Random tensors and the SYK model

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https://arxiv.org/abs/1812.03008

Quantum Geometry, Field Theory and Gravity
Corfu Summer Institute
September 2019

The Sachdev-Ye-Kitaev (SYK) model

Quantum mechanical model ($1+0$ dimensions) with N Majorana fermions $\psi_i(t)$ with random degree q (even) interaction

$$H = \mathrm{i}^{q/2} J_{i_1 \dots i_q} \psi_{i_1} \cdots \psi_{i_q}, \qquad \{\psi_i, \psi_j\} = \delta_{ij}$$

- Random coupling $J_{i_1 \dots i_q} \to$ quenched disorder (average extensive/connected quantities) with Gaussian distribution

$$\langle J_{i_1 \dots i_q} J_{j_1 \dots j_q} \rangle = \sum_{\text{permutations } \pi} \epsilon(\pi) \frac{\sigma^2}{N^{q-1}} \delta_{i_1 j_{\pi(1)}} \cdots \delta_{i_q j_{\pi(q)}}$$

- Generalisation to a model with flavours (Gross, Rosenhaus)

$$H = \mathrm{i}^{q/2} J_{\underbrace{i_{1,1} \dots i_{1,q_1}}_{\text{flavour 1}}, \dots, \underbrace{i_{f,1} \dots i_{1,q_f}}_{\text{flavour f}}} \prod_{\substack{a=1,\dots,f \\ \text{flavours}}} \psi^a_{i_{a,1}} \cdots \psi^a_{i_{a,q_a}}$$

Model in condensed matter (Sachdev, Ye, Georges, Parcollet) and AdS_2/CFT_1 at large N (Kitaev, Maldacena, Stanford, Polchinski, Rosenhaus, Gross, ...). For an introduction, see lecture by V. Rosenhaus `https://arxiv.org/abs/1807.03334`

Lagrangian formulation and Feynman rules

Path integral formulation with Grassmann variables

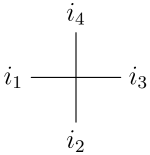
$$L = \int dt \left\{ \psi_i \partial_t \psi_i - \mathrm{i}^{q/2} J_{i_1 \dots i_q} \psi_{i_1} \cdots \psi_{i_q} \right\}$$

Expand for a fixed realisation of the disorder, then average

- Free propagator

$$i \;\frac{}{}\; j \quad \to \quad G_0(t,t') = \langle T \psi_i(t) \psi_j(t') \rangle = \frac{\delta_{ij}}{2} \mathrm{sgn}(t - t')$$

- Interaction vertex (for $q = 4$)



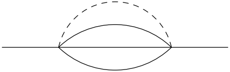
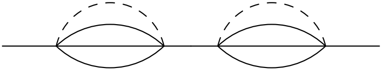
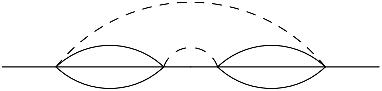
$$\to \quad J_{i_1 \dots i_4}$$

- Sum over internal indices

Large N limit

Quenched disorder : evaluate connected graphs at fixed J and then average over J :

$\langle J_{i_1 \dots i_q} \, J_{j_1 \dots j_q} \rangle = \sum_{\text{permutations } \pi} \epsilon(\pi) \frac{\sigma^2}{N^{q-1}} \, \delta_{i_1 j_{\pi(1)}} \cdots \delta_{i_q j_{\pi(q)}}$

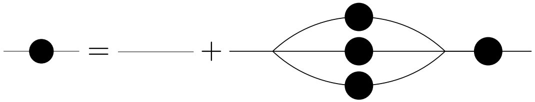
-  $\rightarrow \quad \frac{N^3}{N^3} = 1$
-  $\rightarrow \quad \frac{N^3 \times N^3}{N^3 \times N^3} = 1$
-  $\rightarrow \quad \frac{N \times N^3}{N^3 \times N^3} = \frac{1}{N^2}$

In the large N limit "melonic" graphs contribute : vertices should come by pairs related by $q-1$ dressed edges

Similar results for the 4 point function summing ladder (geometric series) : expression in terms of hypergeometric function (nearly conformal)

Schwinger-Dyson equations

Graphical recursive construction at large N (only melonic graphs)



$$G(t,t') = G_0(t,t') + \int dudv\, G_0(t,u) G^{q-1}(u,v) G(v,t')$$

$$G = G_0 + G_0 \star G^{q-1} \star G$$

At large N in the IR ($N \gg J|t-t'| \gg 1$), assume $G \ll G_0$

$$\boxed{G_*(t,t') \propto \frac{\mathrm{sgn}(t-t')}{|t-t'|^{2\Delta}}}$$

with Δ anomalous dimension of ψ in the IR (trivial in the UV)

$$0 = 0 + 2 - (q-1)\times 2\Delta - 2\Delta \quad \Rightarrow \quad \Delta = \frac{1}{q}$$

Effective action for bilocal fields

- Quenched disorder : introduce replicas $\log(Z) = \lim_{n\to 0} \frac{Z^n - 1}{n}$
  $\psi_i(t) \to \psi_i^r(t)$ with $r = 1, \dots, n$
- New variables O(N) invariant bilocal variables
  $G^{rr'}(t,t') = \frac{1}{N}\sum_i \psi_i^r(t)\psi_i^{r'}(t')$ with $\Sigma^{rr'}(t,t')$ Lagrange multiplier
- Perform Gaussian integrals over J and ψ
- Assume replicas diagonal solutions $G^{rr'} = G\delta^{rr'}$, $\Sigma^{rr'} = \Sigma\delta^{rr'}$
  Non diagonal solution are also possible (Arafeva, ..)

$$\boxed{\langle \log \mathcal{Z} \rangle_J = \int [DG][D\Sigma] \exp N S_{\text{eff}}[G,\Sigma]}$$

with O(N) invariant effective action for bilocals ($\star$ = convolution)

$$S_{\text{eff}}[G,\Sigma] = \frac{1}{2}\log\det\left(\partial - \Sigma\right)_\star + \frac{1}{2}\iint dtdt'\Sigma(t,t')G(t,t') + \sigma^2 G^q(t,t')$$

Large N limit and saddle point approximation

Path integral with effective action $\int [DG][D\Sigma] \exp NS_{\text{eff}}[G,\Sigma]$

$$S_{\text{eff}}[G,\Sigma] = \frac{1}{2}\log\det\left(\partial - \Sigma\right)_\star + \frac{1}{2}\iint dtdt'\Sigma(t,t')G(t,t') + \sigma^2 G^q(t,t')$$

Saddle point $\rightarrow$ Schwinger-Dyson equation

$$G(t,t') = \Big[\delta(t-t')\partial_t - \Sigma(t,t')\Big]_\star^{-1}, \qquad \Sigma(t,t') \;\; = J\Big[G(t,t')\Big]^{q-1}$$

Eliminating $\Sigma(t,t')$ with $G_0(t,t') = \left[\delta(t-t')\partial_t\right]_\star^{-1}$ free propagator

$$G(t,t') = G_0(t,t') + \int dudv\, G_0(t,u)G^{q-1}(u,v)G(v,t')$$

$$G = G_0 + G_0 \star G^{q-1} \star G$$

In the IR, we may simply drop $\delta(t-t')\partial_t$

$$\delta(t-t') = \int du\, G^{q-1}(t,u)G(u,t') \Leftrightarrow -1 = G^{q-1} \star G$$

Reparametrisation invariance

$$S_{\text{eff}}[G,\Sigma] = \frac{1}{2}\log\det\left(\partial_t - \Sigma\right)_\star + \iint dt dt' \Sigma(t,t')G(t,t') + JG^q(t,t')$$

In the IR (drop ∂_t), S_{eff} invariant under reparametrisation $t \to f(t)$ with $\Delta = \frac{1}{q}$ anomalous dimension

$$\psi(t) \to \left|\frac{df}{dt}\right|^{\Delta}\psi(f(t))$$
$$G(t,t') \to \left|\frac{df}{dt}\right|^{\Delta}\left|\frac{df}{dt'}\right|^{\Delta} G(f(t),f(t'))$$
$$\Sigma(t,t') \to \left|\frac{df}{dt}\right|^{1-\Delta}\left|\frac{df}{dt'}\right|^{1-\Delta} \Sigma(f(t),f(t'))$$

$G_*(t,t') \propto \dfrac{\text{sgn}(t-t')}{|t-t'|^{2\Delta}}$ only $\text{SL}_2(\mathbb{R})$ invariant $t \to f(t) = \dfrac{at+b}{ct+d}$

Spontaneous (and explicit by ∂_t) breaking of reparametrisation invariance $\to$ Schwarzian action for pseudo Goldstone modes (Kitaev, Witten, Standford)

Non Gaussian averages and random tensors

Non Gaussian disorder ($V_N(J)$ perturbation) $\rightarrow J_{i_1 \cdots q}$ random tensor

$$\langle \ldots \rangle_J = \frac{\int dJ \, \ldots \, \exp - \left\{ \frac{N^{q-1}}{2\sigma^2} J^2 + V_N(J) \right\}}{\int dJ \exp - \left\{ \frac{N^{q-1}}{2\sigma^2} J^2 + V_N(J) \right\}}$$

Average of interaction term expressed as

$$\left\langle \exp \left\{ J_{i_1 \ldots i_q} \sum_r \int dt \psi^r_{i_1}(t) \cdots \psi^r_{i_q}(t) \right\} \right\rangle = \exp \left\{ \frac{N^{q-1}}{2\sigma^2} K^2 - V'_N(K) \right\}$$

with normalised background effective action $V'_N(K)$ for K

$$K_{i_1 \ldots i_q} = \sum_r \int dt \, \psi^r_{i_1}(t) \cdots \psi^r_{i_q}(t)$$

$$V'_N(K) = -\log \int dJ \exp - \left\{ \frac{N^{q-1}}{2\sigma^2} J^2 + V_N(K + J) \right\}$$

A (very) short overview of random tensors

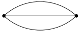
- Generalisation of random matrices $M_{ij} \to T_{ijk...}$ with U(N) or O(N) invariant interactions
- Generating function for higher dimensional random geometries

$$\int_{\text{rank } D \text{ tensors}} dT \exp -S_N(T) = \sum_{\substack{D\text{-valent Feynman graphs} \\ \Leftrightarrow\ D\text{-dimensional triangulations}}} W(\text{triangulation})$$

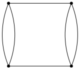
- Perturbative expansion of non symmetric (coloured) complex $T, \overline{T}$ models (Gurau, Rivasseau, Bonzom, Riello, Tanasa, ...) and some real models (Carrozza, Tanasa) dominated by melonic Feynman graphs with well defined large N limit.
- Reformulation of the SYK model without quenched disorder (Witten, Gurau, Klebanov, Tarnopolski, ...) : $\psi_{ijk...}(t)$ fundamental degrees of freedom instead of $\psi_i(t)$ (here the coupling is the random tensor instead)

Invariant interactions

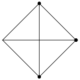
U(N) or O(N) invariant interactions constructed using graphs : tensors at vertices and indices contracted along the edges

- Dipole 

$$(T \cdots T)_\Gamma = T_{ijk} T_{ijk}$$

- Quartic melon 

$$(T \cdots T)_\Gamma = T_{ijk} T_{ijl} T_{mnk} T_{mnj}$$

- Tetrahedron 

$$(T \cdots T)_\Gamma = T_{ijk} T_{klm} T_{mjn} T_{nli}$$

For non symmetric complex models : black and white vertices $(T, \overline{T})$ and label $1, 2, .., D$ edges at each vertex (place of index)

Existence of a large N limit

Find suitable exponents δ_Γ in such a way that

$$V'_N(T') = -\log \int dT \, \exp - \left\{ \frac{N^{q-1}}{2\sigma^2} T^2 + V_N(T' + T) \right\}$$

with potentials expanded over graphs

$$V_N(T) = \sum_\Gamma N^{\delta_\Gamma} \lambda_\Gamma N^{\delta_\Gamma} (T \cdots T)_\Gamma, \quad V'_N(T') = \sum_\Gamma \lambda'_\Gamma N^{\delta_\Gamma} (T \cdots T)_\Gamma$$

such that all $\lim_{N \to +\infty} \lambda'_\Gamma(\lambda_\Gamma)$ exists). Positive answer for

- Complex non symmetric tensors (Gurau, Rivasseau) with $\delta_\Gamma = q - 1 - \# \{\text{connected components of } \Gamma\}$
- Real and complex non symmetric tensors with tetrahedral interaction (Carrozza, Tanasa)
- Real antisymmetric and symmetric traceless tensors with tetrahedral interaction (Benedetti, Carrozza, Gurau)

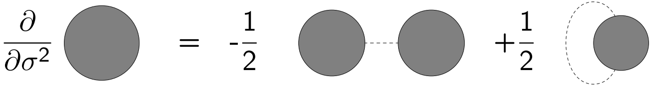
Large N limit from a Polchinski like equation

Effective action for a complex non symmetric rank q tensor

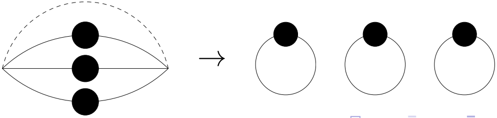
$$V'_N(T', \bar{T}') = -\log \int dT d\bar{T} \, \exp - \left\{ \frac{N^{q-1}}{2\sigma^2} T\bar{T} + V_N(T' + T, \bar{T}' + \bar{T}) \right\}$$

obeys a Polchinski like equation (fast modes integration in QFT)

$$\frac{\partial V'_N}{\partial \sigma^2} = \frac{1}{N^{q-1}} \sum_{1 \le i_1, \dots, i_q \le N} \left(\frac{\partial^2 V'_N}{\partial T_{i_1,\dots,i_q} \partial \bar{T}_{i_1,\dots,i_q}} - \frac{\partial V'_N}{\partial T_{i_1,\dots,i_q}} \frac{\partial V'_N}{\partial \bar{T}_{i_1,\dots,i_q}} \right)$$



- Derivation with respect to T and $\bar{T}$ removes a pair of vertices
- Corresponding free lines are reattached by index contraction
- Leading contribution when creation of connected components

(melonic graphs) 

Gaussian universality for bilocal fields

Inserting the bilocal field $G(t,t') = \frac{1}{N}\sum_i \psi_i^r(t)\psi_i(t')$, the scaling is $N^{\delta_\Gamma - 1 - v(q-1)+e} = N^{\delta_\Gamma - 1 - v(q/2-1)}$ $(2e = qv$ for q-valent graphs)

$$S'_{\text{eff}}(G,\Sigma) = S_{\text{eff}}(G,\Sigma) + \sum_{\Gamma} N^{\delta_\Gamma - 1 - v(q/2-1)} \lambda'_\Gamma \int \prod_v dt_v \prod_{e=vv'} G(t_v, t_{v'})$$

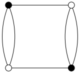
- Gaussian universality : for $q > 2$, for all know scalings leading to a large N limit, only the dipole contributes at leading order, with a modified covariance
- For $q = 2$ (matrix model), all single trace interactions survive
- Corrections are invariant under reparametrisation
- Similar to previous results on p-spin glasses (Bonzom, Gurau, Smerlak)

Details for a complex coloured model

Model with q complex field $\rightarrow$ random coupling = complex non symmetric tensor (simplest combinatorics)

$$H = \mathrm{i}^{\frac{q}{2}} \sum_{i_1,\ldots,i_q} \bar{J}_{i_1,\ldots,i_q} \psi^1_{i_1} \cdots \psi^q_{i_q} + \mathrm{i}^{\frac{q}{2}} \sum_{i_1,\ldots,i_q} J_{i_1,\ldots,i_q} \bar{\psi}^1_{i_1} \cdots \bar{\psi}^q_{i_q}$$

Example of a quartic melonic interaction,



$$\rightarrow \quad \lambda N^3 J_{i_1 i_2 i_3 k} \bar{J}_{i_1 i_2 i_3 l} \bar{J}_{j_1 j_2 j_3 k} J_{j_1 j_2 j_3 l}$$

The modified covariance is computed as

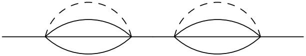
$$(\sigma')^2 = \frac{-1 + \sqrt{1 + 4\lambda\sigma^4}}{\sigma^2}$$

Leading order non Gaussian correction to the effective action

$$\propto \frac{1}{N^2} \int dt_1 dt_2 dt_3 dt_4 G^3(t_1, t_2) G(t_1, t_3) G(t_2, t_4) G^3(t_3, t_4)$$

Short summary

- SYK : quantum mechanical models with N Majorana fermions and quenched disorder
- Solvable in the large N limit with nearly conformal invariance in the IR (only "melonic" graphs survive)



- Random tensors : higher dimensional generalisations of random matrices $M_{ij} \to T_{ijk\dots}$ with large N limit dominated by melonic graphs
- Disorder : coupling constant considered as a random tensor

$$\left\langle \exp \int J_{i_1 \dots i_q} \psi_{i_1} \cdots \psi_{i_q} \right\rangle_J$$

- Gaussian universality : average over non Gaussian disorder equivalent to a Gaussian one