# A Safe Beginning for the Universe?

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K.S.S., Phys.Rev. D16 (1977) 953
E. S. Fradkin and A. A. Tseytlin, Nucl. Phys. B201 (1982) 469
I. G. Avramidi and A. O. Barvinsky, Phys. Lett. 159B (1985) 269
M. Niedermaier, PRL 103, 101303 (2009) & Nucl. Phys. B 833 (2010) 226

J-L. Lehners & K.S.S., arXiv 1909.01169

# Renormalizable Gravity

One-loop quantum corrections to General relativity in 4-dimensional spacetime produce ultraviolet divergences of curvature-squared structure.

G. 't Hooft and M. Veltman, *Ann. Inst. Henri Poincaré* **20**, 69 (1974)

Inclusion of $\int d^4x\sqrt{-g}(-\alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \beta R^2)$ terms ab initio into the gravitational action leads to a renormalizable $D = 4$ theory, but at the eventual price of a loss of *unitarity* owing to the ghost modes arising from the $\alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ term, where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor. K.S.S., *Phys. Rev.* **D16**, 953 (1977)
[In $D = 4$ spacetime dimensions, this (Weyl)$^2$ term is equivalent, up to a topological total derivative *via* the Gauss-Bonnet theorem, to the combination $-\alpha(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2)$].

The "classical" gravitational action from which we will start is thus

$$I = \int d^4x \sqrt{-g}(\gamma R - \Lambda - \alpha C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \beta R^2) \, .$$

By expanding this action about flat space ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ for $\Lambda = 0$) and separating the dynamics into the separate spins, one deduces the particle content of the flat-space linearized theory:

*positive-energy* massless spin-two,

*negative-energy* massive spin-two with $m_2^2 = \gamma/(2\alpha)$ and

*positive-energy* massive spin-zero with $m_0^2 = \gamma/(6\beta)$ .

K.S.S., Gen.Rel.Grav. 9 (1978) 353

Despite the apparent ultra-high energy nonphysical behavior, quadratic-curvature gravities have been explored in a variety of contexts:

- *Cosmology:* Starobinsky's model for inflation was based on a $\int d^4x\sqrt{-g}(M_{\rm Pl}^2 R + \beta R^2)$ model.

  A.A. Starobinsky 1980; Mukhanov & Chibisov 1981

  This early model has been quoted as a good fit to CMB fluctuation data from the Planck satellite.

  J. Martin, C. Ringeval and V. Vennin, arXiv:1303.3787

  Note, however, a curious feature of the Starobinsky inflation model: in order for it to work, the $R^2$ coefficient must be very *large*, $\beta \sim 10^{10}$, giving $m_0 \sim 10^{-5} M_{\rm Pl}$. How does such a large coefficient arise?
- The *asymptotic safety scenario* considers a non-Gaussian renormalization-group fixed point for dimensionless versions of Newton's constant and the cosmological constant

  S. Weinberg 1976, M. Reuter 1996, M. Niedermaier 2009 &10

# Asymptotic Safety

One approach to asymptotic safety focuses on the renormalizable set of operators in Einstein-plus-quadratic-curvature gravity and argues that the existence of a non-Gaussian fixed point for dimensionless versions of Newton's constant and the cosmological constant can be determined directly from perturbation theory.

Niedermaier 2009 &10

A key feature of the Einstein-plus-quadratic-curvature system is that the $(\mathrm{Weyl})^2$ term is *asymptotically free* in the sense that for $\alpha = 1/g_2^2$, one finds $g_2 \to 0$ at large momenta, as in Yang-Mills theory. E.S. Fradkin & A.A. Tseytlin 1981, 1982; I.G Avramidy & A.O. Barvinsky 1985 So the interactions permitting decay into the negative-energy states turn off as one approaches the regime where this becomes kinematically possible.

This is also true for the $R^2$ term, although in the large-momentum asymptotic limit, the $\beta$ coefficient becomes negative, giving a tachyonic $m_0^2$ at ultra-high scales. I.G Avramidy & A.O. Barvinsky 1985

# Renormalization group equations

Niedermaier's approach to asymptotic safety is based upon one-loop renormalization group flows.

Let $\gamma = \mu^2/\tilde{g}$ and $\Lambda = \mu^4 \lambda$ for the ordinary Einstein-Hilbert and cosmological terms and also $\alpha = 1/(2\sigma)$ and $\beta = \omega/(3\sigma)$ for the quadratic curvature terms. The four coefficients $\tilde{g}\,, \lambda\,, \sigma$ and $\omega$ are all dimensionless.

The parameter $\mu$ is the renormalization scale, so at a low-energy reference scale one may take $\gamma_0 = \mu_0^2/\tilde{g}(\mu_0) = M_{\rm Pl}^2$, *i.e.* $\mu_0 = \sqrt{\tilde{g}}\, M_{\rm Pl}$, where $M_{\rm Pl} = \sqrt{16\pi G}$.

The renormalization group equations are then

$$\mu\frac{d}{d\mu}\tilde{g} = f_{\tilde{g}}(\tilde{g},\lambda,\sigma,\omega) \qquad \mu\tfrac{d}{d\mu}\lambda = f_{\lambda}(\tilde{g},\lambda,\sigma,\omega)$$

$$\mu\frac{d}{d\mu}\sigma = -\frac{133}{160\pi^2}\sigma^2 \qquad \mu\tfrac{d}{d\mu}\omega = -\tfrac{25+1098\omega+200\omega^2}{960\pi^2}\sigma$$

where $f_{\tilde{g}}$ and $f_{\lambda}$ are certain functions of all the couplings.

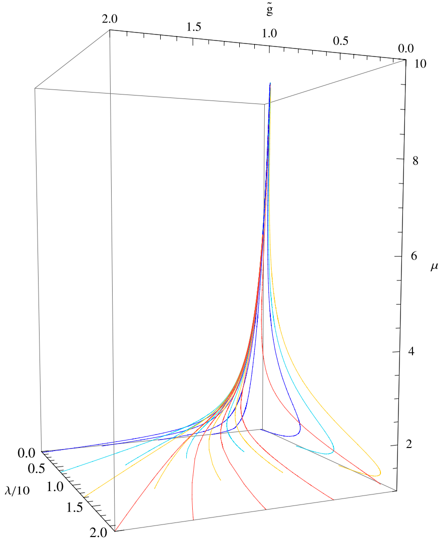
Niedermaier's result was that these dimensionless couplings tend as $\mu \to \infty$ to a fixed point $(\tilde{g}_*\,,\lambda_*\,,\sigma_*\,,\omega_*)$ with

$$\sigma_* = 0 \qquad \omega_* \approx 0.0228\,,$$

with the $\sigma_*$ value reproducing the asymptotic freedom result for the curvature-squared terms in the action, and

$$\tilde{g}_* \approx .42 \qquad \lambda_*/10 \approx 1.18$$

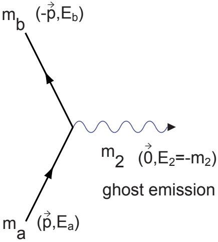
for the pure higher-derivative gravity system without matter. The fixed-point values can change when matter is included.




*Renormalization-group trajectories in coupling-constant space ending on a non-Gaussian fixed point with finite dimensionless $\tilde{g}_*$ and dimensionless cosmological constant $\lambda_*$.*

# When should one be afraid of ghosts?

Simple analogous kinematics with a scalar ghost in its rest frame




Let $m_2 \gg m_a\,,\ m_b$

$$
\begin{aligned}
|\vec{p}\,|^2 &= \frac{1}{4m_2^2}\left(m_2^2-(m_a+m_b)^2(m_2^2-(m_a-m_b)^2\right) \\
&\simeq \frac{m_2^2}{4}-\tfrac{1}{2}(m_a^2+m_b^2)+\cdots
\end{aligned}
$$

requires at least $|\vec{p}\,| \sim m_2/2$ for ghost production

# Catching up to the ghost as it runs

Approaching the fixed point, take $\gamma \sim \frac{\mu^2}{\tilde{g}_*}$ and since $\mu \frac{d\sigma}{d\mu} = -c\sigma^2$ (with $c = \frac{133}{160\pi^2} \sim 0.08$) one has a running ghost $m_2^2(\mu) = \frac{\gamma(\mu)}{2\alpha(\mu)} \sim \frac{\mu^2}{c\tilde{g}_* \ln(\mu/\mu_0)}$.

For the relevant momentum scale, take $\mu \sim |\vec{p}| \sim \frac{1}{2} m_2(\mu)$. Then since $\mu_0 = M_{\rm Pl}\sqrt{\tilde{g}(\mu_0)}$ and taking $\tilde{g}(\mu_0) \sim \tilde{g}_* \sim 0.42$ for $\tilde{g}(\mu)$ slowly moving, one finds

$$\mu \sim \mu_0 \exp(\frac{1}{4c\tilde{g}_*}) \sim e^7 M_{\rm Pl} \sim 10^3 M_{\rm Pl}$$

The main point to take home is that it is hard to catch up with the running spin-two ghost. So, whatever the ultimate fate of the $\int R - \Lambda - C^2 + R^2$ effective theory, it should remain valid and unbothered by ghosts for a very large range of super-Planckian scales.

# Finite Euclidean action

Given the renormalizability, asymptotic safety and at least temporary safety from ghosts, one can now trust the $\int R - \Lambda - C^2 + R^2$ theory up to super-Planckian energies. If one then considers transition amplitudes in the early universe, calculated in the Euclidean formulation as a sum over paths weighted by the action, then one obtains a contribution to this sum only if the action is well-defined and not divergent.

This motivates the following condition, motivated by quantum mechanics: all physical solutions relevant to the early universe should have *finite action*.

We assume that the spatial volume of the universe is finite, approaching zero volume as the time coordinate $t$ tends to zero. Then at fixed (but arbitrarily large) scale $\mu$, we impose the requirement that the time integral in the action does not diverge as $t \to 0$.

# Anisotropies

Now consider a Bianchi type IX metric as a model early-universe metric. This will allow us to investigate the fate of general anisotropies under the finite-action requirement.

$$ds^2_{IX} = -dt^2 + \sum_m \left(\frac{l_m}{2}\right)^2 \sigma_m^2$$

where $\sigma_1 = \sin\psi\, d\theta - \cos\psi \sin\theta\, d\varphi$, $\sigma_2 = \cos\psi\, d\theta + \sin\psi \sin\theta\, d\varphi$, and $\sigma_3 = -d\psi + \cos\theta\, d\varphi$ are differential forms on $\mathcal{S}^3$ with coordinate ranges $0 \leq \psi \leq 4\pi$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$. One can then rescale

$$l_1 = a\, e^{\frac{1}{2}\left(\beta_+ + \sqrt{3}\beta_-\right)}, \quad l_2 = a\, e^{\frac{1}{2}\left(\beta_+ - \sqrt{3}\beta_-\right)}, \quad l_3 = a\, e^{-\beta_+}$$

in which $a$ represents the spatial volume and the $\beta_i$ parametrize the shapes of the spatial slices. When $\beta_- = \beta_+ = 0$ one recovers the isotropic case.

The Einstein-Hilbert action in these coordinates is given by

$$\begin{aligned} S_{EH} &= \int d^4x\sqrt{-g}\,\frac{R}{2} \\ &= 2\pi^2\int dt\,a\left(-3\dot{a}^2+\frac{3}{4}a^2(\dot{\beta}_+^2+\dot{\beta}_-^2)-U(\beta_+,\beta_-)\right) \end{aligned}$$

where the anisotropy parameters evolve subject to the effective potential

$$\begin{aligned} U(\beta_+,\beta_-) = -2\left(e^{2\beta_+}+e^{-\beta_+-\sqrt{3}\beta_-}+e^{-\beta_++\sqrt{3}\beta_-}\right) \\ +\left(e^{-4\beta_+}+e^{2\beta_+-2\sqrt{3}\beta_-}+e^{2\beta_++2\sqrt{3}\beta_-}\right) \end{aligned}$$

Simplifying the on-shell action using the Friedman equation $3H^2=3(\dot{a}/a)^2=\frac{3}{4}\dot{\beta}_+^2+\frac{3}{4}\dot{\beta}_-^2+\frac{1}{a^2}U(\beta_+,\beta_-)$, the on-shell Einstein-Hilbert action becomes

$$S_{EH}^{on-shell}=\int d^4x\sqrt{-g}\,\frac{R}{2}=-4\pi^2\int dt\,a\,U(\beta_+,\beta_-)$$

Now take as an illustrative example a solution with $\beta_- = 0$. The anisotropy potential reduces to $U(\beta_+, 0) = e^{-4\beta_+} - 4e^{-\beta_+}$ and the asymptotic solution is given by

$$a(t) = a_0 t^{1/3} \qquad e^{\beta_+} = b_+ t^{-2/3}$$

As $t \to 0$ the anisotropy diverges, *i.e.* $\beta_+ \to \infty$. The on-shell Einstein-Hilbert action can then be approximated by

$$S_{EH}^{on-shell} = -4\pi^2 \int_{t_0}^{t_1} dt\, a\, U(\beta_+, 0) \approx 16\pi^2 \frac{a_0}{b_+} \int_{t_0}^{t_1} dt\, t$$

where the approximate expression holds for small $t$. This action nonetheless remains finite as $t_0 \to 0$, so one obtains no obstruction to a highly anisotropic solution of this kind in pure Einstein theory.

# Consequences of the curvature-squared terms

The absence of any limitations on the initial anisotropy in pure Einstein theory persists when one considers a more general power-law ansatz

$$a \propto t^s \qquad e^{\sqrt{3}\beta_-} \propto t^m \qquad e^{\beta_+} \propto t^p$$

provided $s \geq 1/3$, which is characteristic of the general chaotic mixmaster solutions that one obtains.

Now consider, however, what happens to the action when the quadratic-curvature terms are included. Requiring $t_0 \to 0$ finiteness of the integrated $(\mathrm{Weyl})^2$ and $R^2$ terms leads to the conditions

$$s \geq \frac{1}{3}\,,\ -\frac{1}{2}(1+s) \leq p \leq \frac{1}{4}(1+s)\,,\ -\frac{1}{2}(1+s) \leq p \pm m \leq 1+s\,.$$

These conditions are only satisfied if $s = 1\,,\ p = 0\,,\ m = 0$, *i.e.* if the anisotropy parameters $\beta_\pm$ become constant near $t = 0$. However, even then one finds that the action still diverges logarithmically unless $\beta_+ = \beta_- = 0$. The finite action requirement thus suppresses the anisotropies altogether at the earliest times.

# Inhomogeneities

In order to investigate inhomogeneities, consider a metric of the Lemaître-Tolman-Bondi class,

$$ds^2 = -dt^2 + \frac{A'^2}{F^2} dr^2 + A^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where now the scale factor $A$ depends both on time and on the radial coordinate $r$, *i.e.* $A = A(t, r)$ and $A' = \partial A/\partial r$.

The function $F(r)$ describes the inhomogeneity in the $r$ coordinate. For simplicity, spherical symmetry is maintained in the remaining spatial directions. When $F(r) = 1$, a field redefinition returns the metric to the standard flat Robertson-Walker form.

As in the discussion of anisotropies, now make an ansatz that the scale factor has a power-law time dependence in the approach to $A = 0$, *i.e.* suppose that $A(t,r) \sim t^s$ near $t = 0$, where $s \geq 0$.

The Einstein-Hilbert action then has two different types of terms: those related to the time evolution of the scale factor, such as $\int_{t_0} dt \frac{\dot{A}^2 A'}{F} \sim \int_{t_0} dt\, t^{3s-2} \sim t_0^{3s-1}$, which require $s \geq 1/3$ for convergence, and those related to the inhomogeneity, such as $\int_{t_0} dt A' F \sim \int_{t_0} dt\, t^s \sim t_0^{s+1}$. The latter are always convergent for $s \geq 1/3 > 0$.

So initial inhomogeneities do not lead to a divergence of the Einstein-Hilbert action and would consequently be acceptable in the context of pure Einstein theory.

Now redo the analysis including the quadratic-curvature terms. Similarly to what happens in the anisotropy analysis, both the integrated $R^2$ and $(\text{Weyl})^2$ terms lead to temporal scalings of the forms

$$\int_{t_0} dt\, t^{3s-4}, \int_{t_0} dt\, t^{s-2}, \int_{t_0} dt\, t^{-s}$$

where the last term arises from inhomogeneity terms like $\int dt \frac{A' F^4}{A^2 F}$. Convergence near $t_0 = 0$ then requires

$$s > 1\,, \quad s < -1$$

where the second requirement arises from the inhomogeneity contributions, and is clearly in conflict with the first requirement. Thus, near $t_0 = 0$ the inhomogeneity must be damped out so that $F(r) \to 1$ and the solution tends to a form equivalent to the standard flat Robinson-Walker metric.

Accordingly, only initially homogeneous early universes are allowed, with a scale factor undergoing an accelerated expansion, $A \sim t^s$, with $s \geq 1$.

# Conclusions

- Renormalizable $\int R - \Lambda - C^2 + R^2$ gravity is asymptotically free for the $(\text{Weyl})^2$ and $R^2$ terms and asymptotically safe for the dimensionless $\tilde{g}$, $\lambda$ coefficients of the Einstein-Hilbert and cosmological terms. This gives an effective theory that one may trust up to energy scales well beyond the Planck scale.
- Imposing the requirement of finite Euclidean action as $t_0 \to 0$ does not give any restrictions on the initial form of the metric in pure Einstein theory, but the inclusion of the $(\text{Weyl})^2$ and $R^2$ terms in the action does give restrictions on the initial metric that require vanishing initial anisotropy and inhomogeneity at $t_0 = 0$.
- These initial conditions for the early universe give a vanishing initial Weyl curvature tensor, thus dynamically deriving Roger Penrose's Weyl curvature hypothesis, which he proposed as an explanation for the low initial entropy of the universe which is required to derive the second law of thermodynamics.

R. Penrose, Ann. N.Y. Acad. Sci. 571 (1989) 249