On the counting tensor model observables as U(N) and O(N) classical invariants

Joseph Ben Geloun

LIPN, Paris Nord

Based on

JHEP **1711**, 092 (2017) and
Ann. Inst. HP D Comb. Phys. Interact. **1**, 77-138 (2014) with S. Ramgoolam (QMUL)

"On the counting O(N) invariants," arXiv:1907.04668 [math-ph],

with R. C. Avohou (Jerusalem U & ICMPA, Benin), N Dub (LIPN)

September 18, 2019
Humboldt Kolleg "Frontiers in Physics: From Electroweak to the Planck Scales"
EISA & Corfu Summer Institute

Outline

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Tensor models

• Extend Matrix models [Ambjørn et al. '90; Sasakura '90, Boulatov '92; Ooguri '92]: Models for quantum gravity/random discrete geometry in any D

• Admit a (new) large 't Hooft N expansion: melonic diagrams (talk by Dario) [Gurau '10 '11; Gurau, Rivasseau '11, Bonzom, Gurau, Riello, Rivasseau '11]

• Extend to Group Field Theory [Oriti '06] for gravity

• Recent development: TM have the same large N limit that the Sachdev-Ye-Kitaev-model (condensed matter model, integrable, test for AdS/CFT correspondence/holography). [Witten '16, Gurau, 16' .....]

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Observables and Correlators in TM

• Key objects for tensor models: interactions/observables $\equiv$ invariants of classical Lie groups, $U(N), O(N)$.

• Observables $\equiv$ contractions of tensor fields.

• Correlators compute in terms of Feynman graphs and their combinatorics.

The calculations heavily relies on diagrammatics and combinatorics.
Do we have an algebraic way of representing the same computations?

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A formulation of QFT in symmetric group language and its representation theory

Why embedding your results in a different formulation?

→ to shed a different light on your results

→ to discover new and genuine effects

→ to bridge theories and therefore discover new correspondences between theories which from the outset look rather different (new bijections between different-looking objects)

At the computational level:

→ to gain confidence when implementing computations by softwares; computations could have been otherwise very difficult to handle by hand. (In particular, for permutation groups, there are quite a lot of resources!)

→ to guide our intuition with computational experiments ...

→ and to ask new questions

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Switching to the symmetric group and its representation theory

$\Rightarrow$ Theoretical Physics

Matrix models: integrable models, 2D gravity, Riemann surfaces, String theory.

$\rightarrow$ Understanding the half-BPS sector of $N = 4$ SYM. [de Mello Koch & Ramgoolam, Rodrigues, Mattioli, Diaz, ...]

$\rightarrow$ Highlighting new correspondences between countings in QFT, Matrix Model, and String theory

$\rightarrow$ Quantum information processing

$\Rightarrow$ Math: Combinatorics, algebra .....

$\Rightarrow$ Linguistic [Kartsaklis, Ramgoolam, Sadrzadeh].

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$\Rightarrow$ Revelation of hidden structures: Combinatorics and algebra

$\rightarrow$ Exact enumeration the observables/invariants

$\rightarrow$ Connection to topological field theory (TFT): a geometrical interpretation of the counting

$\rightarrow$ Generation an algebra of observables with interesting properties (semi-simplicity, orthogonal bases, graduation)

$\rightarrow$ Simplification and discovery new integer sequences [OEIS]

$\rightarrow$ Discover computable sectors for correlators in TM

$\Rightarrow$ Link with Theoretical Computer Science
$\rightarrow$ Computational Complexity Theory

- Counting of tensor invariants relates to the Kronecker coefficient
- recurrent theme for the problem NP vs P.

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Goals

- Invitation to

Goals

- Invitation to TENSOR FUN ZONE

~~EAT.~~ ~~PLAY.~~ ~~DRINK.~~ FUN!

SIMPLY COMPUTE

Outline

Building blocks: Complex and real tensors

• A covariant complex or real tensor $T_{p_1,\dots,p_d}$ (distinguished indices) with transformation rule

$$T^R_{p_1,\dots,p_d} = \sum_{q_k} R^{(1)}_{p_1 q_1} \dots R^{(d)}_{p_d q_d} T_{q_1,\dots,q_d} \,, \qquad R^{(a)} \in U(N_a),\ O(N_a) \tag{1}$$

• Tensor contractions = unitary or orthogonal invariants

$$\begin{aligned} complex: \qquad S^{\rm int}_{\mathbf{b}}(T,\bar T) &= \mathrm{Tr}_{\mathbf{b}}(\bar T \cdot T \dots \bar T \cdot T) \\ real: \qquad S^{\rm int}_{\mathbf{b}}(T) &= \mathrm{Tr}_{\mathbf{b}}(T \cdot T \cdot \dots \cdot T) \end{aligned} \tag{2}$$

• If you are doing QG: T is viewed as a $(d-1)$-simplex. $S^{\rm int}_{\mathbf{b}}$ "is" a gluing of simplexes and represents a d-polytope geometry (Dario's talk)

• In the following, all illustrations are made at fixed rank $d = 3$ but generally extends in any d.

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Unitary invariants

$$S^{\text{int}}_{\mathbf{b}}(T,\bar{T}) = \mathrm{Tr}_{\mathbf{b}}(\bar{T}\cdot T \ldots \bar{T}\cdot T)$$

- Coding unitary invariants: **b** bi-partite colored graphs

- Rank $D=1$,Vectors: $||\phi|| = \sum_a |\phi_a|^2$, 1 invariant.
- Rank $D=2$, Matrices: $\mathrm{Tr}[(M^\dagger M)^n]$, $\forall n \geq 1$, cyclic graphs.

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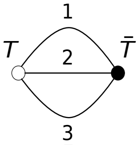

$\mathrm{Tr}_2(\bar{T} T)$

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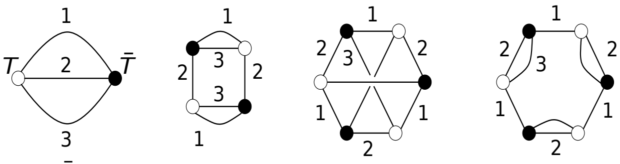

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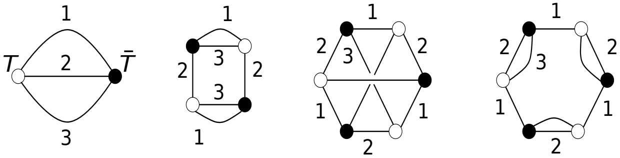

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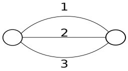
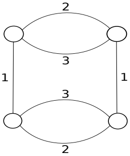
Orthogonal invariants

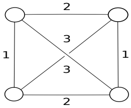
$$S_{\mathbf{b}}^{\text{int}}(T) = \mathrm{Tr}_{\mathbf{b}}(T \cdot T \ldots T \cdot T) \tag{3}$$

- Coding orthogonal invariants **b** colored graphs



$\mathrm{Tr}_2(T^2)$





This one is new!

Tensor correlators

- Gaussian measure

$$d\mu(T,\bar{T}) \equiv \prod_{i_k} dT_{i_1 i_2 \dots i_d}\, d\bar{T}_{i_1 i_2 \dots i_d}\, e^{-\sum_{i_k} T_{i_1 i_2 \dots i_d} \bar{T}_{i_1 i_2 \dots i_d}} \tag{4}$$

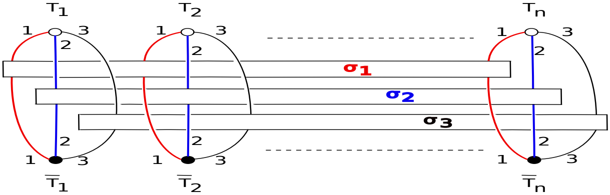
- Correlators:

$$O_{\mathbf{b}}(T,\bar{T}) = \mathrm{Tr}_{\mathbf{b}}(T\cdot\bar{T}\dots T\cdot\bar{T})$$

$$\langle O_{\mathbf{b}}(T,\bar{T})\rangle = \int d\mu(T,\bar{T}) O_{\mathbf{b}}(T,\bar{T})\,. \tag{5}$$

Outline

[BG, Ramgoolam, AIHP D '14] Illustration in rank 3:



- Counting permutation triples $(\sigma_1, \sigma_2, \sigma_3) \in (S_n \times S_n \times S_n)$ up to the equivalence

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\gamma_1 \sigma_1 \gamma_2,\ \gamma_1 \sigma_2 \gamma_2,\ \gamma_1 \sigma_3 \gamma_2), \qquad \gamma_i \in S_n\,. \tag{6}$$

- Counting eléments of the double quotient $\mathrm{Diag}(S_n) \backslash (S_n \times S_n \times S_n) / \mathrm{Diag}(S_n)\,.$

Counting orbits: Apply Burnside's lemma

$$|H_1 \backslash G / H_2| = \frac{1}{|H_1||H_2|} \sum_{h_1 \in H_1} \sum_{h_2 \in H_2} \sum_{g \in G} \delta(h_1 g h_2 g^{-1}) \tag{7}$$

- Number of invariants

$$Z_3(n) = \frac{1}{(n!)^2} \sum_{\sigma_{1,2,3} \in S_n} \sum_{\gamma_1, \gamma_2 \in S_n} \delta(\gamma_1 \sigma_1 \gamma_2^{-1} \sigma_1^{-1}) \delta(\gamma_1 \sigma_2 \gamma_2^{-1} \sigma_2^{-1}) \delta(\gamma_1 \sigma_3 \gamma_2^{-1} \sigma_3^{-1}) \tag{8}$$

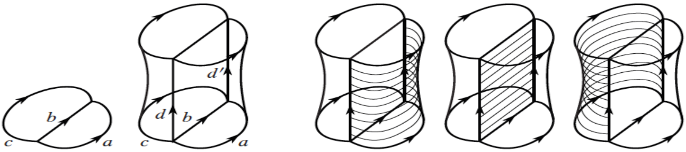
$\rightarrow$ Programming in Gap, and Mathematica [OEIS: A110143 (isomorphism of graph coverings)] Illustration at rank $d = 3$

$$1; 4; 11; 43; 161; 901; 5579; 43206; 378360; 3742738, \ldots \tag{9}$$

Topological Field Theory

$\rightarrow$ TFT_2 on toric lattice

$$Z_3(n) = \frac{1}{(n!)^2} \sum_{\sigma_{1,2,3} \in S_n} \sum_{\gamma_1, \gamma_2 \in S_n} \delta(\gamma_1 \sigma_1 \gamma_2^{-1} \sigma_1^{-1}) \delta(\gamma_1 \sigma_2 \gamma_2^{-1} \sigma_2^{-1}) \delta(\gamma_1 \sigma_3 \gamma_2^{-1} \sigma_3^{-1}) \tag{10}$$



G_3 $\quad$ $G_3 \times S^1$

After some manipulations (gauge fixing one σ_i and introduce another variable), one arrives at

$$Z_3(n) = \frac{1}{n!} \sum_{\tau_0, \tau_1, \tau_2 \in S_n} \sum_{\gamma \in S_n} \delta(\gamma \tau_1 \gamma^{-1} \tau_1^{-1}) \delta(\gamma \tau_2 \gamma^{-1} \tau_2^{-1}) \delta(\gamma \tau_0 \gamma^{-1} \tau_0^{-1}) \delta(\tau_0 \tau_1 \tau_2) \tag{11}$$

3 generators with a single relation, that is the fundamental group of S^2 with 3 punctures.

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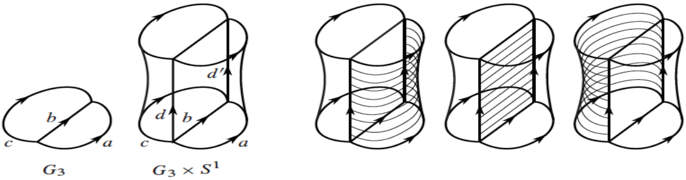
$\rightarrow$ TFT$_2$ on toric lattice

$$Z_3(n) = \frac{1}{(n!)^2} \sum_{\sigma_{1,2,3}\in S_n} \sum_{\gamma_1,\gamma_2\in S_n} \delta(\gamma_1\sigma_1\gamma_2^{-1}\sigma_1^{-1})\delta(\gamma_1\sigma_2\gamma_2^{-1}\sigma_2^{-1})\delta(\gamma_1\sigma_3\gamma_2^{-1}\sigma_3^{-1}) \tag{10}$$



After some manipulations (gauge fixing one σ_i and introduce another variable), one arrives at

$$Z_3(n) \;=\; \frac{1}{n!} \sum_{\tau_0,\tau_1,\tau_2\in S_n} \sum_{\gamma\in S_n} \delta(\gamma\tau_1\gamma^{-1}\tau_1^{-1})\delta(\gamma\tau_2\gamma^{-1}\tau_2^{-1})\delta(\gamma\tau_0\gamma^{-1}\tau_0^{-1})\delta(\tau_0\tau_1\tau_2) \tag{11}$$

3 generators with a single relation, that is the fundamental group of S^2 with 3 punctures.

Representation of the symmetric group: basics

- Irreps of symmetric group S_n are labelled by Young diagrams or $R \vdash n$ partition of n.

$$n = 7, \quad R = (1,2,4) = \ydiagram{4,2,1} \tag{12}$$

- $D^R_{ij}(\sigma) = \langle R, j|\sigma|R, i\rangle$ the real matrix representation of σ in the irrep $R \vdash n$ (dimension $d(R)$)

$$\text{Orthogonality}: \quad \sum_{\sigma \in S_n} D^R_{ij}(\sigma) D^S_{kl}(\sigma) = \frac{n!}{d(R)}\, \delta^{RS}\, \delta_{ik}\delta_{jl}\ ;$$

$$\text{Clebsch} - \text{Gordan}: \quad \sum_{\sigma \in S_n} D^{R_1}_{i_1 j_1}(\sigma) D^{R_2}_{i_2 j_2}(\sigma) D^{R_3}_{i_3 j_3}(\sigma) = \frac{n!}{d(R_3)} \sum_{\tau} C^{R_1,R_2;R_3,\tau}_{i_1,i_2;i_3}\, C^{R_1,R_2;R_3,\tau}_{j_1,j_2;j_3}$$

$$\tau \in [\![1, \mathrm{C}(R_1, R_2, R_3)]\!]$$

Revisiting the counting

- A small calculation

$$
\begin{aligned}
Z_3(n) &= \frac{1}{(n!)^2} \sum_{\sigma_i \in S_n} \sum_{\gamma_1, \gamma_2 \in S_n} \delta(\gamma_1 \sigma_1 \gamma_2^{-1} \sigma_1^{-1}) \delta(\gamma_1 \sigma_2 \gamma_2^{-1} \sigma_2^{-1}) \delta(\gamma_1 \sigma_3 \gamma_2^{-1} \sigma_3^{-1}) \\
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&= \sum_{R_1, R_2, R_3 \vdash n} (\mathrm{C}(R_1, R_2, R_3))^2 \qquad (13)
\end{aligned}
$$

where the symbol

$$
\mathrm{C}(R_1, R_2, R_3) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^{R_1}(\sigma) \chi^{R_2}(\sigma) \chi^{R_3}(\sigma) \qquad (14)
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is the Kronecker coefficient.

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$\rightarrow$ multiplicity of the one-dimensional (trivial) representation in the tensor product $R_1 \otimes R_2 \otimes R_3$.

- Number of invariants $\equiv$ dimension of vector space $\mathcal{K}(n)$?

- Link with Computational Complexity theory:

$\rightarrow$ Finding a combinatorial rule to characterize them in general (Munurghan 1938, Stanley 2000)

What we find:

$$\sum_{R_1, R_2, R_3 \vdash n} (C(R_1, R_2, R_3))^2 = \#Rank - 3 \text{ tensor model observables} \qquad (15)$$

Challenge

Find a proper refinement of that problem to make progress in the above problematic.

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$\mathcal{K}(n)$, the double coset graph algebra

- Group algebra $\mathbb{C}(S_n)$, i.e. an element of which writes $a = \sum_{\sigma \in S_n} \lambda_\sigma \sigma, \;\; \lambda_\sigma \in \mathbb{C}$
- Double coset formulation in $\mathbb{C}(S_n)^{\otimes 3}$: Consider the orbits

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- Define $\mathcal{K}(n) \subset \mathbb{C}(S_n)^{\otimes 3}$ is the vector space over $\mathbb{C}$

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The graph algebra

- Convenient normalization

$$A_{\sigma_1,\sigma_2,\sigma_3} = \frac{1}{(n!)^2} \sum_{\gamma_1,\gamma_2\in S_n} \gamma_1\sigma_1\gamma_2 \otimes \gamma_1\sigma_2\gamma_2 \otimes \gamma_1\sigma_3\gamma_2 \tag{18}$$

- Multiplication

$$A_{\sigma_1,\sigma_2,\sigma_3}A_{\sigma_4,\sigma_5,\sigma_6} = \frac{1}{n!}\sum_{\tau\in S_n} A_{\sigma_1\tau\sigma_4,\,\sigma_2\tau\sigma_5,\,\sigma_3\tau\sigma_6} \tag{19}$$

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$\rightarrow$ The product of $\mathcal{K}(n)$ is associative and admits a unit.
$\rightarrow$ $\mathcal{K}(n)$ is an associative unital subalgebra of $\mathbb{C}(S_n)^{\otimes 3}$ which is semi-simple with the pairing

$$\boldsymbol{\delta}_3(\otimes_{i=1}^{3}\sigma_i; \otimes_{i=1}^{3}\sigma_i') = \prod_{i=1}^{3} \delta(\sigma_i \sigma_i'^{-1}) \tag{20}$$

Wedderburn-Artin theorem explains the sum of squares:

$$\sum_{R_1, R_2, R_3 \vdash n} (C(R_1, R_2, R_3))^2 \tag{21}$$

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$\mathcal{K}(n)$ decomposes in matrix blocks

- Introduce the Fourier basis of $\mathbb{C}(S_n)$

$$Q^R_{ij} = \frac{\kappa_R}{n!} \sum_{\sigma \in S_n} D^R_{ij}(\sigma)\sigma \tag{22}$$

$$\underbrace{\sum_{i_l,j_l,k} C^{R_1,R_2;R_3,\tau}_{i_1,i_2;i_3} C^{R_1,R_2;R_3,\tau'}_{j_1,j_2;j_3}}_{\textit{Make it legs/momentum invariant}} \underbrace{\sum_{\sigma_1,\sigma_2} \rho_L(\sigma_1)\rho_R(\sigma_2)}_{\textit{Make it } L,R \textit{ invariant}} \underbrace{Q^{R_1}_{i_1j_1} \otimes Q^{R_2}_{i_2j_2} \otimes Q^{R_3}_{i_3j_3}}_{\textit{Ordinary base of } C(S_n)^{\otimes 3}} = Q^{R_1,R_2,R_3}_{\tau,\tau'} \tag{23}$$

- The set $\{Q^{R_1,R_2,R_3}_{\tau,\tau'}\}$ forms an orthogonal matrix base of $\mathcal{K}(n)$.

Multiply like matrices $\quad Q^{R,S,T}_{\tau_1,\tau_2} Q^{R',S',T'}_{\tau'_2,\tau_3} = \delta^{RR'}\delta^{SS'}\delta^{TT'}\delta_{\tau_2\tau'_2} Q^{R,S,T}_{\tau_1,\tau_3}$ (24)

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Correlators

At rank $d = 3$, consider the Gaussian model

$$\mathcal{Z} = \int d\Phi d\bar{\Phi} \; e^{-\frac{1}{2}\sum_{i_l} \Phi_{i_1 i_2 i_3} \bar{\Phi}_{i_1 i_2 i_3}} \tag{25}$$

- The Wick theorem

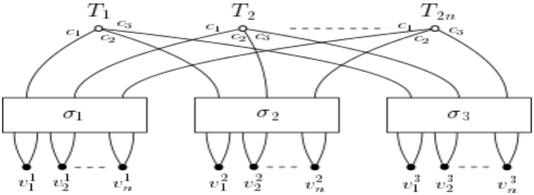
$$\langle \mathcal{O}_{\sigma_1,\sigma_2,\sigma_3} \rangle = \sum_{\mu \in S_n} N^{\mathbf{c}(\mu\sigma_1)+\mathbf{c}(\mu\sigma_2)+\mathbf{c}(\mu\sigma_3)} = N^{\#\mathrm{F}aces} \tag{26}$$

$\mathbf{c}(\alpha)$ is the number of cycles of α.

Outline

Counting orthogonal invariants

Illustration in rank 3:



- Counting permutation triples $(\sigma_1, \sigma_2, \sigma_3) \in (S_{2n} \times S_{2n} \times S_{2n})$ up to the equivalence

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\gamma_1 \sigma_1 \gamma,\ \gamma_2 \sigma_2 \gamma,\ \gamma_3 \sigma_3 \gamma), \qquad \gamma_i \in S_n[S_2], \gamma \in S_n. \quad (27)$$

- Elément of the double quotient $S_n[S_2] \times S_n[S_2] \times S_n[S_2] \backslash (S_{2n} \times S_{2n} \times S_{2n}) / \mathrm{Diag}(S_{2n})$.
- Number of invariants 1; 5; 16; 86; 448; 3580; 34981; 448628; 6854130; 121173330

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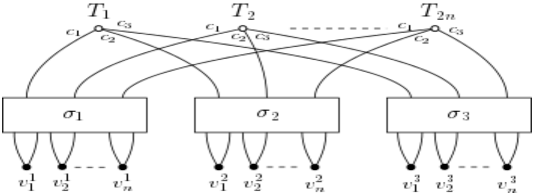

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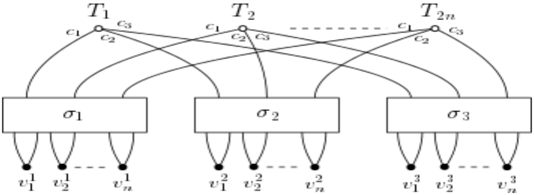

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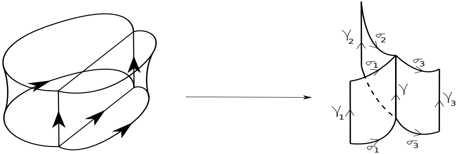
Resulting TFT and algebra

- TFT



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- $\mathcal{K}_o(n)$ is an associative unital semi-simple algebra. By WA, it is decomposable in matrix blocks.
- The dimension in representation:

$$Z_{o;3}(n) = \sum_{R_i \vdash 2n;\; R_i \;\; even} C(R_1, R_2, R_3) \tag{29}$$

- Invariant orthogonal base (NOT Wedderburn-Artin base)

$$Q^{R_1,R_2,R_3;\tau} = \kappa_{\vec{R}} \sum_{p_l\;,q_l} C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3} B^{R_1;\,tr}_{p_1} B^{R_2;\,tr}_{p_2} B^{R_3;\,tr}_{p_3} Q^{R_1}_{p_1 q_1} \otimes Q^{R_2}_{p_2 q_2} \otimes Q^{R_3}_{p_3 q_3} \tag{30}$$

$$B^{R;\,r,\nu_r}_{i;\,m_r} = \langle R, i\,| r, m_r, \nu_r\rangle = \langle r, m_r, \nu_r\,| R, i\rangle \;. \tag{31}$$

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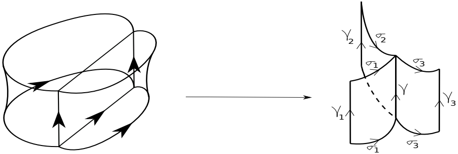

- The algebra $\mathcal{K}_o(n)$

$$\mathcal{K}_o(n) = \mathrm{Span}_{\mathbb{C}}\Big\{ \sum_{\gamma_i \in S_n[S_2];\gamma \in S_{2n}} \gamma_1\sigma_1\gamma \otimes \gamma_2\sigma_2\gamma \otimes \gamma_3\sigma_3\gamma,\ \ \sigma_1, \sigma_2, \sigma_3 \in S_{2n} \Big\} \tag{28}$$

- $\mathcal{K}_o(n)$ is an associative unital semi-simple algebra. By WA, it is decomposable in matrix blocks.
- The dimension in representation:

$$Z_{o;3}(n) = \sum_{R_i \vdash 2n;\ R_i \ \ even} C(R_1, R_2, R_3) \tag{29}$$

- Invariant orthogonal base (NOT Wedderburn-Artin base)

$$Q^{R_1,R_2,R_3;\tau} = \kappa_{\vec{R}} \sum_{p_l\ ,q_l} C^{R_1,R_2;R_3,\tau}_{q_1,q_2;q_3} B^{R_1;\,tr}_{p_1} B^{R_2;\,tr}_{p_2} B^{R_3;\,tr}_{p_3} Q^{R_1}_{p_1 q_1} \otimes Q^{R_2}_{p_2 q_2} \otimes Q^{R_3}_{p_3 q_3} \tag{30}$$

$$B^{R;\,r,\nu_r}_{i;\,m_r} = \langle R, i\,|r, m_r, \nu_r\rangle = \langle r, m_r, \nu_r\,|R, i\rangle\ . \tag{31}$$

Outline

Conclusion

	Unitary TM	Orthogonal TM
Counting observables	(d=3) 1; 4; 11; 43; 161;	(d=3) 1; 5; 16; 86; 448;
TFT_2	branched covers of the 2-sphere	Covers of torus with defects
Algebraic structure	graded unitary semi simple	graded unitary semi simple
Invariant ortho. rep. base	$\surd$	$\surd$
Wedderburn-Artin decomp	$\surd$	X
1-pt and 2-pt correlators	$\surd$	$\surd$

• Possible applications:

- Computable sectors can be found; extract physics needs more work;
- Re-express melons in terms of permutations;
- The success of applying this method on Matrices rests on the connection with strings. Finding first the dual of tensor models, and all the mathematics will ready to be used.

• Application to Theoretical Computer Science: Master refined countings (to tackle challenges like finding a combinatorial interpretation of the Kronecker).

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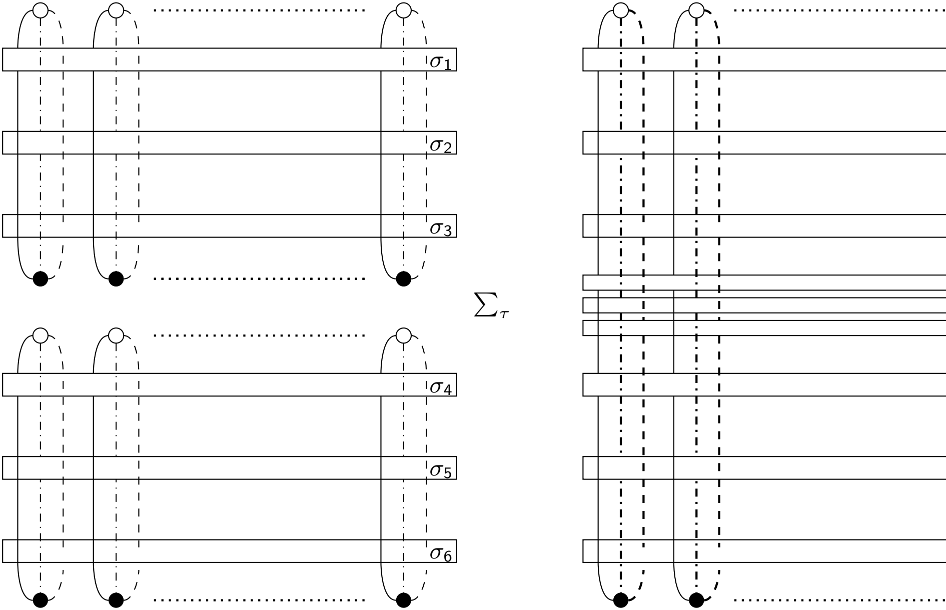
Multiplication of graphs: Complex TM



Correlators

At rank $d = 3$, consider the Gaussian model

$$\mathcal{Z} = \int d\Phi d\bar{\Phi}\; e^{-\frac{1}{2}\sum_{i_l} \Phi_{i_1 i_2 i_3}\bar{\Phi}_{i_1 i_2 i_3}} \tag{33}$$

$$\mathcal{O}_{\sigma_1,\sigma_2,\sigma_3} = \sum_{i_l,j_l,k_l} \Phi_{i_1 j_1 k_1}\Phi_{i_2 j_2 k_2}\dots\Phi_{i_n j_n k_n}\; \bar{\Phi}_{i_{\sigma_1(1)} j_{\sigma_2(1)} k_{\sigma_3(1)}}\bar{\Phi}_{i_{\sigma_1(2)} j_{\sigma_2(2)} k_{\sigma_3(2)}}\dots\bar{\Phi}_{i_{\sigma_1(n)} j_{\sigma_2(n)} k_{\sigma_3(n)}} \tag{34}$$

- The Wick theorem

$$\begin{aligned}
\langle \mathcal{O}_{\sigma_1,\sigma_2,\sigma_3}\rangle &= \frac{1}{\mathcal{Z}}\int d\Phi d\bar{\Phi}\; e^{-\frac{1}{2}\sum_{i,j,k}\Phi_{ijk}\bar{\Phi}_{ijk}}\mathcal{O}_{\sigma_1,\sigma_2,\sigma_3} \\
&= \sum_{i_l,j_l,k_l}\sum_{\mu\in S_n} \delta_{i_1 i_{\mu(\sigma_1(1))}}\delta_{i_2 i_{\mu(\sigma_1(2))}}\dots\delta_{i_n i_{\mu(\sigma_1(n))}} \\
&\times\delta_{j_1 j_{\mu(\sigma_2(1))}}\delta_{j_2 j_{\mu(\sigma_2(2))}}\dots\delta_{j_n j_{\mu(\sigma_2(n))}}\;\delta_{k_1 k_{\mu(\sigma_3(1))}}\delta_{k_2 k_{\mu(\sigma_3(2))}}\dots\delta_{k_n k_{\mu(\sigma_3(n))}} \\
&= \sum_{\mu\in S_n} N^{\mathbf{c}(\mu\sigma_1)+\mathbf{c}(\mu\sigma_2)+\mathbf{c}(\mu\sigma_3)}
\end{aligned} \tag{35}$$

$\mathbf{c}(\alpha)$ is the number of cycles of α.

Gaussian correlators in orthogonal TM

- Gaussian measure

$$d\nu(T) = \prod_{j_l} dT_{j_1 j_2 \dots j_d} \, e^{-O_2(T)} \,, \qquad O_2(T) = \sum_{j_k} (T_{j_1 j_2 \dots j_d})^2 \,. \tag{36}$$

- The Wick theorem for an observable $\mathcal{O}_{\sigma_1,\sigma_2,\sigma_3}$:

$$\langle \mathcal{O}_{\sigma_1,\sigma_2,\sigma_3} \rangle = \sum_{\mu \in S_n} N^{\mathbf{c}(\mu\widetilde{\sigma}_1)+\mathbf{c}(\mu\widetilde{\sigma}_2)+\mathbf{c}(\mu\widetilde{\sigma}_3)} \tag{37}$$

where $\widetilde{\sigma} = \sigma^{-1}\xi\sigma \,,\ \xi = (12)(34)\dots(2n-1,2n)$.