# Spectrum of anomalous dimensions in the hypercubic theory

**Jahmall Bersini**

Rudjer Bošković Insitute (IRB), Zagreb, Croatia

O. Antipin and J. Bersini, arXiv:1903.04950

September 9, 2019

# $O(N)$ critical model with cubic anisotropy

## $O(N)$-symmetric scalar critical model with cubic anisotropy

$$\mathcal{S}_{\mathcal{GL}} = \int D^{4-\epsilon}x \left( \frac{(\partial\phi_i)^2}{2} + \frac{g_1}{4!}(\phi_i\phi_i)^2 + \frac{g_2}{4!}\sum_i \phi_i^4 \right) \tag{1}$$

- The $\phi^4$ term explicitly breaks the $O(N)$ symmetry and the action is invariant only under the symmetry group of an $N$-dimensional hypercube $H_N \subset O(N)$.
- This model describes the critical properties of cubic magnets (like Iron) and certain structural phase transitions such as the cubic to tetragonal transition in $SrTiO_3$.
- Present-day results:

  Computation of anomalous dimensions $(\gamma_\phi, \gamma_{m^2})$ and beta functions to six loop order in the $\epsilon$-expansion.

  Non-perturbative investigations via an exact RGE or resorting to conformal bootstrap

# The aim of this work

## Goals

- Find the spectrum of composite operators with an arbitrary number of fields $n$ and no derivatives (no spin).
- Use this information to compute all their anomalous dimensions at the 1-loop order.
- Analyze the main features of the spectrum of anomalous dimensions.

# Fixed Points

In 4-$\epsilon$ dimensions, the renormalized model predicts four fixed points, which at the 1-loop level read:

$$(g_1^G, g_2^G) = (0,0)\,, \qquad (g_1^I, g_2^I) = (4\pi)^2\left(0, \frac{\epsilon}{3}\right)\,,$$

$$(g_1^O, g_2^O) = (4\pi)^2\left(\frac{3\ \epsilon}{N+8}, 0\right)\,, (g_1^H, g_2^H) = (4\pi)^2\left(\frac{\epsilon}{N}, \frac{(N-4)}{3N}\epsilon\right)$$

For $N = 4$ the cubic fixed point coincides with the $O(N)$ symmetric one, for $N \to \infty$ and $N = 2$ it correspond to the Ising one, while for $N = 1$ the cubic theory reduces to the free one.

These limits will provide the cross-checks for our results.

# Describing the method

Conformal symmetry + Equations of motion (EOM)

EOM:

$$\Box\phi_i = \frac{1}{3!}\left(g_1\phi_i\phi^2 + g_2\phi_i^3\right), \qquad \phi^2 \equiv \sum_i \phi_i^2 \tag{2}$$

Conformal symmetry:

$$\langle O_i(x)O_j(y)O_k(z)\rangle = \frac{C_{ijk}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}|y-z|^{\Delta_j+\Delta_k-\Delta_i}|z-x|^{\Delta_k+\Delta_i-\Delta_j}} \tag{3}$$

The key idea is:

$$\langle\Box\phi_i S_n S_{n+1}\rangle = \frac{1}{3!}\langle(g_1\phi_i\phi^2 + g_2\phi_i^3)S_n S_{n+1}\rangle \tag{4}$$

$S_n$ is a composite scaling operator of order $n$, i.e. a product of $n$ fields transforming under an irreducible representation of $H_N$.

S. Rychkov and Z. M. Tan arXiv:1505.00963 [hep-th]
A. Codello, M. Safari, G. P. Vacca and O. Zanusso arXiv:1809.05071 [hep-th]

# Describing the method

Conformal symmetry + Equations of motion (EOM)

The key idea is:

$$\langle \Box \phi_i S_n S_{n+1} \rangle = \frac{1}{3!} \langle (g_1 \phi_i \phi^2 + g_2 \phi_i^3) S_n S_{n+1} \rangle \tag{5}$$

This implies an eigenvalue equation for the anomalous dimensions $\gamma_{S_n}$:

$$\boxed{\mathcal{D} S_n = \gamma_{S_n} S_n} \tag{6}$$

$$\mathcal{D} = \frac{1}{3N} \left( \frac{\phi^2 \partial^2}{2} + (\phi \cdot \partial)^2 - \phi \cdot \partial + \frac{N-4}{2} \sum_i \phi_i^2 \partial_i^2 \right) \tag{7}$$

$$\Delta_{S_n} = n(1 - \frac{\epsilon}{2}) + \gamma_{S_n} \epsilon + \mathcal{O}(\epsilon^2) \tag{8}$$

# Irreducible representations of hypercubic group $H_N$

$$\boxed{H_N = \mathcal{S}_N \ltimes \mathcal{Z}_2^N} \tag{9}$$

We label the two irreps of $\mathcal{Z}_2$ as [1] and [2]. The irreducible representations of $\mathcal{Z}_2^N$ are:

$$[2]^{\otimes\alpha} \otimes [1]^{\otimes\beta} \ , \qquad \alpha + \beta = N \ . \tag{10}$$

Example Irreps of $\mathcal{Z}_2^3$:

$$[2]^{\otimes 3},\ [2]^{\otimes 2} \otimes [1],\ [2] \otimes [1]^{\otimes 2},\ [1]^{\otimes 3} \tag{11}$$

In accordance with these representations, the symmetric group $S_N$ is divided into direct products $\mathcal{S}_\alpha \times \mathcal{S}_\beta$ and then the irreps of $\mathcal{S}_N \ltimes \mathcal{Z}_2^N$ are generated by multiplying those of $\mathcal{Z}_2^N$ with the corresponding direct product.

# Irreducible representations of hypercubic group $H_N$

$$\boxed{H_N = \mathcal{S}_N \ltimes \mathcal{Z}_2^N} \tag{12}$$

Example Irreps of $\mathcal{Z}_2^3$:

$$[2]^{\otimes 3},\ [2]^{\otimes 2} \otimes [1],\ [2] \otimes [1]^{\otimes 2},\ [1]^{\otimes 3} \tag{13}$$

Irreps of $H_3$:

$$([\mathbf{2}]^{\otimes \mathbf{3}} \otimes \mathcal{S}_\mathbf{3}) : (\ydiagram{3}, \emptyset), (\ydiagram{2,1}, \emptyset), (\ydiagram{1,1,1}, \emptyset); \quad ([\mathbf{2}]^{\otimes \mathbf{2}} \otimes [\mathbf{1}] \otimes \mathcal{S}_\mathbf{2} \times \mathbf{S_1}) : (\ydiagram{2}, \ydiagram{1}), (\ydiagram{1,1}, \ydiagram{1});$$

$$([\mathbf{1}]^{\otimes \mathbf{3}} \otimes \mathcal{S}_\mathbf{3}) : (\emptyset, \ydiagram{3}), (\emptyset, \ydiagram{2,1}), (\emptyset, \ydiagram{1,1,1}); \quad ([\mathbf{2}] \otimes [\mathbf{1}]^{\otimes \mathbf{2}} \otimes \mathcal{S}_\mathbf{1} \times \mathbf{S_2}) : (\ydiagram{1}, \ydiagram{2}), (\ydiagram{1}, \ydiagram{1,1})$$

A smart way to label the irreps of $H_N$ is in terms of double-partitions of N, $(\alpha, \beta)$, which can be represented as ordered pairs of Young diagrams with $\alpha$ and $\beta$ boxes, respectively.

# Scaling operators in the hypercubic model

Goal: Build non-derivative scaling operators with $n$ fields.
First step: Find the corresponding bi-tableaux by computing the tensor product of the defining representation $n$ times. As result we will have:

1. Bi-tableaux which do not appear at smaller $n \Rightarrow$ **Unique operator**
2. bi-tableaux that already appeared at the levels $n-2$, $n-4$, ..., $\Rightarrow$ more than one operator $\Rightarrow$ **Operator mixing**

## Unique operators

Left partition $\left( \begin{array}{l} \boxed{\mu_1}\boxed{\mu_2}\cdots\boxed{\mu_s} \\ \boxed{i}\boxed{j} \\ \boxed{k} \end{array}, \emptyset \right) = (\phi^4_{[k}\phi^2_i\phi^0_{\mu_1]})\cdot(\phi^2_{[j}\phi^0_{\mu_2]})\cdot\phi^0_{\mu_3}\phi^0_{\mu_4}\dots\phi^0_{\mu_s} = \phi^4_{[k}\phi^2_i\phi^0_{\mu_1]}\phi^2_{[j}\phi^0_{\mu_2]} \quad i\neq j\neq k\neq \mu_1\neq\mu_2$

Right partition $\left( \boxed{\mu_1}\boxed{\mu_2}\boxed{\mu_3}\dots\boxed{\mu_s}, \begin{array}{l} \boxed{i}\boxed{k} \\ \boxed{j} \end{array} \right) = \underbrace{\phi^0_{\mu_1}\phi^0_{\mu_2}\dots\phi^0_{\mu_s}}_{\text{Left}} \times \underbrace{\phi_k(\phi^3_i\phi_j-\phi^3_j\phi_i)}_{\text{Right}} = \phi_k(\phi^3_i\phi_j-\phi^3_j\phi_i) \qquad i\neq j\neq k$

$$\boxed{H_{n,\{m_i\},\{p_i\}} = \prod_{i=1}^{k}(\phi^{m_i}_{[\mu^i{}_1}\phi^{m_i-2}_{\mu^i{}_2}\phi^{m_i-4}_{\mu^i{}_3}\dots\phi^{M}_{\mu^i{}_{q_i}]})^{p_i} \qquad \mu^i_1\neq\mu^i_2\neq\dots\neq\mu^i_{q_i} \qquad (14)}$$

# Scaling operators in the hypercubic model

## Operator mixing

1. Write the *unique* composite operator according to the rules above and then multiply the result with the appropriate power of $\phi^2$ needed to reach $n$.

$$\left(\ \ydiagram{4,1}\dots,\ \emptyset\ \right) = (\phi^2)^2(\phi_i^2 - \phi_j^2) \qquad i \neq j \qquad (n = 6)$$

2. "Distribute" the $\phi^2$'s through the rest of the tensor in all possible ways:

$$\left(\ \ydiagram{4,1}\dots,\ \emptyset\ \right) = \begin{cases} (\phi^2)^2(\phi_i^2 - \phi_j^2) \\ (\phi^2)(\phi_i^4 - \phi_j^4) \\ (\phi_i^6 - \phi_j^6) \end{cases} \qquad i \neq j$$

3. Finally, we have to take into account the mixing between powers of $\phi^2$ and the other $H_N$-scalars. In our example $(\phi^2)^2$ will mix with $\sum_k \phi_k^4$:

$$\left(\ \ydiagram{4,1}\dots,\ \emptyset\ \right) = \begin{cases} \sum_k \phi_k^4(\phi_i^2 - \phi_j^2) \\ (\phi^2)^2(\phi_i^2 - \phi_j^2) \\ (\phi^2)(\phi_i^4 - \phi_j^4) \\ (\phi_i^6 - \phi_j^6) \end{cases} \qquad i \neq j$$

# The hypercubic tower

Inserting the compact expression for the unique scaling operators into the eigenvalue equation:

$$\mathcal{D}H_n = \gamma_{H_n} H_n \tag{15}$$

$$H_{n,\{m_i\},\{p_i\}} = \prod_{i=1}^{k} (\phi^{m_i}_{[\mu^i{}_1} \phi^{m_i-2}_{\mu^i{}_2} \phi^{m_i-4}_{\mu^i{}_3} ... \phi^{M}_{\mu^i{}_{q_i}]})^{p_i} \qquad \mu^i_1 \neq \mu^i_2 \neq ... \neq \mu^i_{q_i} \tag{16}$$

We obtain a compact formula for their anomalous dimensions $\gamma_{H_n}$:

$$\boxed{\gamma_{H_n} = \frac{1}{6N} \left( 2n(n-1) + (N-4) \sum_{i=1}^{k} p_i [m_i(m_i-1) + (m_i-2)(m_i-3) + ...] \right)} \tag{17}$$

# The hypercubic tower

## Examples

$k = 1,\ m_i = 1,\ p_i = p = n$

This is given by the family of bi-tableaux with one row:

$$\Big( \underbrace{\square\square\square\square\ldots}_{N-p}\ ,\ \underbrace{\square\square\square\square\ldots}_{p} \Big)$$

which corresponds to:

$$\phi_{\mu_1}\phi_{\mu_2}\ldots\phi_{\mu_p}, \quad \mu_1 \neq \mu_2 \neq \ldots \neq \mu_p \tag{18}$$

Their dimensions and anomalous dimensions at the level $n$ are:

$$dim = \begin{pmatrix} N \\ n \end{pmatrix} \qquad \gamma_n = \frac{n(n-1)}{3N}\ . \tag{19}$$

# The hypercubic tower

## Examples

$k = 3,\ m_i = \{1, 2, 3\}$ and $p_i = \{p_1, p_2, p_3\}$

This corresponds to the family of bi-tableaux with two rows.

$$\underbrace{\phi_{\mu_1}\phi_{\mu_2}\dots\phi_{\mu_{p_1}}}_{p_1 \text{ terms}}\underbrace{(\phi^2_{\mu_{p_1+1}} - \phi^2_{\mu_{p_1+2}})(\phi^2_{\mu_{p_1+3}} - \phi^2_{\mu_{p_1+4}})\dots(\phi^2_{\mu_{p_1+2p_2-1}} - \phi^2_{\mu_{p_1+2p_2}})}_{p_2 \text{ terms}} \times$$

$$\times \underbrace{(\phi^3_{\mu_{p_1+2p_2+1}}\phi_{\mu_{p_1+2p_2+2}} - \phi^3_{\mu_{p_1+2p_2+2}}\phi_{\mu_{p_1+2p_2+1}})\dots.(\phi^3_{\mu_{q-1}}\phi_{\mu_q} - \phi^3_{\mu_q}\phi_{\mu_{q-1}})}_{p_3 \text{ terms}}, \qquad \mu_1 \neq \mu_2 \neq \dots \neq \mu_q$$

Their dimensions and anomalous dimensions at the level $n$ are:

$$dim = \binom{N}{p_2}\binom{N-p_2}{2p_3+p_1}\binom{2p_3+p_1}{p_3+p_1}\frac{(N-p_1-2p_3-2p_2+1)(p_1+1)}{(N-p_1-2p_3-p_2+1)(p_1+p_3+1)} \tag{20}$$

$$\gamma_{n,p_1,p_2} = \frac{4n(n-1)+(N-4)(3n-2p_2-3p_1)}{12N} \tag{21}$$

# Operator mixing

Apart from the unique operators, all the operators of order $n$ will be be generated in the way previously explained.

Example: all the scaling operators with $n = 5$ fields.

$$dim = \begin{pmatrix} N \\ 5 \end{pmatrix} \qquad \phi_i\phi_j\phi_k\phi_l\phi_m \qquad \gamma = \frac{20}{3N}$$

$$dim = \frac{N(N-1)(N-4)}{2} \qquad \phi_i(\phi_j^2 - \phi_k^2)(\phi_l^2 - \phi_m^2) \qquad \gamma = \frac{2(N+6)}{3N}$$

$$dim = \frac{N(N-1)(N-2)}{3} \qquad \phi_i(\phi_j^3\phi_k - \phi_k^3\phi_j) \qquad \gamma = \frac{3N+8}{3N}$$

$$dim = \frac{N(N-1)(N-2)(N-4)}{6} \qquad \phi_i\phi_j\phi_k(\phi_l^2 - \phi_m^2) \qquad \gamma = \frac{16+N}{3N}$$

$$dim = 2 \times \frac{N(N-1)(N-2)}{6} \qquad \begin{pmatrix} \phi^2\phi_i\phi_j\phi_k \\ \phi_i^3\phi_j\phi_k + perm \end{pmatrix} \qquad \gamma = \frac{30+5N \pm \sqrt{(46+N)(N-2)}}{6N}$$

$$dim = 3 \times N(N-2) \qquad \begin{pmatrix} \phi^2\phi_i(\phi_j^2 - \phi_k^2) \\ \phi_i^3(\phi_j^2 - \phi_k^2) \\ \phi_i(\phi_j^4 - \phi_k^4) \end{pmatrix} \qquad \gamma = \frac{1}{3N}\begin{pmatrix} 3(N+6) & 3 & 6 \\ 2(N-4) & 4(N+1) & 0 \\ 4(N-4) & 0 & 2(3N-2) \end{pmatrix}$$

$$dim = 4 \times N \qquad \begin{pmatrix} \phi_i\left(\phi^2\right)^2 \\ \phi_i^5 \\ \phi^2\phi_i^3 \\ \phi_i\sum_{j=1}^N \phi_j^4 \end{pmatrix} \qquad \gamma = \frac{1}{3N}\begin{pmatrix} 4(N+5) & 0 & 3 & 6 \\ 0 & 10(N-2) & 6(N-4) & 4(N-4) \\ 4(N-4) & 10 & 5(N+2) & 4 \\ 4(N-4) & 0 & 0 & 2(3N-2) \end{pmatrix}$$

All the indices of the operators listed here have to be understood being different.

# General features of the spectrum

## Unique operators

$$H_{n,\{m_i\},\{p_i\}} = \prod_{i=1}^{k} (\phi^{m_i}_{[\mu^i{}_1} \phi^{m_i-2}_{\mu^i{}_2} \phi^{m_i-4}_{\mu^i{}_3} ... \phi^{M}_{\mu^i{}_{q_i}]})^{p_i} \qquad \mu^i_1 \neq \mu^i_2 \neq ... \neq \mu^i_{q_i}$$

$$\gamma_{H_n} = \frac{1}{6N} \left( 2n(n-1) + (N-4) \sum_{i=1}^{k} p_i [m_i(m_i-1) + (m_i-2)(m_i-3) + ...] \right)$$

## Scalar sector

It gives the lowest dimensional irreps at every even $n$. The $H_N$-scalars of order $n$ are formed by products and powers of all the operators of the form:

$$\sum_i \phi_i^2 = \phi^2, \ \sum_i \phi_i^4, \ \sum_i \phi_i^6, ..., \sum_i \phi_i^n \tag{22}$$

As a consequence, the number of scalars at a given order $n$ is given by the number of partitions of $\frac{n}{2}$.

Furthermore, at every level $n$ there is at least one $\gamma_n(N=1) = 0$ needed in order to obtain the free Gaussian theory. This is satisfied also by the vector ($dim = N$) irreps, which appear at every odd $n$.

# General features of the spectrum

## Weighted sum of anomalous dimensions

Further insights on the spectrum of anomalous dimensions can be gained by looking at:

$$W_n = \sum_{S_n} d_{S_n} \gamma_{S_n} \tag{23}$$

where $d_{S_n}$ and $\gamma_{S_n}$ are the dimensions and the anomalous dimensions of the composite operators $S_n$, respectively, and the sum runs over all the irreducible representations at the level $n$.

The values of $W_n$ exhibit an interesting pattern:

$$\begin{aligned} W_2 &= \frac{2}{3}(N-1), & W_3 &= \frac{2}{3}(N-1)(N+2), \\ W_4 &= \frac{1}{3}(N-1)(N+2)(N+3), & W_5 &= \frac{1}{9}(N-1)(N+2)(N+3)(N+4) \end{aligned} \tag{24}$$

which is indicative of a general formula for $W_n$.

# Conclusions

- The critical $H_N$ theory has interesting physical applications.
- We formalized the Representation Theory of $H_N$ and found the operator content of the hypercubic model.
- This allows the computation of 1-loop spectrum of anomalous dimensions at the cubic fixed point.
- The spectrum exhibits intriguing features which deserve further investigations.

Thank you!

# Irreducible representations of hypercubic group $H_N$

The defining ($N-$dimensional) representation:

$$\phi_i = (N-1,1) = (\underbrace{\square\square\ldots\square}_{N-1},\ \boxed{i}\ ) \tag{25}$$

Dimension of an irrep $(\alpha, \beta)$:

$$dim(\alpha,\beta) = \begin{pmatrix} N \\ \alpha \end{pmatrix} \times dim(\alpha) \times dim(\beta) \tag{26}$$

Decomposition of the tensor product:

$$(N-1,1) \otimes (\alpha,\beta) = \sum_{\alpha^+,\ \beta^-} (\alpha^+,\beta^-) \oplus \sum_{\alpha^-,\ \beta^+} (\alpha^-,\beta^+) \tag{27}$$

$$(\underbrace{\square\square\ldots}_{N-1},\square)\otimes(\underbrace{\square\square\ldots}_{N-1},\square) = (\underbrace{\square\square\ldots}_{N},\emptyset)\oplus(\underbrace{\begin{smallmatrix}\square\square\ldots\\ \square\end{smallmatrix}}_{N},\emptyset)\oplus(\underbrace{\square\square\ldots}_{N-2},\square\square)\oplus(\underbrace{\square\square\ldots}_{N-2},\begin{smallmatrix}\square\\ \square\end{smallmatrix})$$