Noncommutative Scalar Field from Angular Twist

Corfu Summer Institute

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Content

Literature

- M. D. Ćirić, N. Konjik and A. Samsarov, Class. Quant. Grav. **35** (2018) no.17, 175005 doi:10.1088/1361-6382/aad201 [arXiv:1708.04066 [hep-th]].
- M. Dimitrijevic Ciric, N. Konjik, M. A. Kurkov, F. Lizzi and P. Vitale, Phys. Rev. D **98** (2018) no.8, 085011 doi:10.1103/PhysRevD.98.085011 [arXiv:1806.06678 [hep-th]].
- M. D. Ćirić, N. Konjik and A. Samsarov, arXiv:1904.04053 [hep-th].

NC space-time from the angular twist

Twist is used to deform a symmetry Hopf algebra
Twist $\mathcal{F}$ is invertible bidifferential operator from the universal enveloping algebra of the symmetry algebra

We work in 4D and deform the space-time by the following twist

$$\mathcal{F} = \mathrm{e}^{-\frac{i}{2}\theta_{ab}X^a \bigotimes X^b}$$

$[X^a, X^b] = 0$, a,b=1,2 $\quad X_1 = \partial_0$ and $X_2 = x\partial_y - y\partial_x$

$$\mathcal{F} = \mathrm{e}^{\frac{-ia}{2}(\partial_0 \otimes (x\partial_y - y\partial_x) - (x\partial_y - y\partial_x)\otimes \partial_0)}$$

Commutation relations between coordinates are:

$[\hat{x}^0, \hat{x}] = ia\hat{y},$ All other commutation relations are zero

$[\hat{x}^0, \hat{y}] = -ia\hat{x}$

Our approach: deform space-time by an Abelian twist to obtain commutation relations

Angular twist in curved coordinates $X_1 = \partial_0$ and $X_2 = \partial_\varphi$

-supose that metric tensor $g_{\mu\nu}$ does not depend on t and φ coordinates

-Hodge dual becomes same as in commutative case

- Deformation of differential calculus

$$f \star g = \mu \mathcal{F}^{-1}(f \otimes g) \qquad \omega_1 \wedge_\star \omega_2 = \wedge \mathcal{F}^{-1}(\omega_1 \otimes \omega_2).$$

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- Action of the vector field on the differential forms is given by the Lie derivative along that vector field $X_A \triangleright \omega = \ell_{X_A} \omega$
- Twisted Hopf algebra is

$$\begin{aligned} [t^a, t^b] &= i f^{ab}_{\ c} t^c, \\ \Delta_{\mathcal{F}}(t^a) &= \mathcal{F} \Delta(t^a) \mathcal{F}^{-1}, \\ \epsilon(t^a) &= 0, \quad S_{\mathcal{F}}(t^a) = \mathrm{f}^\alpha S(\mathrm{f}_\alpha) S(t^a) S(\bar{\mathrm{f}}^\beta) \mathrm{f}_\beta. \end{aligned}$$

where is

$$\mathcal{F} = \mathrm{f}^\alpha \otimes \mathrm{f}_\alpha, \quad \mathcal{F}^{-1} = \bar{\mathrm{f}}^\alpha \otimes \bar{\mathrm{f}}_\alpha,$$

Angular noncommutativity

- Product of two plane waves is

$$e^{-ip\cdot x} \star e^{-iq\cdot x} = e^{-i(p+_\star q)\cdot x}$$

where is $p +_\star q = R(q_0)p + R(-p_0)q$ and

$$R(t) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\left(\frac{at}{2}\right) & \sin\left(\frac{at}{2}\right) & 0 \\ 0 & -\sin\left(\frac{at}{2}\right) & \cos\left(\frac{at}{2}\right) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Angular noncommutativity

- $e^{-ip\cdot x} \star e^{-iq\cdot x} \star e^{-ir\cdot x} = e^{-i(p+_\star q+_\star r)\cdot x}$ gives

$$p +_\star q +_\star r = R(r_0 + q_0)p + R(-p_0 + r_0)q + R(-p_0 - q_0)r$$

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- General case

$$p^{(1)} +_\star \dots +_\star p^{(N)} = \sum_{j=1}^{N} R\left(-\sum_{1\leq k<j} p_0^{(k)} + \sum_{j<k\leq N} p_0^{(k)}\right) p^{(j)}$$

- Conservation of momentum is broken!

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- Suppose that a field (or a state) ϕ_p is an eigenvector of the momentum operator P_μ with the eigenvalue p_μ: $P_\mu \phi_p = p_\mu \phi_p$
- Then $P_\mu(\phi_p \star \phi_q) = \mu_\star\{\Delta^{\mathcal{F}} P_\mu(\phi_p \otimes \phi_q)\} = (p +_\star q)(\phi_p \star \phi_q)$

Particle decay

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- We will look at kinematically decay of one particle in the rest to the two other particles. Momentum law conservation is

$$p +_\star (-q) +_\star (-r) = R(-q_0 - r_0)\cdot p - R(-p_0 - r_0)\cdot q - R(-p_0 + q_0)\cdot r$$

Particle decay

- These four equations are

$$M = E_q + E_r$$

$$0 = q_z + r_z$$

$$0 = \cos\big(\frac{a}{2}(M+E_r)\big)q_x - \sin\big(\frac{a}{2}(M+E_r)\big)q_y + \cos\big(\frac{aE_r}{2}\big)r_x - \sin\big(\frac{aE_r}{2}\big)r_y$$

$$0 = \cos\big(\frac{a}{2}(M+E_r)\big)q_y + \sin\big(\frac{a}{2}(M+E_r)\big)q_x + \cos\big(\frac{aE_r}{2}\big)r_y + \sin\big(\frac{aE_r}{2}\big)r_x$$

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- NC effects are obviousy in the xy-plane

Particle decay

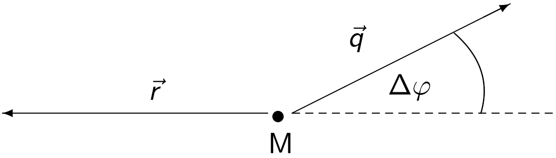

Figure: Decay of particle in rest with its mass M in xy-plane.

$\Delta\varphi = \frac{aM}{2}$

Scalar $U(1)_\star$ gauge theory

If a one-form gauge field $\hat{A} = \hat{A}_\mu \star dx^\mu$ is introduced to the model through a minimal coupling, the relevant action becomes

$$
\begin{aligned}
S[\hat{\phi}, \hat{A}] &= \int \left(d\hat{\phi} - i\hat{A} \star \hat{\phi}\right)^+ \wedge_\star *_H \left(d\hat{\phi} - i\hat{A} \star \hat{\phi}\right) \\
&\quad - \int \frac{\mu^2}{4!} \hat{\phi}^+ \star \hat{\phi} \epsilon_{abcd}\, e^a \wedge_\star e^b \wedge_\star e^c \wedge_\star e^d \\
&= \int d^4x \sqrt{-g} \star \left(g^{\mu\nu} \star D_\mu \hat{\phi}^+ \star D_\nu \hat{\phi} - \mu^2 \hat{\phi}^+ \star \hat{\phi}\right)
\end{aligned}
$$

After expanding action and varying with respect to Φ^+ EOM is

$$g^{\mu\nu}\left(D_\mu D_\nu\phi - \Gamma^\lambda_{\mu\nu}D_\lambda\phi\right) - \frac{1}{4}\theta^{\alpha\beta}g^{\mu\nu}\Big(D_\mu(F_{\alpha\beta}D_\nu\phi) - \Gamma^\lambda_{\mu\nu}F_{\alpha\beta}D_\lambda\phi$$
$$-2D_\mu(F_{\alpha\nu}D_\beta\phi) + 2\Gamma^\lambda_{\mu\nu}F_{\alpha\lambda}D_\beta\phi - 2D_\beta(F_{\alpha\mu}D_\nu\phi)\Big) = 0$$

Scalar field in the Reissner–Nordström background

RN metric tensor is

$$g_{\mu\nu} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & -\frac{1}{f} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{bmatrix}$$

with $f = 1 - \frac{2MG}{r} + \frac{Q^2 G}{r^2}$ which gives two horizons (r_+ and r_-)
Q-charge of RN BH
M-mass of RN BH
Non-zero components of gauge fields are $A_0 = -\frac{qQ}{r}$ i.e. $F_{r0} = \frac{qQ}{r^2}$
q-charge of scalar field

EOM for scalar field in RN space-time

$$\Big(\frac{1}{f}\partial_t^2 - \Delta + (1-f)\partial_r^2 + \frac{2MG}{r^2}\partial_r + 2iqQ\frac{1}{rf}\partial_t - \frac{q^2Q^2}{r^2f}\Big)\phi$$
$$+\frac{aqQ}{r^3}\Big((\frac{MG}{r} - \frac{GQ^2}{r^2})\partial_\varphi + rf\partial_r\partial_\varphi\Big)\phi = 0$$

where a is $\theta^{t\varphi}$

Assuming ansatz $\phi_{lm}(t,r,\theta,\varphi) = R_{lm}(r)e^{-i\omega t}Y_l^m(\theta,\varphi)$ we got equation for radial part

$$fR''_{lm} + \frac{2}{r}(1-\frac{MG}{r})R'_{lm} - \Big(\frac{l(l+1)}{r^2} - \frac{1}{f}(\omega - \frac{qQ}{r})^2\Big)R_{lm}$$
$$-ima\frac{qQ}{r^3}\Big((\frac{MG}{r} - \frac{GQ^2}{r^2})R_{lm} + rfR'_{lm}\Big) = 0 \qquad (1)$$

NC QNM solutions

QNM

-Detection of the gravitational waves can help better understanding of structure of space-time

-Dominant stage of the perturbed BH are dumped oscillations of the geometry or matter fields (Quasinormal modes) -special solution of equation

-damped oscillations of a perturbed black hole

A set of the boudary condition which leads to this solution is the following: at the horizon, the QNMs are purely incoming, while in the infinity the QNMs are purely outgoing

Continued fraction method

To get form

$$\frac{d^2\psi}{dy^2} + V\psi = 0$$

y must be

$$y = r + \frac{r_+}{r_+ - r_-}\Big(r_+ - iamqQ\Big)\ln(r - r_+) - \frac{r_-}{r_+ - r_-}\Big(r_- - iamqQ\Big)\ln(r - r_-)$$

y is modified Tortoise RN coordinate

Asymptotic form of the eq. (1)

$$R(r) \to \begin{cases} Z^{out} e^{i\Omega y} y^{-1 - i\frac{\omega qQ - \mu^2 M}{\Omega} - amqQ\Omega} & \text{za } y \to \infty \\ \\ Z^{in} e^{-i\left(\omega - \frac{qQ}{r_+}\right)\left(1 + iam\frac{qQ}{r_+}\right)y} & \text{za } y \to -\infty \end{cases}$$

Combining assyptotic forms, we get general solution in the form

$$R(r) = e^{i\Omega r}(r - r_-)^{\epsilon} \sum_{n=0}^{\infty} a_n \left(\frac{r - r_+}{r - r_-}\right)^{n+\delta} \tag{2}$$

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$$\delta = -i\frac{r_+^2}{r_+ - r_-}\left(\omega - \frac{qQ}{r_+}\right), \qquad \epsilon = -1 - iqQ\frac{\omega}{\Omega} + i\frac{r_+ + r_-}{2\Omega}\left(\Omega^2 + \omega^2\right),$$

$$\Omega = \sqrt{\omega^2 - \mu^2}$$

Putting general form (2) to eq (1) we get 6-term recurrence relations for a_n:

$$\begin{aligned}
A_n a_{n+1} + B_n a_n + C_n a_{n-1} + D_n a_{n-2} + E_n a_{n-3} + F_n a_{n-4} = 0, \\
A_3 a_4 + B_3 a_3 + C_3 a_2 + D_3 a_1 + E_3 a_0 = 0, \\
A_2 a_3 + B_2 a_2 + C_2 a_1 + D_2 a_0 = 0, \\
A_1 a_2 + B_1 a_1 + C_1 a_0 = 0, \\
A_0 a_1 + B_0 a_0 = 0,
\end{aligned}$$

$$
\begin{aligned}
A_n &= r_+^{\mathbf{3}}\alpha_n,\\
B_n &= r_+^{\mathbf{3}}\beta_n - iamqQ(r_+ - r_-)r_+(n+\delta) - \frac{1}{2}iamqQ(r_+ + r_-)r_+\\
&\quad + iamqQr_+r_- - 3r_+^{\mathbf{2}}r_-\alpha_{n-\mathbf{1}},\\
C_n &= r_+^{\mathbf{3}}\gamma_n + 3r_+r_-^{\mathbf{2}}\alpha_{n-\mathbf{2}} + iamqQ(r_+ - r_-)(2r_+ + r_-)(n+\delta-1)\\
&\quad - iamqQ(r_+ - r_-)r_+\epsilon \qquad + \frac{1}{2}iamqQ(r_+ + r_-)(2r_+ + r_-)\\
&\quad - 3iamqQr_+r_- + amqQ\Omega(r_+ - r_-)^{\mathbf{2}}r_+ - 3r_+^{\mathbf{2}}r_-\beta_{n-\mathbf{1}}+,\\
D_n &= -r_-^{\mathbf{3}}\alpha_{n-\mathbf{3}} + 3r_+r_-^{\mathbf{2}}\beta_{n-\mathbf{2}} - 3r_+^{\mathbf{2}}r_-\gamma_{n-\mathbf{1}} + iamqQ(r_+^{\mathbf{2}} - r_-^{\mathbf{2}})\epsilon + 3iamqQr_+r_-\\
&\quad - amqQ\Omega(r_+ - r_-)^{\mathbf{2}}r_- - iamqQ(r_+ - r_-)(r_+ + 2r_-)(n+\delta-2)\\
&\quad - \frac{1}{2}iamqQ(r_+ + r_-)(r_+ + 2r_-),\\
E_n &= 3r_+r_-^{\mathbf{2}}\gamma_{n-\mathbf{2}} - r_-^{\mathbf{3}}\beta_{n-\mathbf{3}} + iamqQ(r_+ - r_-)r_-(n+\delta-3)\\
&\quad - iamqQ(r_+ - r_-)r_-\epsilon + \frac{1}{2}iamqQ(r_+ + r_-)r_- iamqQr_+r_-,\\
F_n &= -r_-^{\mathbf{3}}\gamma_{n-\mathbf{3}},
\end{aligned}
$$

$$\alpha_n = (n+1)\Big[n+1-2i\frac{r_+}{r_+-r_-}(\omega r_+ - qQ)\Big],$$

$$\beta_n = \epsilon + (n+\delta)(2\epsilon - 2n - 2\delta) + 2i\Omega(n+\delta)(r_+ - r_-) - l(l+1) - \mu^2 r_-^2$$
$$+ \frac{2\omega r_-^2}{r_+ - r_-}(\omega r_+ - qQ) - \frac{2r_-^2}{(r_+ - r_-)^2}(\omega r_+ - qQ)^2 + 4\omega r_-(\omega r_+ - qQ)$$
$$- \frac{2r_-}{r_+ - r_-}(\omega r_+ - qQ)^2 + (r_+ - r_-)\Big[i\Omega + 2\omega(\omega r_+ - qQ) - \mu^2(r_+ + r_-)\Big],$$

$$\gamma_n = \epsilon^2 + (n+\delta-1)(n+\delta-1-2\epsilon) + \Big(\omega r_- - \frac{r_-}{r_+ - r_-}(\omega r_+ - qQ)\Big)^2$$

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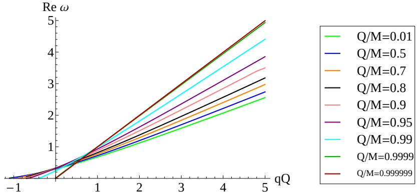
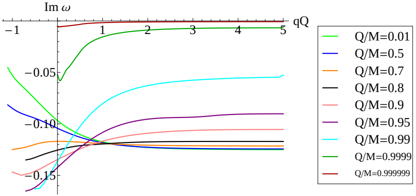
- 3-term relation

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0,$$
$$\alpha_0 a_1 + \beta_0 a_0 = 0$$

gives following equation

$$0 = \beta_0 - \cfrac{\alpha_0\gamma_1}{\beta_1 - \cfrac{\alpha_1\gamma_2}{\beta_2 - \cfrac{\alpha_2\gamma_3}{\beta_3 - \cdots \cfrac{\alpha_n\gamma_{n+1}}{\beta_{n+1} - \cdots}}}}$$


Re ω
qQ
Q/M=0.01
Q/M=0.5
Q/M=0.7
Q/M=0.8
Q/M=0.9
Q/M=0.95
Q/M=0.99
Q/M=0.9999
Q/M=0.999999


Im ω
qQ
Q/M=0.01
Q/M=0.5
Q/M=0.7
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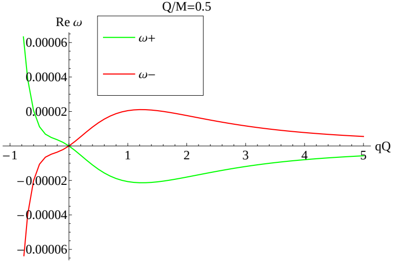
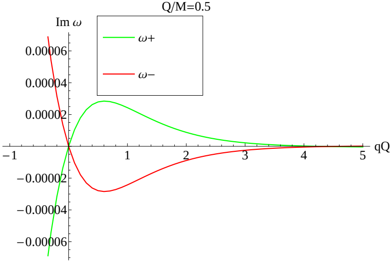

Q/M=0.5
Re ω
ω+
ω−
qQ


Q/M=0.5
Im ω
ω+
ω−
qQ

Outlook

- We constructed Angular twist which induces angular noncommutativity
- Angular NC scalar and vector gauge theory is constructed
- EOM is solved with QNM boundary conditions for scalar field coupled to RN geometry
- But this is toy model!
- Plan for future is to calculate gravitational QNMs and to compare it with results from LIGO, VIRGO, LISA...
- Plan to apply modified momentum conservation law to some measurable process in SM