of freedom, Zamolodchikov's C-theorem.

Outline

- The theories of interest
- Effective actions: Self-interacting theories.
- Exact β-functions and anomalous dimensions.

$$\boxed{\text{Perturbative info} + \text{symmetry} + \text{analyticity} \Longrightarrow \text{exact info}}$$

- A more powerful and easier to apply method.

$$\boxed{\text{Effective action} + \text{geometry of couplings} \Longrightarrow \text{exact info}}$$

- The anomalous dims of $d_{a_1 \dots a_m} J_+^{a_1} \dots J_+^{a_m}$ and $d_{abc} J_+^a J_+^b J_-^c$.
- Concluding remarks.

The theories of interest

Old work: 2-dim fermions in $SU(N)$ [Dashen-Frishman 73 & 75]

$$\mathcal{L} = \overbrace{\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi}^{\text{free action}} - \frac{g_v}{2}\overbrace{J^a_{\mu}J^{a\mu}}^{\text{interaction}} \quad , \quad J^a_{\mu} = \bar{\psi}t^a\gamma_{\mu}\psi\,, \;\; a = 1, \ldots, N^2 - 1\ .$$

- A trivial scale invariant point at $g_v = 0$ and $\Delta_{\psi} = \frac{1}{2}$.
- A non-trivial scale invariant point at

$$g_v = \frac{4\pi}{N+1}: \qquad \Delta_{\psi} = \frac{1}{2} + \frac{N-1}{N}\ (\text{new dimension})\ .$$

Admits a large N limit, by keeping $gN = \text{fixed}$.

Bosonic generalization

- Replace free fermions with the WZW action for a group G
- Keep the interaction bilinear in currents the same

$$S_{k,\lambda} = S_k(g) + \frac{k}{\pi}\int d^2\sigma\ \lambda_{ab}J^a_+J^b_- \ , \quad a, b = 1, 2, \ldots, \dim G\ .$$

A the CFT point $J^a_{\pm}$ obey two commuting current algebras.

Effective action for λ-deformations [KS 13]

Idea of the construction

WZW and PCM actions for $g, \tilde{g} \in G$. Gauge the global symmetry

$$g \to \Lambda^{-1} g \Lambda \,, \qquad \tilde{g} \to \Lambda^{-1} \tilde{g} \,, \qquad \Lambda \in G$$

and we consider the action

$$S_{k,\kappa^2}(g, \tilde{g}, A_\pm) = S_k(g, A_\pm) + S_{\rm PCM}(\tilde{g}, A_\pm) \,.$$

- The gauged WZW action is

$$S_k(g, A_\pm) = S_k(g) + \frac{k}{\pi} \int \mathrm{Tr} \Big(A_- \partial_+ g g^{-1} - A_+ g^{-1} \partial_- g + A_- g A_+ g^{-1} - A_- A_+ \Big) \,.$$

- The gauged PCM action is

$$S_{\rm PCM}(\tilde{g}, A_\pm) = -\frac{\kappa^2}{\pi} \int \mathrm{Tr}(\tilde{g}^{-1} \widetilde{D}_+ \tilde{g} \tilde{g}^{-1} \widetilde{D}_- \tilde{g}) \,, \quad D_\pm \tilde{g} = \partial_\pm \tilde{g} - A_\pm \tilde{g} \,.$$

The effective action

After gauge fixing $\tilde{g} = \mathbb{1}$ and integrating out the gauge fields we obtain the action

$$\boxed{S_{k,\lambda}(g) = S_k(g) + \frac{k}{\pi}\int J_+^a(\lambda^{-1}\mathbb{I} - D^T)^{-1}_{ab} J_-^b} \ ,$$

where the parameter $\lambda^{-1} = 1 + \kappa^2/k$, so that $0 < \lambda < 1$ and

$$J_+^a = -i\mathrm{Tr}(t^a \partial_+ g g^{-1}) \ , \quad J_-^a = -i\mathrm{Tr}(t^a g^{-1}\partial_- g) \ ,$$
$$D_{ab} = \mathrm{Tr}(t_a g t_b g^{-1}) \ .$$

- Since D is orthogonal this action is non-singular.
- Singularities appear only when $\lambda \to \pm 1$. Letting also $k \to \infty$

$$(1-\lambda)k = \text{finite} \ , \qquad (1+\lambda)^3 k = \text{finite} \ ,$$

Non-Abelian & Pseudo-dual lims [KS 13, Georgiou-KS-Siampos 16].

Some properties

- Generalization to $\lambda\delta_{ab} \to \lambda_{ab}$ straightforward.
- For $\lambda \ll 1$

$$S_{k,\lambda}(g) = S_k(g) + \frac{k}{\pi}\int d^2\sigma\ \lambda_{ab} J_+^a J_-^b + \dots\ .$$

- The full action $S_{k,\lambda}(g)$ has a duality-type symmetry [Itsios-KS-Siampos 14]

$$\boxed{k \to -k\ , \qquad \lambda \to \lambda^{-1}\ , \qquad g \to g^{-1}}\ .$$

  It should be reflected as a symmetry in physical quantities.

- Integrable for special forms of λ_{ab}, eg. [KS 13, Hollowood-Miramontes-Schmidtt 14]

$$\lambda_{ab} = \lambda\delta_{ab}\ .$$

  Not integrable at any finite order in λ.
  More integrable cases [KS-Siampos 14, KS-Siampos-Thompson 15].

Exact β-function and anomalous dimensions

Three distinct approaches:

CFT and symmetry approach [Georgiou-Siampos-KS 15 & 16]

Perturbative information to $\mathcal{O}(\lambda^2)$ the symmetry and analyticity are enough to determine β^λ and γ_J.

- We compute perturbatively n-point functions and use conformal perturbation theory.
- We obtain the β-function and anomalous dimension of the currents

$$\beta^\lambda = \frac{d\lambda}{d\ln\mu^2} = -\frac{c_G}{2k}(\lambda^2 - 2\lambda^3) + \mathcal{O}(\lambda^4) \ ,$$
$$\gamma_J = \frac{c_G}{k}(\lambda^2 - 2\lambda^3) + \mathcal{O}(\lambda^4) \ .$$

c_G: quadratic Casimir in the adjoint, i.e. $f_{acd} f_{bcd} = c_G \delta_{ab}$.

- The action is singular at $\lambda = \pm 1$. Hence, β^λ and γ_J may have poles at $\lambda = \pm 1$.
- The β-function & anomalous dims should be invariant under

$$k \to -k \ , \qquad \lambda \to \frac{1}{\lambda} \ , \qquad (\text{for } k \gg 1) \ .$$

- We obtain

$$\beta^\lambda = -\frac{c_G}{2k}\frac{\lambda^2}{(1+\lambda)^2} \ , \quad \gamma_J = \frac{c_G}{k}\frac{\lambda^2}{(1-\lambda)(1+\lambda)^3} \ .$$

- Agreement with perturbation theory to $\mathcal{O}(\lambda^3)$ and $\mathcal{O}(\lambda^4)$.
- Similarly for current 3-point functions and correlators of primary fields [Georgiou-KS-Siampos 16].

Gravitational approach [Itsios-KS-Siampos 14, KS-Siampos 14, KS-Siampos-Sagkrioti 18]

One uses the RG flow eqs. for a general σ-model [Ecker-Honerkamp 71, Friedan 80]

$$\frac{d(G_{\mu\nu}+B_{\mu\nu})}{d\ln\mu^2} = R_{\mu\nu} + \cdots \ .$$

- From the σ-model/geometrical data one finds the same β^λ as before.
- Advantage: Extendable for the β-function for general λ_{ab} [KS-Siampos 14]

  $$\frac{d\lambda_{ab}}{dt} = \frac{1}{2k}\mathrm{Tr}\left(\mathcal{N}_a(\lambda)\mathcal{N}_b(\lambda^T)\right) \ .$$

  The matrices $\mathcal{N}_a(\lambda)$ have non-linear dependence on λ_{ab}.
- Disadvantage: Not immediately possible to use it for computing anonalous dimensions of operators.

Questions and limitations of previous approaches

- What about the anomalous dimensions of general composite operators of the type

$$\boxed{\mathcal{O}^{(m,n)} = S_{a_1\dots a_m;b_1\dots b_n} J_+^{a_1} \dots J_+^{a_m} J_-^{b_1} \dots J_-^{b_n}}\ ,$$

  where the S-tensor belongs to some irrep of G.

- How are the operators dressed/modified in the process of the deformation?

- Clearly the previous approaches are limited.

- How can we use the effective action to perform such computations?

- How to deal with possible operator mixing.

Gauging and geometry in coupling space approach

[Georgiou-Panopoulos-Sagkrioti-KS, 19]

Gauging approach with an extra gauge invariant source term

$$\frac{ks}{\pi}\int S_{a_1\dots a_m;b_1\dots b_n}(\tilde{g}^{-1}D_+\tilde{g})^{a_1}\dots(\tilde{g}^{-1}D_+\tilde{g})^{a_m}(\tilde{g}^{-1}D_-\tilde{g})^{b_1}\dots(\tilde{g}^{-1}D_-\tilde{g})^{b_n}\ ,$$

where s is a coupling.

- This action is still gauge invariant, so we gauge fix as $\tilde{g}=\mathbb{1}$.
- At the end set $s=0$ (should be consistent with the β-function); the λ-deformed action is recovered.
- The anomalous dimension of $\mathcal{O}^{(m,n)}$ will be extracted.

The dressed operators

- Upshot after integrating out the gauge fields

$$S = S_{k,\lambda}(g) - \frac{ks}{\pi}\int d^2\sigma\, \mathcal{O}_\lambda^{(m,n)} + \mathcal{O}(s^2)\ .$$

- The dressed operators are

$$\mathcal{O}_\lambda^{(m,n)} = S_{a_1\dots a_m;b_1\dots b_n} A_+^{a_1}\dots A_+^{a_m} A_-^{b_1}\dots A_-^{b_n}\ .$$

  with the classical gauge fields (dressed currents)

$$A_+ = i\left(\lambda^{-1}\mathbb{1} - D\right)^{-1} J_+\ ,\quad A_- = -i\left(\lambda^{-1}\mathbb{1} - D^T\right)^{-1} J_-\ .$$

- Obviously, as $\lambda \to 0$ we have that $A_\pm \sim J_\pm$. Therefore

$$\mathcal{O}_\lambda^{(m,n)} \sim S_{a_1\dots a_m;b_1\dots b_n} J_+^{a_1}\dots J_+^{a_m} J_-^{b_1}\dots J_-^{b_n}\ .$$

Strategy for computing the $\mathcal{O}_\lambda^{(m,n)}$ anomalous dimensions

- For small values of λ and s

$$S = S_k(g) + \frac{k}{\pi}\int d^2\sigma\Big(\lambda J_+^a J_-^a + \tilde{\lambda}\mathcal{O}^{(m,n)}\Big) + \cdots \ ,$$

  where $\tilde{\lambda} \sim s\lambda^{m+n}$. This is of the form of a general perturbation with operators

$$\lambda^i \mathcal{O}_i \ , \quad \text{beta functions } \beta^i \ .$$

- Using the background field method compute the β^λ and $\beta^{\tilde{\lambda}}$ exactly in λ and to leading order in $\tilde{\lambda}$.
- There is metric in the space of couplings G_{ij} defined via

$$G_{ij}(\lambda) = x^{d_i+d_j}\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle \ .$$

- Assuming renormalizability and using the Callan-Symanzik eq. the anomalous dimension matrix is [Kutasov 89]

$$\gamma_i{}^j = \partial_i\beta^j + G^{jm}\left(G_{in}\partial_m\beta^n + \beta^n\partial_n G_{im}\right) \ .$$

Consider the two couplings: $\mathcal{O}_\lambda = J_+^a J_-^a$ and $\mathcal{O}_{\tilde{\lambda}} = \mathcal{O}^{(m,n)}$.

- $\mathcal{O}_{\tilde{\lambda}} = \mathcal{O}^{(m,n)}$ breaks Lorentz and/or scale invariance.
- Operator mixing and an anomalous dimension matrix ${\gamma_i}^j$.
- When $\tilde{\lambda} \to 0$, no mixing and only ${\gamma_\lambda}^\lambda, {\gamma_{\tilde{\lambda}}}^{\tilde{\lambda}} \neq 0$.
- It turns out that $\beta^{\tilde{\lambda}} = \mathcal{O}(\tilde{\lambda})$.
- In the limit $\tilde{\lambda} \to 0$, the anomalous dimensions is

$$\boxed{\gamma_{\mathcal{O}_{\tilde{\lambda}}} = {\gamma_{\tilde{\lambda}}}^{\tilde{\lambda}} = 2\partial_{\tilde{\lambda}}\beta^{\tilde{\lambda}} + \beta^\lambda \partial_\lambda \ln g^{(0)}_{\tilde{\lambda}\tilde{\lambda}}} , \quad g^{(0)}_{ii}(\lambda) = G^{(0)}_{ij}(\lambda, 0) \ .$$

- Hence, when $\tilde{\lambda} = 0$, we have the λ-deformed theory and as a bonus the anomalous dimension of $\mathcal{O}_{\tilde{\lambda}}$.
- In our case the metric has

$$g^{(0)}_{\tilde{\lambda}\tilde{\lambda}} \sim \frac{1}{(1-\lambda^2)^{m+n}} \ .$$

Computable with free field contractions (strict $k \to \infty$ limit).

New examples

- Arbitrary number of same chirality operators

$$\mathcal{O}^{(m,0)} = d_{a_1 \dots a_m} J_+^{a_1} \dots J_+^{a_m} \;, \quad \gamma_{\mathcal{O}_\lambda^{(m,0)}} = 0 \;, \quad m = 2, 3, \dots \;.$$

  where $d_{a_1 \dots a_m}$ the symmetric rank-m tensor of $SU(N)$.
  At CFT point dim $(m, 0)$ [Bais-Bouwknegt-Surridge-Schoutens 88].
  Quite general and robust; related to chiral conservation laws.

- Mixed chirality operators I

$$\gamma_{\mathcal{O}_\lambda^{(m,0)} \mathcal{O}_\lambda^{(0,n)}} = 0 \;, \quad m, n = 2, 3, \dots \;.$$

  The anomalous dimension of the product of a product of a chiral and an anti-chiral operator vanishes.

- Mixed chirality operators II

$$\mathcal{O}^{(2,1)} = d_{abc} J_+^a J_+^b J_-^c \ .$$

At CFT point dim $(2,1)$ [Bais-Bouwknegt-Surridge-Schoutens 88].

$$\gamma_{\mathcal{O}_\lambda^{(2,1)}} = \gamma_{\mathcal{O}_\lambda^{(1,1)}} = -\frac{2c_G}{k} \frac{\lambda(1-\lambda(1-\lambda))}{(1-\lambda)(1+\lambda)^3} \ .$$

Unexpected degeneracy ....

- Perturbative checks, a laborious task ... with full agreement. Checks: $\gamma_{\mathcal{O}^{(m,0)}}$ and $\gamma_{\mathcal{O}^{(2,1)}}$ to orders 2 and 3, respectively.

- Vanishing at $\mathcal{O}(\lambda^2)$ implies that $\gamma_{\mathcal{O}} = 0$ (to $\mathcal{O}(1/k)$).

Remarks and future directions

Presented models of interacting current algebra theories.

- Focused on self-interacting theories, i.e. interactions of the type $J_+ J_-$.
- Generalization with many theories self- and mutually interacting, i.e. of the type $J_{1+} J_{2-}$, [Georgiou-KS, 18]. New IR fixed points.
- One may compute operators anomalous dimensions and β-functions:
  - Based on leading order perturbative results and symmetries.
  - Based on the geometrical data of the effective action σ-model.
- Cannot treat arbitrary operators this way.

New method and results based on the effective action and geometry in coupling space; not a panacea.

- In general one is faced with the problem of operator mixing. Not the case in the examples given above.
  However, for instance

$$\text{e.g.} \quad d_{abcde} J_+^a J_+^b J_+^c J_-^d J_-^e \quad \text{and} \quad (d_{abc} J_+^a J_+^b J_-^c) J_+^d J_-^d \ ,$$

  could mix.
- There is no mixing with descendant-type operators of the form

$$d_{abcd} J_+^a J_+^b J_+^c \partial_\pm J_-^d \ ,$$

  to $\mathcal{O}(1/k)$, but at higher ones.
- Indications that the spectrum of anomalous dimensional matrix has degeneracies, i.e. $\gamma_{\mathcal{O}_\lambda^{(2,1)}} = \gamma_{\mathcal{O}_\lambda^{(1,1)}}$.
- Indications for underlying spin chain for the anomalous dimensions of composite current operator [Georgiou-KS, in progress]. Is it integrable?