Global Properties of Warped Solutions in General Relativity with Electromagnetic field and Cosmological Constant

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$(\mathbb{M}, g)$ - space-time = four dimensional manifold with Lorentzian signature metric

What is the problem to be solved in a gravity theory ?

Global solution is a pair $(\mathbb{M}, g)$ where $\mathbb{M}$ is a manifold and g is a Lorentzian signature metric such that two conditions are fulfilled:

1) metric g satisfies Einstein's equations,
2) manifold $\mathbb{M}$ is maximally extended along extremals (geodesics)
   = any extremal can be either continued to infinite value of the canonical parameter or it ends up at a singular point at a finite value of the canonical parameter

The problem is very hard to solve:

1) solution of Einstein's equations in some coordinate system
2) solution of equations for geodesics
3) analysis of geodesics for completeness
4) extension of a manifold

Warped product metric in general relativity

4D space-time is a product of two surfaces: $\mathbb{M} = \mathbb{U} \times \mathbb{V}$

$$x^i = \left\{ x^\alpha, y^\mu \right\} \in \mathbb{M}, \quad i, j, \ldots = 0,1,2,3$$ - coordinates on 4D space-time

$$x^\alpha \in \mathbb{U}, \quad \alpha, \beta, \ldots = 0,1$$ - coordinates on a Lorentzian surface

$$y^\mu \in \mathbb{V}, \quad \mu, \nu, \ldots = 2,3$$ - coordinates on a Riemannian surface

$$\hat{g}_{ij} = \begin{pmatrix} k(y) g_{\alpha\beta}(x) & 0 \\ 0 & m(x) h_{\mu\nu}(y) \end{pmatrix}$$ - 4D metric

$g_{\alpha\beta}(x),\ m(x)$ - 2D metric and a scalar (dilaton) field on $\mathbb{U}$

$h_{\mu\nu}(y),\ k(y)$ - 2D metric and a scalar (dilaton) field on $\mathbb{V}$

No symmetry assumptions on 4D metric

Solution for electromagnetic field

The action
$$S := \int d\hat{x}\sqrt{|\hat{g}|}\left(\hat{R} - 2\Lambda - \frac{1}{4}\hat{F}^2\right)$$

$d\hat{x} := dx^0 dx^1 dx^2 dx^3$ $\qquad$ $\hat{g} := \det \hat{g}_{ij}$ - determinant of the metric

$\hat{R}$ - scalar curvature for metric $\hat{g}_{ij}$ $\qquad$ Λ - cosmological constant

$\hat{F}_{ij} := \partial_i \hat{A}_j - \partial_j \hat{A}_i$ - electromagnetic field strength $\qquad$ $\hat{F}^2 := \hat{F}_{ij}\hat{F}^{ij}$

Equations of motion for electromagnetic field
$$\partial_j\left(\sqrt{|\hat{g}|}\hat{F}^{ji}\right) = 0$$

$$\hat{g} = k^2 m^2 gh, \qquad g := \det g_{\alpha\beta}, \qquad h := \det h_{\mu\nu}$$

Ansatz for electromagnetic field $\hat{A}_i := \left(A_\alpha(x), A_\mu(y)\right)$ $\qquad$ $\hat{F}_{ij} = \begin{pmatrix} F_{\alpha\beta} & 0 \\ 0 & F_{\mu\nu} \end{pmatrix}$

$$F_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha, \qquad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$i = \alpha: \quad F_{\alpha\beta} = \frac{2Q}{|m|}\varepsilon_{\alpha\beta}, \quad Q = \text{const}$$
$$i = \mu: \quad F_{\mu\nu} = \frac{2P}{|k|}\varepsilon_{\mu\nu}, \quad P = \text{const}$$

- a general solution

Warped solutions of Einstein's equations

4D Einstein's equations

$$\hat{R}_{ij} - \frac{1}{2}\hat{g}_{ij}\hat{R} + \hat{g}_{ij}\Lambda = -\frac{1}{2}\hat{T}_{ij}$$

$$\hat{T}_{ij} := -\hat{F}_{ik}\hat{F}_j{}^k + \frac{1}{4}\hat{F}^2$$

$$\hat{T}_{ij} = \begin{pmatrix} \hat{T}_{\alpha\beta} & 0 \\ 0 & \hat{T}_{\mu\nu} \end{pmatrix} \qquad \hat{T}_{\alpha\beta} = \frac{2g_{\alpha\beta}}{km^2}\left(Q^2 + P^2\right), \quad \hat{T}_{\mu\nu} = -\frac{2h_{\mu\nu}}{k^2 m}\left(Q^2 + P^2\right)$$

For simplicity we put $P = 0$

Warped solutions of Einstein's equations

4D Einstein's equations $$\hat{R}_{ij} - \frac{1}{2}\hat{g}_{ij}\hat{R} + \hat{g}_{ij}\Lambda = -\frac{1}{2}\hat{T}_{ij}$$

$$R_{\alpha\beta} + \frac{\nabla_\alpha \nabla_\beta m}{m} - \frac{\nabla_\alpha m \nabla_\beta m}{2m^2} + g_{\alpha\beta}\left(\frac{\nabla^2 k}{2m} - k\Lambda + \frac{Q^2}{m^2 k}\right) = 0,$$

$$R_{\mu\nu} + \frac{\nabla_\mu \nabla_\nu k}{k} - \frac{\nabla_\mu k \nabla_\nu k}{2k^2} + h_{\mu\nu}\left(\frac{\nabla^2 m}{2k} - m\Lambda - \frac{Q^2}{k^2 m}\right) = 0,$$

$$\frac{\nabla_\alpha m \nabla_\mu k}{mk} = 0.$$ ⇒ Three cases:

A: $k = \text{const}, \quad m = \text{const},$

B: $k = \text{const}, \quad \nabla_\alpha m \neq 0,$

C: $\nabla_\mu k \neq 0, \quad m = \text{const}.$

Symmetry of 4D metric is the consequence of the equations of motion

- Spontaneous symmetry emergence

Constant curvature surfaces

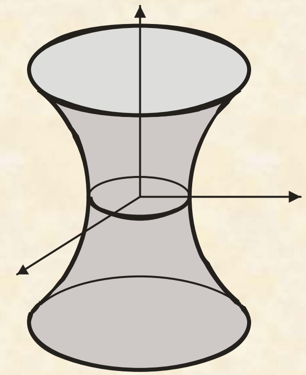
$R^{(2)} = -2K = \text{const}$

$ds^2 = dt^2 - dx^2 - dy^2$ - 3D Minkowskian space-time $\mathbb{R}^{1,2}$

$t^2 - x^2 - y^2 = -1$ - one-sheeted hyperboloid

Lorentzian metric $\text{sign}\, g_{\alpha\beta} = (+-)$

$\mathbb{L}^2 \quad K = -1$

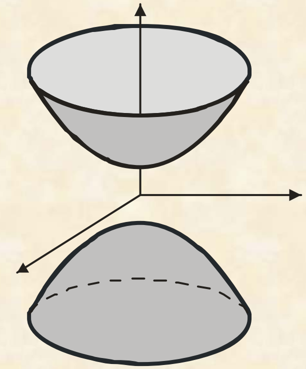


$ds^2 = -dt^2 + dx^2 + dy^2$ - 3D Minkowskian space-time $\mathbb{R}^{1,2}$

$-t^2 + x^2 + y^2 = -1$ - two-sheeted hyperboloid (two copies of Lobachevsky plane)

Euclidean metric $\text{sign}\, g_{\alpha\beta} = (++)$

$\mathbb{H}^2 \quad K = -1$

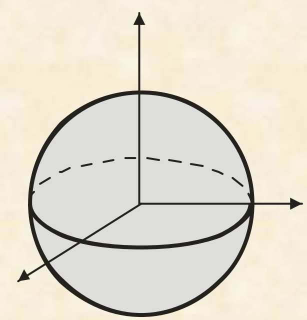


$ds^2 = dx^2 + dy^2 + dz^2$ - 3D Euclidean space-time $\mathbb{R}^3$

$x^2 + y^2 + z^2 = 1$ - sphere

Euclidean metric $\text{sign}\, g_{\alpha\beta} = (++)$

$\mathbb{S}^2 \quad K = 1$



For zero curvature we have either Euclidean $\mathbb{R}^2$ or Minkowskian $\mathbb{R}^{1,1}$ plane

Solutions with constant curvature surfaces (Case A)

Rescaling of coordinates: $k=1, \quad m=-1$

$$ds^2 = \frac{dt^2 - dx^2}{\left[1 + \frac{K^g}{4}\left(t^2 - x^2\right)\right]^2} - \frac{dy^2 + dz^2}{\left[1 + \frac{K^h}{4}\left(y^2 + z^2\right)\right]^2}$$

$$K^g = Q^2 - \Lambda, \quad K^h = Q^2 + \Lambda$$

Four essentially different cases:

$$\Lambda < -Q^2 : \quad K^g > 0, \quad K^h < 0, \qquad \mathbb{M} = \mathbb{L}^2 \times \mathbb{H}^2$$

$$\Lambda = -Q^2 : \quad K^g > 0, \quad K^h = 0, \qquad \mathbb{M} = \mathbb{L}^2 \times \mathbb{R}^2$$

$$-Q^2 < \Lambda < Q^2 : \quad K^g > 0, \quad K^h > 0, \qquad \mathbb{M} = \mathbb{L}^2 \times \mathbb{S}^2$$

$$\Lambda = Q^2 : \quad K^g = 0, \quad K^h > 0, \qquad \mathbb{M} = \mathbb{R}^{1,1} \times \mathbb{S}^2$$

$$\Lambda > Q^2 : \quad K^g < 0, \quad K^h > 0, \qquad \mathbb{M} = \mathbb{L}^2 \times \mathbb{S}^2$$

Each metric has 6 Killing vectors

Symmetry of 4D metric is the consequence of the equations of motion
= Spontaneous symmetry emergence

Spatially symmetric solutions (Case B) $k=1$

The full system of equations:

$$\nabla_\alpha \nabla_\beta m - \frac{\nabla_\alpha m \nabla_\beta m}{2m} - \frac{1}{2} g_{\alpha\beta} \left[\nabla^2 m - \frac{(\nabla m)^2}{2m} \right] = 0,$$

$$R^h + \nabla^2 m - 2m\Lambda - \frac{2Q^2}{m} = 0, \quad \Rightarrow \quad R^h(y),\ m(x)$$

$$R^h = -2K^h = \text{const}$$

$$R^g + \frac{\nabla^2 m}{m} - \frac{(\nabla m)^2}{2m^2} - 2\Lambda + \frac{2Q^2}{m^2} = 0.$$

The space-time:

$\mathbb{M} = \mathbb{U} \times \mathbb{S}^2,\ K^h = 1$

$\mathbb{M} = \mathbb{U} \times \mathbb{R}^2,\ K^h = 0$

$\mathbb{M} = \mathbb{U} \times \mathbb{L}^2,\ K^h = -1$

The symmetry group:

$\mathbb{SO}(3)$

$\mathbb{IO}(2)$

$\mathbb{SO}(1,2)$

Symmetry of 4D metric is the consequence of the equations of motion
= Spontaneous symmetry emergence

Spacially symmetric solutions (Case B)

A general solution in Schwarzschild coordinates

$$ds^2 = \left(K^h - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3} \right) dt^2 - \frac{dr^2}{K^h - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}} - r^2 d\Omega_h$$

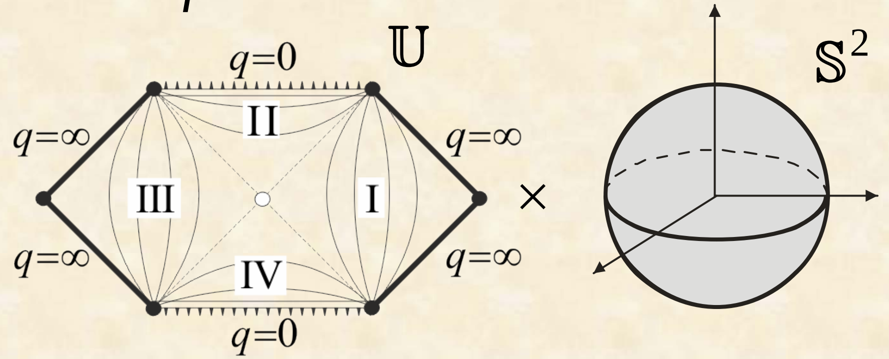
$K^h = \pm 1, 0$ $\quad$ M - arbitrary constant of integration

The Schwarzschild solution $\quad K^h = 1, \quad Q = 0, \quad M > 0, \quad \Lambda = 0$

$$ds^2 = \left(1 - \frac{2M}{r} \right) dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2 \left(d\vartheta^2 + \sin^2\vartheta d\varphi \right)$$

The space-time: $\mathbb{M} = \mathbb{U} \times \mathbb{S}^2$

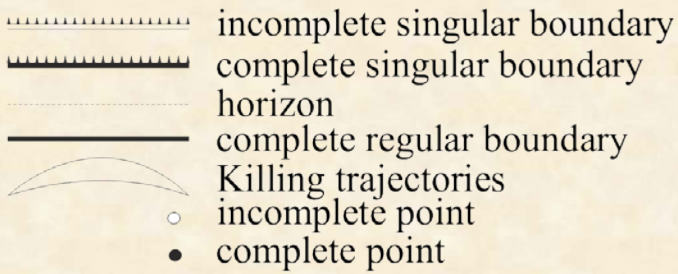
The symmetry group: $\mathbb{SO}(3)$

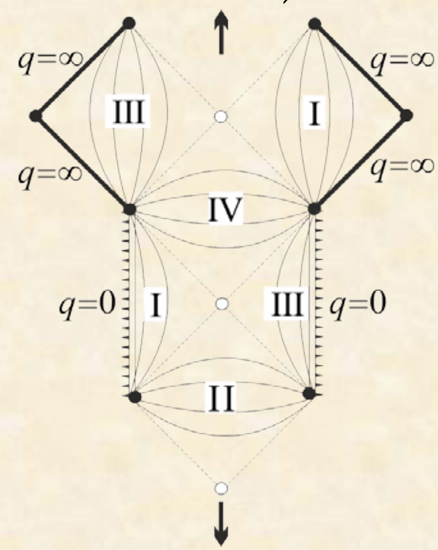


Spacially symmetric solutions (Case B)

The Reissner-Nordstròm solution $Q < M, \quad \Lambda = 0$

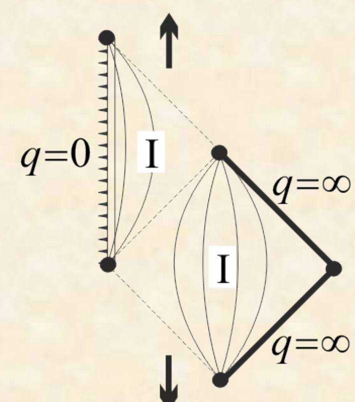
$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 - \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} - r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi\right)$$





Extremal black hole $Q = M, \quad \Lambda = 0$

$$ds^2 = \left(1 - \frac{M}{r}\right)^2 dt^2 - \frac{dr^2}{\left(1 - \frac{M}{r}\right)^2} - r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi\right)$$



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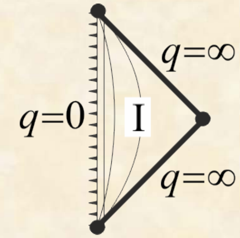
The space-time: $\mathbb{M} = \mathbb{U} \times \mathbb{S}^2$

The symmetry group: $\mathbb{SO}(3)$

Spacially symmetric solutions (Case B)

Naked singularity $Q > M, \quad \Lambda = 0$



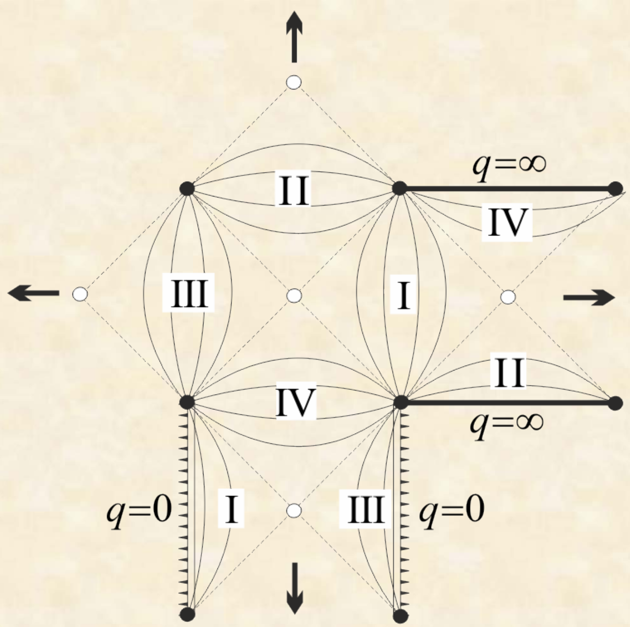
$$g_{00} = 1 - \frac{2M}{q} + \frac{Q^2}{q^2} - \frac{\Lambda q^2}{3} =: \frac{\varphi(q) + 3Q^2}{3q^2}$$

$$\varphi(q) := -q(\Lambda q^3 - 3q + 6M) = 0 \quad \Rightarrow \quad q_{2,1} = -\sqrt{\frac{2}{\Lambda}} \cos\left(\frac{\alpha}{3} \pm \frac{\pi}{3}\right), \quad q_3 = \sqrt{\frac{2}{\Lambda}} \cos\frac{\alpha}{3}$$

$$\cos\alpha := -3M\sqrt{\frac{\Lambda}{2}}$$

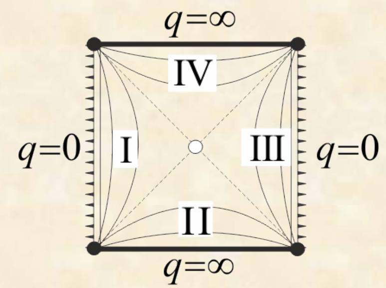
Three horizons

$$-\varphi_3 < 3Q^2 < -\varphi_2$$

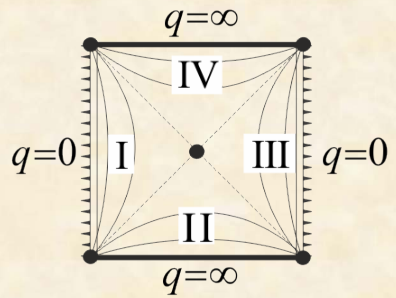


Spacially symmetric solutions (Case B)

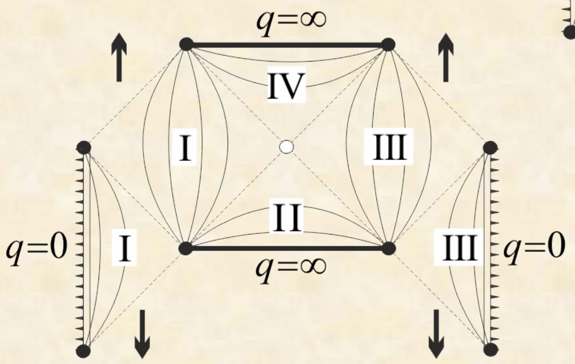
One simple horizon and timelike singularity



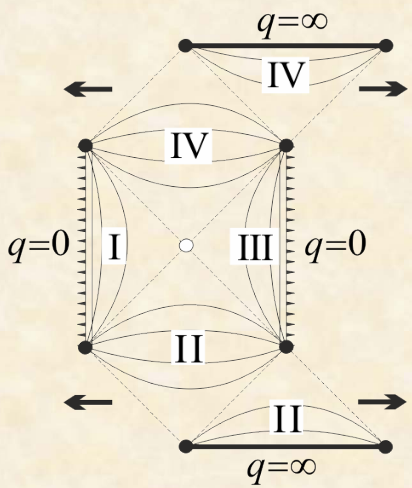
Triple horizon



Two horizons with double minima

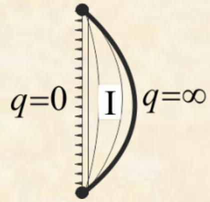


Two horizons with double maximum

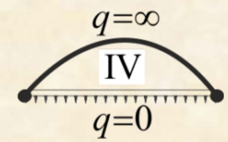


Spacially symmetric solutions (Case B)

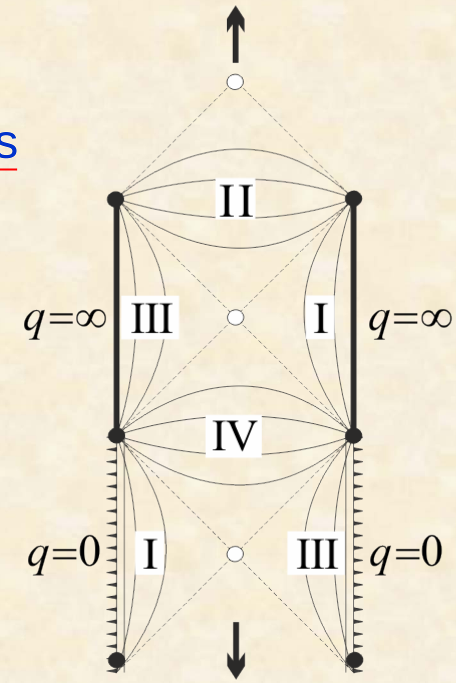
Timelike singularity



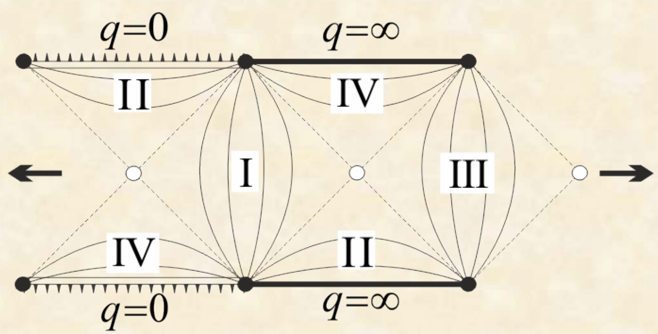
Spacelike singularity



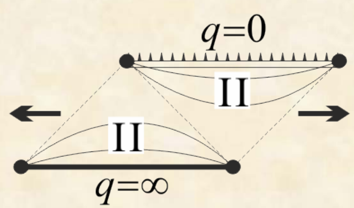
Timelike singularity and two horizons



Spacelike singularity with two horizons

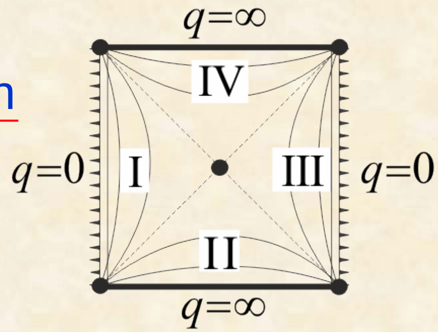


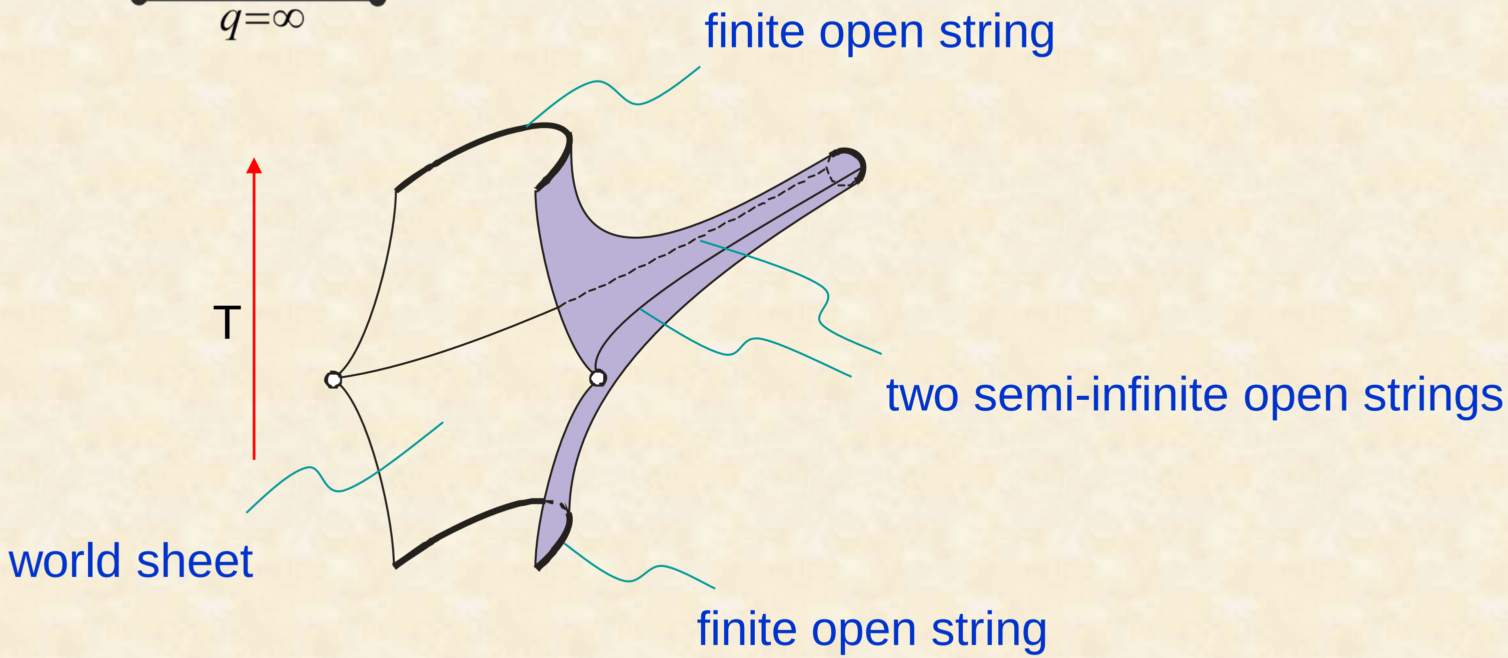
Spacelike singularity with double horizon



Changing topology of space in time

Triple horizon





Conclusion

1) All global warped product solutions of General relativity with electromagnetic field and cosmological constant are found and classified in cases A and B.

2) Totally we get 37 topologically different solutions.

3) There is a solution describing changing of topology of space in time.