An exact symmetry in λ-deformed σ-models

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ongoing work with
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Introduction and motivation

Exact β-functions and non-perturbative symmetries

1. In a renormalizable field theory, its quantum behaviour is encoded within the RG flows
$$\beta^\lambda = \frac{d\lambda}{d\ln\mu^2}$$
which are usually determined perturbatively.
2. Can we obtain the all-loop β-function?
3. Any non-perturbative symmetries?
4. New fixed points towards the IR?

We study the above in the non-Abelian Thirring model

FOCAL POINTS

- The non-Abelian Thirring model
- The effective action
- The one and two loop β-function
- Conclusion and Outlook

Non-Abelian Thirring model

Consider the WZW action Witten (1983):

$$S_{\mathrm{WZW},k}(g) = -\frac{k}{2\pi}\int \mathrm{d}^2\sigma\, \mathrm{Tr}\left(g^{-1}\partial_+ g\, g^{-1}\partial_- g\right) + \frac{k}{12\pi}\int_B \mathrm{Tr}\left(g^{-1}\mathrm{d}g\right)^3 ,$$

invariant under the left-right current algebra symmetry: $g \mapsto \Omega^{-1}(\sigma_+)\, g\, \Omega(\sigma_-)$.

The non-abelian Thirring model is defined through

$$S = S_{\mathrm{WZW},k}(g) + k\,\frac{\lambda_{ab}}{\pi}\int \mathrm{d}^2\sigma\, J_+^a\, J_-^b$$

The currents and the adjoint action are defined through

$$J_+^a = -i\,\mathrm{Tr}(t_a\,\partial_+ g\, g^{-1})\,, \quad J_-^a = -i\,\mathrm{Tr}(t_a\, g^{-1}\,\partial_- g)\,, \quad D_{ab} = \mathrm{Tr}(t_a g t_b g^{-1})\,,$$

where $D_{ac}D_{bc} = \delta_{ab}$, $[t_a, t_b] = i f_{abc} t_c$, $\mathrm{Tr}(t_a t_b) = \delta_{ab}$ and $f_{acd} f_{bcd} = c_G\, \delta_{ab}$.

NON-ABELIAN THIRRING MODEL

Symmetries of the non-abelian Thirring model:

$$S = S_{\mathrm{WZW},k}(g) + k\,\frac{\lambda_{ab}}{\pi}\int \mathrm{d}^2\sigma\, J_+^a\, J_-^b$$

1. The left-right current algebra symmetry is broken for a generic matrix λ_{ab}
2. It is invariant under the generalized parity symmetry:

$$\lambda \mapsto \lambda^T\,, \qquad g \mapsto g^{-1}\,, \qquad \sigma^{\pm} \mapsto \sigma^{\mp}$$

3. The operator driving the perturbation is marginally relevant Kutasov (1989)

$$\beta^{\lambda} = -\frac{c_G\,\lambda^2}{2k\,(1+\lambda)^2} + \mathcal{O}\left(\frac{1}{k^2}\right)\,, \quad \lambda_{ab} = \lambda\,\delta_{ab}$$

4. The corresponding ”effective action” is invariant under the inversion of the coupling Kutasov (1989)

$$\lambda \mapsto \lambda^{-1}\,, \quad k \mapsto -k - c_G$$

How about an effective action?

Plan of the talk

The effective action

By a gauging procedure we can construct the following action Sfetsos (2013)

$$S_{k,\lambda}(g) = S_{\mathrm{WZW},k}(g) + \frac{k}{\pi}\int \mathrm{d}^2\sigma\, J_+^a \left(\lambda^{-1} - D^T\right)^{-1}_{ab} J_-^b$$

Interpolating between a WZW at $\lambda_{ab} = 0$ and the non-Abelian T-dual of the PCM at $\lambda_{ab} \longrightarrow \delta_{ab}$.

Properties

1. For $\lambda_{ab} \ll \delta_{ab}$ we get the non-Abelian Thirring model

$$S = S_{\mathrm{WZW},k}(g) + k\,\frac{\lambda_{ab}}{\pi}\int \mathrm{d}^2\sigma\, J_+^a J_-^b$$

2. Invariance under the generalized parity symmetry: $g \mapsto g^{-1}$, $\sigma^{\pm} \mapsto \sigma^{\mp}$
3. Explicit weak-strong duality: $S_{-k,\lambda^{-1}}(g^{-1}) = S_{k,\lambda}(g)$
4. Interesting limits around $\lambda_{ab} = \pm\delta_{ab}$ – non-Abelian T-dual of PCM and pseudo-chiral model

One loop

Consider a 1+1-dimensional non-linear σ-model with action

$$S = \frac{1}{2\pi\alpha'} \int d^2\sigma\, E_{\mu\nu}\, \partial_+ X^\mu \partial_- X^\nu \,, \quad E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$$

The one-loop β-functions for $G_{\mu\nu}$ and $B_{\mu\nu}$ read:
Ecker–Honerkamp 71, Friedan 80, Braaten–Curtright–Zachos 85

$$\frac{d\, E_{\mu\nu}}{d \ln \mu^2} = R^{-}_{\mu\nu} + \nabla^{+}_{\nu} \xi_\mu \,,$$

with and the last term corresponds to field redefinitions (diffeomorphisms).

Generalities

- The Ricci tensor and the covariant derivative include torsion terms, i.e. $H = dB$
- The σ-model is renormalizable within the zoo of metrics and 2-forms
- Not given that the RG flows will retain the form at hand of $G_{\mu\nu}$ and $B_{\mu\nu}$

Plan of the talk

An example

Consider the effective action for $g \in SU(2)$ case and $\lambda_{ab} = \mathrm{diag}(\lambda, \lambda, 1)$

$$ds^2 = k\frac{1-\lambda}{1+\lambda}(d\omega^2 + \cot^2\omega d\varphi^2) + 4k\frac{\lambda}{1-\lambda^2}(\cos\varphi d\omega + \sin\varphi \cot\omega d\varphi)^2\,, \quad 0 \leqslant \lambda \leqslant 1$$

describing the λ-deformed $su(2)/u(1)$ coset CFT Sfetsos (2013)

The RG flow at one-loop in $1/k$ expansion reads Itsios–Sfetsos–KS (2014)

$$\beta^\lambda = -\frac{\lambda}{k} + \mathcal{O}\left(\frac{1}{k^2}\right)\,, \quad \xi_\mu = \partial_\mu \Phi\,, \quad \Phi = -2\ln\sin\omega$$

where k does not run with the energy scale.

Properties of the β-function:

1. It is linear on λ, the operator driving the perturbation is relevant with $\Delta = 2 - 2/k$
2. RG flows: UV (coset CFT) $\lambda = 0$ towards the IR (strongly coupled) $\lambda \to 1^-$
3. It respects the weak-strong duality

$$\lambda \to \lambda^{-1}\,, \quad k \to -k\,, \quad k \gg 1$$

How about beyond one-loop order?

TWO LOOP

Consider again the 1+1-dimensional non-linear σ-model with action

$$S = \frac{1}{2\pi\alpha'} \int \mathrm{d}^2\sigma\, E_{\mu\nu}\, \partial_+ X^\mu \partial_- X^\nu\,, \quad E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$$

The two-loop β-functions for $G_{\mu\nu}$ and $B_{\mu\nu}$ read:
Ecker–Honerkamp 71, Friedan 80, Braaten–Curtright–Zachos 85, Metsaev–Tseytlin 87, Hull–Townsend 87, Osborn 90

$$\frac{\mathrm{d}\, E_{\mu\nu}}{\mathrm{d}\ln\mu^2} = R^-_{\mu\nu} + R^-_{\mu\kappa\lambda\rho}\left(R^{-\kappa\lambda\rho}{}_\nu - \frac{1}{2} R^{-\lambda\rho\kappa}{}_\nu\right) + \frac{1}{2}(H^2)^{\kappa\lambda} R^-_{\kappa\mu\nu\lambda} + \nabla^+_\nu \xi_\mu$$

Comments:

1. At first glance as a two-loop result it could be scheme dependent.
2. However, this is the only scheme which gives vanishing β-function for WZW models.
   Metsaev–Tseytlin 87

Back to the example

Consider again the λ-deformed $SU(2)_k/U(1)_k$ coset CFT

$$\mathrm{d}s^2 = k\frac{1-\lambda}{1+\lambda}(\mathrm{d}\omega^2 + \cot^2\omega\,\mathrm{d}\varphi_1^2) + 4k\frac{\lambda}{1-\lambda^2}(\cos\varphi\,\mathrm{d}\omega + \sin\varphi\cot\omega\,\mathrm{d}\varphi)^2\,, \quad 0\leqslant\lambda\leqslant 1$$

Working out the two-loop RG flows we find that the model is NOT renormalizable.

More accurately:

1. The metric does not retain its form under two-loop RG flows.
2. One needs to add counterterms?
3. Doing so, k is running under the energy scale and needs to be redefined
   Hoare, Levine, Tseytlin (2019)

Puzzling... can we take a detour of the problem? Use another (equivalent) effective action?

An effective action

An equivalent yet simpler effective action

$$S_{k,\lambda} = S_{\mathrm{WZW},k}(g_1) + S_{\mathrm{WZW},k}(g_2) + k\,\frac{\lambda_{ab}}{\pi}\int \mathrm{d}^2\sigma\, J^a_{1+}\, J^b_{2-}$$

Georgiou, Sfetsos (2016)

Properties:

1. Interesting limits around $\lambda_{ab} = \pm\delta_{ab}$ – PCM and pseudo-chiral model Nappi (1980)

2. It is canonically equivalent to the λ-deformed action Georgiou, Sfetsos, Siampos (2017)

$$S_{k,\lambda}(g) = S_{\mathrm{WZW},k}(g) + \frac{k}{\pi}\int \mathrm{d}^2\sigma\, J^a_+ \left(\lambda^{-1} - D^T\right)^{-1}_{ab} J^b_-$$

3. Identical β-function and (current, composite current, etc) anomalous dimensions. Georgiou, Sagkrioti, Sfetsos, Siampos (2017)

4. Same Zamolodchikov metric for the composite operator driving the perturbation. Sagkrioti, Sfetsos, Siampos (2018)

5. Weak-strong duality: $S_{-k-c_G,\lambda^{-1}} = S_{k,\lambda}$ Kutasov (1989)

BACK TO THE EXAMPLE

Consider $g_{1,2} \in SU(2)$ and $\lambda_{ab} = \mathrm{diag}(\lambda, \lambda, 1)$.

This case corresponds to a parafermionic deformation of the coset CFT $\frac{SU(2)_k \times SU(2)_k}{U(1)_k}$
Guadagnini, Martellini, Mintchev (1987)

$$S_{\text{coset}} = S_{\text{CFT}} + \frac{k\lambda}{\pi} \int \mathrm{d}^2\sigma\, \mathcal{O}\,, \quad \mathcal{O} = \frac{1}{4}\left(\Psi\bar{\Psi} + \Psi^\dagger \bar{\Psi}^\dagger\right)$$

where the metric and the two-form read

$$\mathrm{d}\ell^2 = \frac{k}{4\pi}\left((\mathrm{d}\psi + \cos\vartheta_1 \mathrm{d}\varphi_1 + \cos\vartheta_2 \mathrm{d}\varphi_2)^2 + \mathrm{d}\vartheta_1^2 + \sin^2\vartheta_1 \mathrm{d}\varphi_1^2 + \mathrm{d}\vartheta_2^2 + \sin^2\vartheta_2 \mathrm{d}\varphi_2^2\right)$$

and

$$B = \frac{k}{4\pi}\,(\mathrm{d}\psi + \cos\vartheta_1 \mathrm{d}\varphi_1) \wedge (\mathrm{d}\psi + \cos\vartheta_2 \mathrm{d}\varphi_2)$$

Zayas–Tseytlin (2000)
In addition, $(\Psi, \bar{\Psi})$ are the parafermion operators

$$\Psi = (\partial_+\vartheta_1 + i\sin\vartheta_1\, \partial_+\varphi_1)\, \mathrm{e}^{-i(\psi/2+\bar{\psi})}\,, \quad \bar{\Psi} = (\partial_-\vartheta_2 + i\sin\vartheta_2\, \partial_-\varphi_2)\, \mathrm{e}^{-i(\psi/2-\bar{\psi})}$$

and their complex conjugates $\Psi^\dagger$ and $\bar{\Psi}^\dagger$ respectively.

Here $\bar{\psi}$ represents a non-local function of (ϑ_i, φ_i), which dresses the operators to ensure conservation $\partial_-\Psi = 0 = \partial_+\bar{\Psi}$

Back to the example

Properties:

1. Its two-loop RG flow reads

$$\beta^{\lambda} = -\frac{\lambda}{k} - \frac{4}{k^2}\frac{\lambda^3}{1-\lambda^2} + \mathcal{O}\left(\frac{1}{k^3}\right) \leqslant 0$$

   and k is not running with the energy scale.

2. The β-function is covariant under the symmetry

$$\lambda \mapsto \lambda^{-1}\,, \qquad k \mapsto -k - c_G$$

   to order $1/k^2$.

Back to the example

3. Using CFT input, the non-perturbative symmetry and well-defined limits around $\lambda = \pm 1$

$$g(\lambda;k) = |x_{12}|^{2(2+\gamma^{(\mathcal{O})})}\langle \mathcal{O}(x_1,\bar{x}_1)\mathcal{O}(x_2,\bar{x}_2)\rangle_\lambda = \frac{1}{(1-\lambda^2)^2}\left(1 + \cancel{\frac{1}{k}\frac{P(\lambda)}{1-\lambda^2}}\right)$$

4. Anomalous dimension of the parafermion bilinear

$$\gamma^{(\mathcal{O})} = 2\partial_\lambda \beta(\lambda;k) + \beta(\lambda;k)\partial_\lambda \ln g(\lambda;k) = -\frac{2}{k}\frac{1+\lambda^2}{1-\lambda^2} - \frac{8}{k^2}\frac{\lambda^2(3+\lambda^2)}{(1-\lambda^2)^2}$$

at the UV CFT point $\lambda = 0$ we find $\Delta = 2 + \gamma^{(\mathcal{O})} = 2 - 2/k$

BACK TO THE EXAMPLE

5. Using the c-theorem

$$\frac{\mathrm{d}C}{\mathrm{d}\ln\mu^2} = \beta^\lambda \partial_\lambda C = 24 g(\lambda;k)\beta^\lambda\beta^\lambda \geqslant 0$$

   we find the C-function

$$C(\lambda) = 5 - \frac{12}{k}\frac{1}{1-\lambda^2} + \frac{24}{k^2}\frac{1-2\lambda^2}{(1-\lambda^2)^2} + \mathcal{O}\left(\frac{1}{k^3}\right)$$

   with $C(0) = c_{\mathrm{UV}} = 2 \times \frac{2k\mathrm{dimG}}{2k+c_G} - 1 = 5 - \frac{12}{k} + \frac{24}{k^2} + \mathcal{O}\left(\frac{1}{k^3}\right)$

6. Analogously we can also work out the C-function for the λ-deformed $SU(2)_k/U(1)_k$

$$C = 2 - \frac{6}{k}\frac{1+\lambda^2}{1-\lambda^2} + \frac{12}{k^2}\frac{1-2\lambda^2-\lambda^4}{(1-\lambda^2)^2},$$

   with $C(0) = c_{\mathrm{UV}} = \frac{2k\mathrm{dimG}}{2k+c_G} - 1 = 2 - \frac{6}{k} + \frac{12}{k^2} + \mathcal{O}\left(\frac{1}{k^3}\right)$

Plan of the talk

The effective action

An example

Conclusion

CONCLUSION

We studied a relevant λ-deformation of the coset CFT $\frac{SU(2)_k \times SU(2)_k}{U(1)_k}$, namely

$$S_{k,\lambda} = S_{\mathrm{WZW},k}(g_1) + S_{\mathrm{WZW},k}(g_2) + k\,\frac{\lambda_{ab}}{\pi}\int \mathrm{d}^2\sigma\, J^a_{1+} J^b_{2-}\,, \quad g_{1,2} \in SU(2)\,, \quad \lambda_{ab} = \mathrm{diag}(\lambda, \lambda, 1)$$

1. Using gravitational techniques, we find its two-loop β-function

   $$\beta^\lambda = -\frac{\lambda}{k} - \frac{4}{k^2}\frac{\lambda^3}{1-\lambda^2} + \mathcal{O}\left(\frac{1}{k^3}\right) \leqslant 0$$

   which is exact in λ and to order $1/k^2$
2. The β-function is invariant under the (exact) symmetry

   $$\lambda \mapsto \lambda^{-1}\,, \qquad k \mapsto -k - c_G$$

   to order $1/k^2$.
3. We worked out the Zamolodchikov metric and the anomalous dimension of the composite operator driving the perturbation.
4. We evaluated its C-function which satisfies Zamolodchikov's c-theorem.
5. This deformation shares the *same* quantum properties as the λ-deformed $SU(2)_k/U(1)_k$

Other deformations?

OTHER DEFORMATIONS

Similarly, we may also consider the isotropic λ-deformed case

$$S_{k,\lambda} = S_{\mathrm{WZW},k}(g_1) + S_{\mathrm{WZW},k}(g_2) + k\,\frac{\lambda_{ab}}{\pi}\int \mathrm{d}^2\sigma\, J^a_{1+}\, J^b_{2-}\,, \quad g_{1,2}\in G\,, \quad \lambda_{ab} = \lambda\delta_{ab}$$

interpolating between a UV $G_k \times G_k$ at $\lambda = 0$ towards a PCM (strongly coupled) $\lambda \longrightarrow 1^-$

1. Using gravitational techniques, we find its two-loop β-function

$$\beta^\lambda = -\frac{c_G}{2k}\,\frac{\lambda^2}{(1+\lambda)^2} + \frac{c_G^2}{2k^2}\,\frac{\lambda^4(1-2\lambda)}{(1-\lambda)(1+\lambda)^5}$$

2. It is invariant under the symmetry

$$\lambda \longrightarrow \frac{1}{\lambda}\left(1 - \frac{c_G}{k}\,\frac{1-\lambda}{1+\lambda}\right), \quad k \longrightarrow -k - c_G$$

   etc

Other extensions involve different levels $k_{1,2}$ for the WZWs, non-trivial IR fixed points

$$\mathrm{UV}_{\lambda=0} : G_{k_1} \times G_{k_2} \quad \Longrightarrow \quad \mathrm{IR}_{\lambda=\lambda_0} : G_{k_2-k_1} \times G_{k_1}\,, \quad \lambda_0 = \sqrt{\frac{k_1}{k_2}} \leqslant 1$$

The corresponding symmetry in this case

$$\lambda \longrightarrow \frac{1}{\lambda}\left(1 - \frac{c_G}{k} f(\lambda,\lambda_0)\right), \quad k_1 \longrightarrow -k_2 - c_G, \quad k_2 \longrightarrow -k_1 - c_G$$

PARAFERMIONS IN $SU(2)_k/U(1)_k$

The λ-deformed $SU(2)_k/U(1)_k$

$$\mathrm{d}s^2 = k\frac{1-\lambda}{1+\lambda}(\mathrm{d}\omega^2 + \cot^2\omega\,\mathrm{d}\varphi_1^2) + 4k\frac{\lambda}{1-\lambda^2}(\cos\varphi\,\mathrm{d}\omega + \sin\varphi\cot\omega\,\mathrm{d}\varphi)^2\,, \quad 0 \leqslant \lambda \leqslant 1$$

It can be understood as a parafermionic perturbation of the coset CFT

$$S = S_{\text{CFT}} + \frac{k\lambda}{\pi}\int \mathrm{d}^2\sigma\left(\Psi\bar{\Psi} + \Psi^\dagger\bar{\Psi}^\dagger\right)$$

where

$$S_{\text{CFT}} = \frac{k}{\pi}\int \mathrm{d}^2\sigma\left(\partial_+\omega\partial_-\omega + \cot^2\omega\,\partial_+\varphi\partial_-\varphi\right)$$

and $(\Psi,\bar{\Psi})$ are the parafermion operators

$$\Psi = (\partial_+\omega + i\cot\omega\,\partial_+\varphi)\,\mathrm{e}^{-i(\varphi+\bar{\varphi})}\,, \quad \bar{\Psi} = (\partial_-\omega + i\cot\omega\,\partial_-\varphi)\,\mathrm{e}^{-i(\varphi-\bar{\varphi})}$$

with $\Psi^\dagger$ and $\bar{\Psi}^\dagger$ their complex conjugates respectively.

Here $\bar{\varphi}$ represents a non-local function of (ω,φ), which dresses the operators to ensure conservation $\partial_-\Psi = 0 = \partial_+\bar{\Psi}$