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Simplicial principal bundles and higher connections

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Outline

- Flight through the simplicial geometry
- Reminder of the Atiyah's approach to connections
- Introduction to higher connections

Flight through the simplicial geometry

Basic ingredients of the simplicial (super)geometry are simplicial (super)manifolds and smooth/super simplicial morphisms

Definition: The **simplicial (super)manifold** is a simplicial object in the category (S)Mfd of (super)manifolds, i.e. a contravariant functor $\mathcal{K}\colon \Delta^{op} \to$ (S)Mfd.

Definition: A simplicial (super)manifold is called **Kan simpicial (super)manifold** when it fullfils *Kan property*:

$\forall i \le n \in \mathbb{N}_0$ the natural morphism $\hom(\Delta^n, \mathcal{K}) \to \hom(\Lambda^n_i, \mathcal{K})$ of manifolds is a surjective submersion

Definition: The **smooth/super simplicial morphism** is a natural transformation between simplicial (super)manifolds which is smooth/super

These ingredients combined give a category of **smooth/super simplicial manifolds** s(S)Mfd and Kan simplicial manifolds form its sub-category

Flight through the simplicial geometry

This category generalizes the category of supermanifolds in the sense of an embedding functor $\mathcal{F} : \mathsf{SMfd} \hookrightarrow \mathsf{sSMfd}$ given on objects as $\mathcal{F}(X) = \mathbf{X}_c$, where $\mathbf{X}_c([n]) = X$ for all $[n] \in \Delta^{op}$ and all $X \in \mathsf{SMfd}$.

Thus, we may define and generalize any concept witnessed in ordinary differential geometry – we will focus on the theory of fiber bundles

Flight through the simplicial geometry

Definition: A **simplicial super Lie group** $\mathscr{G}$ is a group object internalized in the category sSMfd.

Definition: A **simplicial principal bundle** is a pair of simplicial maps (π, α), where $\pi : \mathscr{P} \to \mathscr{M}$ is a *fibration* and $\alpha : \mathscr{P} \times \mathscr{G} \to \mathscr{P}$ the *right action* such that the following is satisfied:

- $\pi_n : \mathscr{P}_n \to \mathscr{M}_n$ is the surjective submersion for all $n \in \mathbb{N}_0$
- α_n: $\mathscr{P}_n \times \mathscr{G}_n \to \mathscr{P}_n$ is principal for all $n \in \mathbb{N}_0$
- Categorical quotient $\mathscr{P}/\mathscr{G}$ (equalizer of mappings $\alpha, \pi_1 : \mathscr{P} \times \mathscr{G} \to \mathscr{P}$) is isomorphic to $\mathscr{M}$

Flight through the simplicial geometry

Considerably large class of examples of simplicial principal bundles are **principal twisted cartesian products**

Definition: Let $\mathscr{X}$ be a Kan simplicial super manifold and $\mathscr{G}$ a simplicial Lie super group. Then, a **twisting function** $\tau : \mathscr{X} \to \mathscr{G}$ is a family of super maps $\{\tau^n : \mathscr{X}_n \to \mathscr{G}_{n-1} \,|\, n \in \mathbb{N}\}$ subject to

$$
\begin{aligned}
(\mathsf{f}_i^{n-1} \circ \tau^n)(x) &= \begin{cases} (\tau^{n-1} \circ \mathsf{f}_1^n)(x)\big((\tau^{n-1} \circ \mathsf{f}_0^n)(x)\big)^{-1} & \text{for } i = 0 \\ (\tau^{n-1} \circ \mathsf{f}_{i+1}^n)(x) & \text{else} \end{cases}, \\
(\tau^{n+1} \circ \mathsf{d}_i^n)(x) &= \begin{cases} 1_{\mathscr{G}_n} & \text{for } i = 0 \\ (\mathsf{d}_{i-1}^{n-1} \circ \tau^n)(x) & \text{else} \end{cases}
\end{aligned}
$$

for all $x \in \mathscr{X}_n$.

Flight through the simplicial geometry

Definition: Let $\mathscr{X}$ and $\mathscr{Y}$ be Kan simplicial manifolds and let $\mathscr{G}$ be a simplicial Lie group. Furthermore, let $\lhd : \mathscr{Y} \times \mathscr{G} \to \mathscr{Y}$ be a right-action of $\mathscr{G}$ on $\mathscr{Y}$ and let $\tau : \mathscr{X} \to \mathscr{G}$ be a twisting function. Then, the **twisted cartesian product**, denoted by $\mathscr{Y} \times_\tau \mathscr{X}$, is the simplicial set

$$(\mathscr{Y} \times_\tau \mathscr{X})_n \;:=\; \mathscr{Y}_n \times \mathscr{X}_n$$

with face and degeneracy maps defined by

$$\mathsf{f}_i^n(y,x) \;:=\; \begin{cases} (\mathsf{f}_0^n(y) \lhd \tau(x), \mathsf{f}_0^n(x)) & \text{for } i = 0 \\ (\mathsf{f}_i^n(y), \mathsf{f}_i^n(x)) & \text{else} \end{cases} ,$$

$$\mathsf{d}_i^n(y,x) \;:=\; (\mathsf{d}_i^n(y), \mathsf{d}_i^n(x))$$

for all $x \in \mathscr{X}_n$, $y \in \mathscr{Y}_n$, and $n \in \mathbb{N}_0$.

Flight through the simplicial geometry

Definition: A twisted cartesian product is called **principal** if and only if $\mathscr{Y} = \mathscr{G}$ and the right $\mathscr{G}$-action is just group multiplication.

This class of simplicial principal bundles has also prominent position in another sense – exactly these bundles can be constructed as categorial pullbacks along classifying maps. [May; Goers & Jardine]

This is why these bundles arise naturally in the construction of *1-jets of Kan simplicial manifolds*

Reminder of the Atiyah's approach to connections

In the sense of Ehresmann, connections on principal fiber bundle $P \to M$ can be globally given by horizontal distribution on tangent bundle TP, i.e. by a morphism of vector bundles $H \to TP$, such that $TP \cong H \oplus V$

Sir M. Atiyah proposed a method of encoding the Ehreshmann connection into the sections of the following exact sequence of Lie algebroids

$$0 \to P \times_G \mathfrak{g} \to TP/\mathsf{G} \xrightarrow{\rho} TM \to 0.$$

Reminder of the Atiyah's approach to connections

This exact sequence of Lie algebroids can be derived by differentiating the exact sequence of Lie groupoids

$$0 \to P \times_{\mathsf{G}} \mathsf{G} \to P \times_{\mathsf{G}} P \xrightarrow{r} \mathsf{Pair}(M) \to 0$$

The Lie groupoid $P \times_{\mathsf{G}} P$ is the **Atiyah-Lie groupoid**

The aim is to reproduce the same construction, but in the different category, category **sSMfd**.

Definition: Let $\mathscr{G}$ be a simplicial Lie group and $\mathscr{X}$ a Kan simplicial manifold. Furthermore, let $\mathscr{P} \to \mathscr{X}$ be a simplicial principal $\mathscr{G}$-bundle over $\mathscr{X}$. The **simplicial Atiyah-Lie groupoid** of $\mathscr{P}$, denoted by $\mathrm{At}(\mathscr{P})$, is the simplicial Lie groupoid

$$\mathrm{At}(\mathscr{P}) := \left\{ \cdots \rightrightarrows \begin{array}{c} \mathscr{P}_2 \times_{\mathscr{G}_2} \mathscr{P}_2 \\ \downdownarrows \\ \mathscr{X}_2 \end{array} \rightrightarrows \begin{array}{c} \mathscr{P}_1 \times_{\mathscr{G}_1} \mathscr{P}_1 \\ \downdownarrows \\ \mathscr{X}_1 \end{array} \rightrightarrows \begin{array}{c} \mathscr{P}_0 \times_{\mathscr{G}_0} \mathscr{P}_0 \\ \downdownarrows \\ \mathscr{X}_0 \end{array} \right\}$$

with $\mathscr{G}_n$ acting diagonally. Face and degeneracy maps are defined

$$\begin{aligned} \mathrm{f}_i^n([(p_0, p_1)]) &:= \big([(\mathrm{f}_i^n(p_0), \mathrm{f}_i^n(p_1))]\big) , \\ \mathrm{d}_i^n([(p_0, p_1)]) &:= \big([(\mathrm{d}_i^n(p_0), \mathrm{d}_i^n(p_1))]\big) , \end{aligned}$$

for all $n \in \mathbb{N}_0$, $0 \leq i \leq n$, and $p_{0,1} \in \mathscr{P}_n$.

To obtain a sequence of L_∞ algebroids analogical to original Atiyah sequence, we need some tool to differentiate simplicial groupoids

Theorem [Ševera '06]: Let us have $\mathscr{K}$ Kan simplicial manifold. Then the **1-jet functor** $\mathrm{hom}(\mathrm{N}(Y \times_X Y \rightrightarrows Y), \mathscr{K}) : \mathrm{SSM}^{op} \to \mathrm{Set}$ is a representable presheaf

If we moreover restrict this functor on the subcategory of surjective submersions of type $\mathbb{R}^{0|n} \times X \to X$, the 1-jet functor turns out to be naturally identifiable with $\mathrm{hom}(\mathbb{R}^{0|n}, \mathbb{R}^{0|n})$-equivariant presheaf on SMfd

How does it provide us a method of differentiating simplicial groupoids?

Claim: A representative of a presheaf $\mathrm{hom}(\mathrm{N}(Y \times_X Y \rightrightarrows Y), N\,\mathrm{BG})$ is an NQ manifold corresponding to the Lie algebra of G.

We can generalise this claim to the category of simplicial groupoids since there is a generalization of the classifying functor B, also known as delooping functor $\bar{W} : \mathrm{sGrpd} \to \mathrm{sSMfd}$

Thus we are interested in finding representatives for 1-jets of presheaves of type $\mathrm{hom}(\mathrm{N}(Y \times_X Y \rightrightarrows Y), \bar{W}(\mathscr{G}))$ for $\mathscr{G}$ being the simplicial Atiyah-Lie groupoid

Definition: Let $\mathscr{G}$ be a simplicial Lie group and $\mathscr{P} \to \mathscr{X}$ a simplicial principal $\mathscr{G}$-bundle over a Kan simplicial manifold $\mathscr{X}$. A **higher connection** on $\mathscr{P}$ is a section of the anchor map

$$\rho : J^1(\bar{W}(\mathrm{At}(\mathscr{P}))) \to J^1(\bar{W}(\mathrm{Pair}(\mathscr{X}))).$$

Example: 2-Connection on principal twisted cartesian product

Let us have a crossed module $\partial : \mathsf{H} \to \mathsf{G}$ as a structure 2-group

and a simplicialy constant base X

As a simplicial Atiyah-Lie groupoid we get the nerve of the double groupoid

$$(\mathcal{P} \times_{\mathcal{G}} \mathcal{P} \rightrightarrows \mathcal{X}) = \left(\left(\begin{array}{c} \mathscr{P}_1 \\ \downdownarrows \\ \mathscr{P}_0 \end{array} \times_{\mathcal{G}} \begin{array}{c} \mathscr{P}_1 \\ \downdownarrows \\ \mathscr{P}_0 \end{array} \right) \rightrightarrows \begin{array}{c} X \\ \downdownarrows \\ X \end{array} \right)$$

After applying the delooping functor $\bar{W}$ on simplicial Atiyah-Lie algebroid we get

$$
\begin{aligned}
\overline{\mathsf{W}}_0(\mathsf{At}(\mathscr{P})) &\cong X\ ,\\
\overline{\mathsf{W}}_1(\mathsf{At}(\mathscr{P})) &\cong \mathscr{P}_0 \times_{\mathscr{G}_0} \mathscr{P}_0\ ,\\
\overline{\mathsf{W}}_2(\mathsf{At}(\mathscr{P})) &\cong (\mathscr{P}_1 \times_{\mathscr{G}_1} \mathscr{P}_1)\,{}_{\mathsf{f}^{\mathrm{v}}_0\circ \mathsf{f}^{\mathrm{h}}_0}\!\times_{\mathsf{f}^{\mathrm{v}}_1} (\mathscr{P}_0 \times_{\mathscr{G}_0} \mathscr{P}_0)
\end{aligned}
$$

Introduction to higher connections

Differentiation procedure gives categorified Atiyah sequence

$$
\begin{array}{ccccccccc}
0 & & & & & & TX & & 0 \\
\Big\downdownarrows & \longrightarrow & \mathsf{ad}(\mathcal{P}) & \longrightarrow & \mathsf{At}(\mathcal{P}) := T\mathcal{P}/\mathcal{G} & \xrightarrow{\rho} & \Big\downdownarrows & \longrightarrow & \Big\downdownarrows \\
0 & & & & & & TX & & 0
\end{array}
$$

where

$$
T\mathcal{P} \quad := \quad \begin{array}{c} T\mathscr{P}_1 \\ {\scriptstyle \mathsf{t}_*}\Big\downdownarrows{\scriptstyle \mathsf{s}_*} \\ T\mathscr{P}_0 \end{array} \qquad \text{and} \qquad \mathsf{ad}(\mathcal{P}) \quad := \quad \begin{array}{c} \mathscr{P}_1 \\ \Big\downdownarrows \\ \mathscr{P}_0 \end{array} \times_{\mathcal{G}} \begin{array}{c} \mathfrak{h} \rtimes \mathfrak{g} \\ \Big\downdownarrows \\ \mathfrak{g} \end{array}
$$

Outlook

1. Interpreting morphisms in dual picture $\mathrm{CE}(\mathrm{At}(P)) \to \mathrm{CE}(TX)$ as well known fields
2. Simplicial Atiyah sequences for non-strict cases
3. Computing Atiyah algebroids for higher groups
4. Generalisation the notion of higher connection to all simplicial principal bundles (not only PTCP)

Thank you :-)