# Exceptional quantum algebra for the standard model of particle physics

Ivan Todorov

Institute for Nuclear Research and Nuclear Energy
Tsarigradsko Chaussee 72, BG-1784 Sofia, Bulgaria
e-mail: ivbortodorov@gmail.com

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# Contents

# Abstract

The exceptional euclidean Jordan algebra $J_3^8$, consisting of $3 \times 3$ hermitian octonion matrices, appears to be tailor made for the internal space of the three generations of quarks and leptons. The maximal rank subgroup of the authomorphism group $F_4$ of $J_3^8$ that respects the lepton-quark splitting is $(SU(3)_c \times SU(3)_{ew})/\mathbb{Z}_3$. Its restriction to the special Jordan $\mathcal{H}_{16}(\mathbb{C}) \otimes \mathcal{H}_{16}(\mathbb{C})$ of $J_2^8$, the subalgebra of hermitian matrices of the associative envelope of its complexification, involves 32 primitive idempotents giving the states of the first generation fermions. The triality relating left and right $Spin(8)$ spinors to 8-vectors corresponds to the Yukawa coupling of the Higgs boson to quarks and leptons.

# Motivation. Alternative approaches

The gauge group of the Standard model (SM),

$$G_{SM} = \frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_6} = S(U(3) \times U(2)) \tag{1}$$

and its (highly reducible) representation for the first generation of basic fermions,

$$\begin{pmatrix} \nu \\ e^- \end{pmatrix}_L \leftrightarrow (\mathbf{1},\mathbf{2})_{-1}, \qquad \begin{pmatrix} u \\ d \end{pmatrix}_L \leftrightarrow (\mathbf{3},\mathbf{2})_{\frac{1}{3}}$$
$$(\nu_R \leftrightarrow (\mathbf{1},\mathbf{1})_0?), \qquad e_R^- \leftrightarrow (\mathbf{1},\mathbf{1})_{-2}, \quad u_R \leftrightarrow (\mathbf{3},\mathbf{1})_{\frac{4}{3}}, \quad d_R(\mathbf{3},\mathbf{1})_{-\frac{2}{3}} \tag{2}$$

(the subscript standing for the value of the weak hypercharge $Y$), look rather baroque for a fundamental symmetry. Unsatisfied, the founding fathers proposed Grand Unified Theories (GUTs) with (semi)simple symmetry groups: $SU(5)$ H. Georgi - S.L. Glashow (1974);
$Spin(10)$ H. Georgi (1975), H. Fritzsch - P. Minkowski (1975);
$Spin(6) \times Spin(4) = \frac{SU(4)\times SU(2)\times SU(2)}{\mathbb{Z}_2}$ J.C. Pati - Abdus Salam (1973).

The first two GUTs, based on simple groups, gained popularity in the beginning, since they naturally accommodated the fundamental fermions:

$$SU(5) : \mathbf{32} = \Lambda\mathbb{C}^5 = \bigoplus_{\nu=0}^{5} \Lambda^\nu, \ \Lambda^1 = \begin{pmatrix} \nu \\ e^- \end{pmatrix}_{-1} \oplus \bar{d}_{\frac{2}{3}} = \bar{\mathbf{5}},$$

$$\Lambda^3 = \begin{pmatrix} u \\ d \end{pmatrix}_{\frac{1}{3}} \oplus \overline{u}_{-\frac{4}{3}} \oplus e^+_2 = \mathbf{10};$$

$$Spin(10) : \mathbf{32} = \mathbf{16}_L \oplus \mathbf{16}_R, \ \mathbf{16}_L = \Lambda^1 \oplus \Lambda^3 \oplus \Lambda^5. \tag{3}$$

However the corresponding adjoint representations **24** (of $SU(5)$) and **45** (of $Spin(10)$) carry, besides the expected eight gluons and four electroweak gauge bosons, unwanted leptoquarks; for instance,

$$\mathbf{24} = (\mathbf{8}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{3})_0 \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{3}, \mathbf{2})_{\frac{5}{3}} \oplus (\bar{\mathbf{3}}, \mathbf{2})_{-\frac{5}{3}}. \tag{4}$$

Moreover, the presence of twelve gauge leptoquarks in (4) yields a proton decay rate that contradicts current experimental bounds.

The Pati-Salam GUT is the only one which does not predict a gauge triggered proton decay (it allows model dependent interactions with scalar fields that would permit such a decay). Accordingly, the Pati-Salam group appears in a preferred reduction of the *Spin*(10) GUT. Intriguingly, a version of this symmetry is also encountered in the noncommutative geometry approach to the SM. Concerning the most popular nowadays *supersymmetric GUTs* the lack of experimental evidence for any superpartner makes us share the misgivings expressed forcefully by Peter Woit and others.

The noncommutative geometry approach, was started 15 years after GUTs, "at the height of the string revolution" and pursued vigorously by Alain Connes, collaborators and followers.

The algebraic approach to quantum theory has, in fact, been initiated back in the 1930's by Pascual Jordan, who axiomatized the concept of *observable algebra*, the prime example of which is the algebra of complex hermitian matrices (or self-adjoint operators in a Hilbert space) equipped with the symmetrized product

$$A \circ B = \frac{1}{2}(AB + BA) \; (= B \circ A). \tag{5}$$

Such a (finite dimensional) Jordan algebra should appear as an "internal" counterpart of the algebra of smooth functions of classical fields. In the case of a special Jordan algebra one can work with the corresponding matrix algebra. In the noncommutative geometry approach to the SM one arrives at the following finite algebra:

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus \mathbb{C}[3] \tag{6}$$

($\mathbb{A}[n]$ standing for the algebra of $n \times n$ matrices with entries in the coordinate ring $\mathbb{A}$). The only hermitian elements of the quaternion algebra $\mathbb{H}$, however, are the real numbers, so $\mathcal{A}_F$ does not appear as the associative envelope of an interesting observable algebra. We shall, by contrast, base our treatment on an appropriate finite dimensional Jordan algebra suited for a quantum theory - permitting, in particular, a spectral decomposition of observables.

# Euclidean Jordan algebras

An euclidean Jordan algebra is a real vector space $J$ with a commutative product $X \circ Y$ satisfying the *formal reality condition*

$$X_1^2 + ... + X_n^2 = 0 \Rightarrow X_1 = ... = X_n = 0 \ (X_i^2 := X_i \circ X_i) \tag{7}$$

and power associativity. These conditions are necessary and sufficient to have spectral decomposition of any element of $J$ and thus treat it as an observable.

In order to introduce spectral decomposition we need the algebraic counterpart of a projector: $e \in J$ satisfying $e^2 = e (\neq 0)$ is called an *idempotent*. Two idempotents $e$ and $f$ are *orthogonal* if $e \circ f = 0$; then multiplication by $e$ and $f$ commute and $e + f$ is another idempotent. The formal reality condition (7) allows to define *partial order* in $J$ saying that $X$ is smaller than $Y$, $X < Y$, if $Y - X$ can be written as a sum of squares. Noting that $f = f^2$ we conclude that $e < e + f$. A non-zero idempotent is called *minimal* or *primitive* if it cannot be decomposed into a sum of (nontrivial) orthogonal idempotents. A *Jordan frame* is a set of orthogonal primitive idempotents $e_1, ..., e_r$ satisfying

$$e_1 + ... + e_r = 1 \quad (e_i \circ e_j = \delta_{ij} e_i). \tag{8}$$

Each such frame gives rise to a complete set of commuting observables. The number of elements $r$ in a Jordan frame is independent of its choice and is called the *rank of J*.

Each $X \in J$ has a *spectral decomposition* of the form

$$X = \sum_{i=1}^{r} \lambda_i e_i, \ \lambda_i \in \mathbb{R}, \ \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_r. \tag{9}$$

For a $X$ for which all $\lambda_i$ in (9) are different the spectral decomposition is unique. Such *regular* $X$ form a dense open set in $J$. The rank of $J$ coincides with the degree of the characteristic polynomial for any $X \in J$:

$$\begin{aligned} F_r(t, X) = t^r - a_1(X)t^{r-1} + ... + (-1)^r a_r(X), \\ a_k(X) \in \mathbb{R}, \ a_k(\alpha X) = \alpha^k a_k(X) \ (\alpha > 0). \end{aligned} \tag{10}$$

The roots of $F_r$ are $(t =)\lambda_1, ..., \lambda_r$ (some of which may coincide). Given a regular $X$ the idempotents $e_i$ can be expressed as polynomial in $X$ of degree $r-1$, determined from the system of equations

$$\begin{aligned} e_1 + ... + e_r = 1, \\ \lambda_1 e_1 + ... + \lambda_r e_r = X, \\ ...... \\ \lambda_1^{r-1} e_1 + ... + \lambda_r^{r-1} e_r = X^{r-1}, \end{aligned} \tag{11}$$

whose Vandermonde determinant is non zero for distinct $\lambda_i$.

We define a trace and an inner product in $J$. The *trace*, $tr(X)$, is a linear functional on $J$ taking value 1 on primitive idempotents:

$$tr(X) = \sum_i \lambda_i (= a_1(X)), \quad tr(1) = r, \tag{12}$$

for $X$ given by (9) (and $a_1(X)$ of (10)). The *inner product*, defined as the trace of the Jordan product, is positive definite:

$$(X, Y) := tr(X \circ Y) \Rightarrow (X, X) > 0 \quad for\ X \neq 0. \tag{13}$$

This justifies the name *euclidean* for a formally real Jordan algebra. The last coefficient, $a_r$, of (10) is the *determinant of* $X$:

$$a_r(X) = det(X) = \lambda_1 ... \lambda_r. \tag{14}$$

If $det(X) \neq 0$ then $X$ is *invertible* and its inverse is given by

$$X^{-1} := \frac{(-1)^r}{det(X)} (X^{r-1} - a_1(X) X^{r-1} + ... + (-1)^{r-1} a_{r-1}(X) 1). \tag{15}$$

The theory of euclidean Jordan algebras is simplified by the fact that any such algebra can be written as a direct sum of *simple* ones.

The finite dimensional simple euclidean Jordan algebras were classified at the dawn of the theory, in 1934, by Jordan, von Neumann and Wigner. They can be labeled by two numbers, the rank r and the degree d (the dimension of off diagonal elements) and split into four infinite series and one exceptional algebra (proven to be non special by A.A. Albert also in 1934):

$$J_r^1 = \mathcal{H}_r(\mathbb{R}),\ r \geq 1; \quad J_r^2 = \mathcal{H}_r(\mathbb{C}),\ r \geq 2;$$
$$J_r^4 = \mathcal{H}_r(\mathbb{H}),\ r \geq 2; \quad J_2^d = JSpin(d+1);$$
$$J_3^8 = \mathcal{H}_3(\mathbb{O}), \quad dim(J_r^d) = \binom{r}{2} d + r \qquad (16)$$

$(dim(\mathcal{H}(\mathbb{R})_r) = \binom{r+1}{2},\ dim(\mathcal{H}(\mathbb{C})_r) = r^2,\ dim(J_2^d) = d+2,\ dim(J_3^8) = 27)$. The first three algebras in the above list consist of familiar hermitian matrices (with entries in associative division rings). The *spin factor* $J_2^d \subset C\ell_{d+1}$ can be thought as the set of $2 \times 2$ matrices of the form

$$X = \xi \mathbf{1} + \hat{x},\ \xi \in \mathbb{R},\ \hat{x}^2 = N(x)\mathbf{1},\ N(x) = \sum_{\mu=0}^{d} x_\mu^2 = -det\hat{x},\ tr\hat{x} = 0. \qquad (17)$$

There are three obvious repetitions in the list (16): the spin factors $J_2^d$ for $d = 1, 2, 4$ coincide with the first items in the three families of matrix algebras in the list (16). We could also write

$$J_2^8 = \mathcal{H}_2(\mathbb{O})(\subset C\ell_9); \tag{18}$$

here (as in $J_3^8$) $\mathbb{O}$ stands for the nonassociative division ring of *octonions*. The spin factor $J_2^8$ (unlike $J_3^8$) is *special* - as a (10-dimensional) Jordan subalgebra of the ($2^9$-dimensional) associative algebra $C\ell_9$.

Remarkably, an euclidean Jordan algebra gives room not only to the observables of a quantum theory, it also contains its states: these are the positive observables.

Each euclidean Jordan algebra $J$ contains a convex, *open cone* $\mathcal{C}$ consisting of all positive elements of $J$ (i.e., all invertible elements that can be written as sums of squares, so that all their eigenvalues are positive). Jordan frames belong to the closure $\bar{\mathcal{C}}$ (in fact, to the boundary) of the open cone, not to $\mathcal{C}$ itself, as primitive idempotents (for $r > 1$) are not invertible.

The *states* are (normalized) positive linear functionals on the space of observables, so they belong to the closure of the dual cone

$$\mathcal{C}^* = \{\rho \in J; (\rho, X) > 0 \ \forall X \in \bar{\mathcal{C}}\}. \tag{19}$$

In fact, the positive cone is *self-dual*, $\mathcal{C} = \mathcal{C}^*$. An element $\rho \in \bar{C} \subset J$ of trace one defines a *state* assigning to any observable $X \in J$ an *expectation value*

$$< X > = (\rho, X) = tr(\rho \circ X), \ \rho \in \bar{\mathcal{C}}, \ \ tr\rho\,(=< 1 >) = 1. \tag{20}$$

The primitive idempotents define *pure states*; they are extreme points in the convex set of normalized states. All positive states (in the open cone $\mathcal{C}$) are (mixed) *density matrices*. There is a distinguished mixed state in $J_r^d$, the normalized unit matrix, called by Baez the *state of maximal ignorance*:

$$< X >_0 = \frac{1}{r} tr(X) \ (r = tr(1)). \tag{21}$$

Any other state can be obtained by multiplying it by a (suitably normalized) observable - thus displaying a *state observable correspondence*.

The cone $\mathcal{C}$ is *homogeneous*: it has a transitively acting symmetry group that defines the *structure group* of the Jordan algebra, $Aut(\mathcal{C}) =: Str(J)$, the product of a central subgroup $\mathbb{R}_+$ of uniform dilations with a (semi)simple Lie group $Str_0(J)$. Here is a list of the corresponding (semi)simple Lie algebras:

$$str_0(J_r^1) = sl(r, \mathbb{R}),\ str_0(J_r^2) = sl(r, \mathbb{C}),\ str_0(J_r^4) = su^*(2r),$$
$$str_0(J_2^d) = so(d+1, 1)(= spin(d+1, 1)),\ \ str_0(J_3^8) = \mathfrak{e}_{6(-26)}. \qquad (22)$$

The stabilizer of the point 1 of the cone is the maximal compact subgroup of $Aut(\mathcal{C})$ whose Lie algebra coincides with the derivation algebra of $J$:

$$\mathfrak{der}(J_r^1) = so(r),\ \mathfrak{der}(J_r^2) = su(r),\ \mathfrak{der}(J_r^4) = usp(2r),$$
$$\mathfrak{der}(J_2^d) = so(d+1)(= spin(d+1)),\ \ \mathfrak{der}(J_3^8) = \mathfrak{f}_4. \qquad (23)$$

We shall argue that the exceptional Jordan algebra $J_3^8$ should belong to the observable algebra of the SM.

# Octonions in $J_3^8$. Quark-lepton symmetry

**Why octonions?**

The *octonions* $\mathbb{O}$ were originally introduced as pairs of quaternions (the "Cayley-Dickson construction"). But it was the decomposition of $\mathbb{O}$ into complex spaces,

$$\mathbb{O} = \mathbb{C} \oplus \mathbb{C}^3, \; x = z + \mathbf{Z}, \; z = x^0 + x^7 e_7, \; \mathbf{Z} = Z^1 e_1 + Z^2 e_2 + Z^4 e_4,$$
$$Z^j = x^j + x^{3j} e_7; \;\; e_j e_{j+1} = e_{j+3(mod7)}, \;\; e_j e_k + e_k e_j = -2\delta_{jk}, \;\; j,k = 1, ...7, \quad (24)$$

that led Feza Gürsey (and his student Günaydin) back in 1973 to apply it to the quarks (then the newly proposed constituents of hadrons). They figured out that the subgoup $SU(3)$ of the automorphism group $G_2$ of the octonions, that fixes the first $\mathbb{C}$ in (24), can be identified with the quark colour group. Gürsey tried to relate the non-associativity of the octnions to the quark confinement - the unobservability of free quarks. Only hesitantly did he propose "as another speculation" that the first $\mathbb{C}$ in (24) "could be related to leptons". Interpreting (24) as a manifestation of the quark-lepton symmetry was only taken seriously in 1987 by A. Govorkov in Dubna.

M. Dubois-Violette pointed out that, conversely, the unimodularity of the quark's colour symmetry yields - through an associated invariant volume form - an essentially unique octonion product with a multiplicative norm. The octonions (just like the quaternions) do not represent an observable algebra. They take part, however, in the exceptional Jordan algebra $J_3^8$ whose elements obey the following Jordan product rules:

$$
\begin{aligned}
X(\xi, x) &= \begin{pmatrix} \xi_1 & x_3 & x_2^* \\ x_3^* & \xi_2 & x_1 \\ x_2 & x_1^* & \xi_3 \end{pmatrix} \\
&= \sum_{i=1}^{3} (\xi_i E_i + F_i(x_i)),\ E_i \circ E_j = \delta_{ij} E_i,\ E_i \circ F_j = \frac{1-\delta_{ij}}{2} F_j, \\
F_i(x) \circ F_i(y) &= (x, y)(E_{i+1} + E_{i+2}),\ F_i(x) \circ F_{i+1}(y) = \frac{1}{2} F_{i+2}(y^* x^*)
\end{aligned}
\tag{25}
$$

(indices being counted mod 3). It incorporates *triality* that will be related to the three generations of basic fermions.

## Quark-lepton splitting of $J_3^8$ and its symmetry

The automorphism group of $J_3^8$ is the compact exceptional Lie group $F_4$ of rank 4 and dimension 52 whose Lie algebra is spanned by the (maximal rank) subalgebra $so(9)$ and its spinorial representation **16**, and can be expressed in terms of $so(8)$ and its three (inequivalent) 8-dimensional representations:

$$\mathfrak{der}(J_3^8) = \mathfrak{f}_4 \cong so(9) + \mathbf{16} \cong so(8) \oplus \mathbf{8}_V \oplus \mathbf{8}_L \oplus \mathbf{8}_R; \qquad (26)$$

here $\mathbf{8}_V$ stands for the 8-vector, $\mathbf{8}_L$ and $\mathbf{8}_R$ for the left and right chiral $so(8)$ spinors. The group $F_4$ leaves the unit element $1 = E_1 + E_2 + E_3$ invariant and transforms the traceless part of $J_3^8$ into itself (under its lowest dimensional fundamental representation **26**).

The lepton-quark splitting (24) of the octonions yields the following decomposition of $J_3^8$:

$$X(\xi, x) = X(\xi, z) + Z,\ X(\xi, z) \in J_3^2 = \mathcal{H}_3(\mathbb{C}),$$
$$Z = (Z_r^j, j = 1, 2, 4,\ r = 1, 2, 3) \in \mathbb{C}[3]. \qquad (27)$$

The subgroup of $Aut(J_3^8)$ which respects this decomposition is the commutant $F_4^\omega \subset F_4$ of the automorphism $\omega \in G_2 \subset Spin(8) \subset F_4$ (of order three):

$$\omega X(\xi, x) = \sum_{i=1}^{3} (\xi_i E_i + F_i(\omega_7 x_i)),\ \omega_7 = \frac{-1 + \sqrt{3} e_7}{2}\ (\omega^3 = 1 = \omega_7^3). \qquad (28)$$

It consists of two $SU(3)$ factors (with their common centre acting trivially):

$$F_4^\omega = \frac{SU(3)_c \times SU(3)_{ew}}{\mathbb{Z}_3} \ni (U, V) :\ X(\xi, x) \to VX(\xi, z)V^* + UZV^*. \qquad (29)$$

We see that the factor, $U$ acts on each quark's colour index $j(= 1, 2, 4)$, so it corresponds to the exact $SU(3)_c$ colour symmetry while $V$ acts on the leptons and on the flavour index $r(= 1, 2, 3)$ and is identified with (an extension of) the broken electroweak symmetry as it will be made clear by restricting the quantum algebra to the first generation of fermions.

# The first generation algebra $J_2^8$ and its euclidean extension

The Jordan subalgebra $J_2^8 \subset J_3^8$, orthogonal, say, to the projector $E_1$,

$$J_2^8(1) = (1 - E_1) J_3^8 (1 - E_1), \tag{30}$$

is special, its associative envelope being $C\ell_9$. Its automorphism group is $Spin(9) \subset F_4$, whose intersection with $F_4^\omega$, that respects the quark-lepton splitting, coincides with - and thus *explains* - *the gauge group of the SM*:

$$G_{SM} = F_4^\omega \cap Spin(9) = S(U(3) \times U(2)) (= \frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_6}). \tag{31}$$

The correct euclidean extension of $J_2^8$ is the Jordan subalgebra of hermitian matrices of the complexificatioin of its associative envelope (the real and hermitian envelopes having both dimension $2 \times 16^2$):

$$J_2^8 \subset C\ell_9 = \mathbb{R}[16] \oplus \mathbb{R}[16] \to J_{16}^2 \oplus J_{16}^2 \subset C\ell_9(\mathbb{C}) \; (J_{16}^2 = \mathcal{H}_{16}(\mathbb{C})). \tag{32}$$

The resulting (reducible) Jordan algebra of rank 32 gives room precisely to the state space (of internal quantum numbers) of fundamental fermions of one generation - including the right handed "sterile" neutrino. In fact, it is acted upon by the simple structure group of $J_2^8$ whose generators belong to the even part of the Clifford algebra $C\ell(9,1)$ isomorphic to $C\ell_9$ and whose Dirac spinor representation splits into two chiral Weyl spinors:

$$Str_0(J_2^8) = Spin(9,1) \subset C\ell^0(9,1)(\cong C\ell_9) \Rightarrow \mathbf{32} = \mathbf{16}_L \oplus \mathbf{16}_R. \quad (33)$$

The introduction of a rank 5 group like $Spin(9,1)$, that is another real form of the GUT's Spin(10), gives room to one more quantum number providing a natural labeling of the fundamental fermions of (any) one generation.

As the quark colour is not observable we only have to distinguish $SU(3)_c$ representations as labels: **3** for a quark triplet, ($\bar{\mathbf{3}}$ for an antiquark) and **1** for an $SU(3)_c$ singlet. The electroweak labels are the hypercharge $Y$ and the (exactly conserved) electric charge $Q(= I_3 + \frac{1}{2}Y)$. The extra quantum number coming from the structure Lie algebra can be identified with the commutant $B - L$ of $su(3)_c$ in the Pati-Salam $su(4) \subset so(9,1)$ (that is the difference between the baryon and the lepton number). We have eight primitive idempotents corresponding to the left and right (anti)leptons and eight (non primitive) chiral (anti)quark idempotents (colour singlets of trace three); for instance,

$$|\nu_L >< \nu_L| \leftrightarrow (\mathbf{1}; Y = -1, Q = 0, B - L = -1),$$
$$\sum_j |d_L^j >< d_L^j| \leftrightarrow (\mathbf{3}; \frac{1}{3}, -\frac{1}{3}, \frac{1}{3}). \tag{34}$$

# Triality and Yukawa coupling

**Associative trilinear form. The principle of triality**

The trace of an octonion $x = \sum_\mu x^\mu e_\mu$ is a real valued linear form on $\mathbb{O}$:

$$tr(x) = x + x^* = 2x^0 = 2Re(x) \ (e_0 \equiv 1). \tag{35}$$

It allows to define an associative and symmetric under cyclic permutations *normed triality* form:

$$2t(x, y, z) = tr((xy)z) = tr(x(yz)) =: tr(xyz) = tr(zxy) = tr(yzx). \tag{36}$$

The normalization factor 2 is chosen to have:

$$|t(x, y, z|^2 \leq N(x)N(y)N(z), \ \ N(x) = xx^*(\in \mathbb{R}). \tag{37}$$

While the norm $N(x)$ and the corresponding scalar product are $SO(8)$-invariant, the trilinear form $t$ corresponds to the invariant product of the three inequivalent 8-dimensional fundamental representations of $Spin(8)$, the 8-vector and the two chiral spinors $S^\pm$.

**Theorem 5.1** (*Principle of triality*). For any $g \in SO(8)$ there exists a pair $(g^+, g^-)$ of elements of $SO(8)$, such that

$$g(xy) = (g^+x)(g^-y), \;\; x, y \in \mathbb{O}. \tag{38}$$

If the pair $(g^+, g^-)$ satisfies (38) then the only other pair which obeys the principle of triality is $(-g^+, -g^-)$.
*Corollary*. If the triple $g, g^+, g^-$ obeys (38) then the form $t$ (36) satisfies the invariance condition

$$t(g^+x, g^-y, g^{-1}z) = t(x, y, z). \tag{39}$$

**Proposition 5.2** The set of triples
$(g, g^+, , g^-) \in SO(8) \times SO(8) \times SO(8)$ satisfying the principle of triality form a group isomorphic to the double cover $Spin(8)$ of $SO(8)$.

An example of a triple $(g^+, g^-, g^{-1})$ satisfying (39) is provided by left-, right- and bi-multiplication by a unit octonion:

$$t(L_u x, R_u y, B_{u^*} z) = t(x, y, z),\ L_u x = ux, R_u y = yu, B_v z = vzv,\ \ uu^* = 1. \tag{40}$$

The maps among $g, g^+, g^-$ belong to the group of outer automorphisms of the Lie algebra $so(8)$ which coincides with the symmetric group $\mathcal{S}_3$ that permutes the nodes of the Dynkin diagram for $so(8)$. In particular, the map permuting $L_u, R_u, B_{u^*}$ belongs to the cyclic group $\mathbb{Z}_3$:

$$\nu : L_u \to R_u \to B_{u^*} \Rightarrow \nu^3 = 1. \tag{41}$$

*Remark.* The associativity law, expressed in terms of left (or right) multiplication, reads

$$L_x L_y = L_{xy},\ R_x R_y = R_{yx}. \tag{42}$$

It is valid for complex numbers and for quaternions; for octonions Eq. (42) only takes place for real multiples of powers of a single element. Left and right multiplications by unit quaternions generate different $SO(3)$ subgoups of the full isometry group $SO(4)$ of quaternions. By contrast, products of upto 7 left multiplications of unit octonions (and similarly of upto 7 $R_u$ or $B_u$) generate the entire $SO(8)$.

**Speculations about Yukawa couplings**

**Proposition 5.3** The subgroup *Spin*(8) of $F_4$ leaves the diagonal projectors $E_i$ in the generic element $X(\xi, x)$ (25) of $J_3^8$ invariant and transforms the off diagonal elements as follows:

$$F_1(x_1) + F_2(x_2) + F_3(x_3) \to F_1(gx_1) + F_2(g^+ x_2) + F_3(g^- x_3). \qquad (43)$$

Thus if we regard $x_1$ as a *Spin*(8) vector, then $x_2$ and $x_3$ should transform as $S^+$ and $S^-$ spinors, respectively. It would be attractive to interpret the invariant trilinear form $t(x_1, x_2, x_3)$ as the internal symmetry counterpart of the Yukawa coupling between a vector and two (conjugate) spinors. Viewing the 8-vector $x_1$ as the finite geometry image of the Higgs boson, the associated Yukawa coupling would be responsible for the appearance of (the first generation) fermion masses.

There are, in fact, three possible choices for the $SO(8)$ vector representation, one for each generation $i$ and for the associated Jordan subalgebra

$$J_2^8(i) = (1 - E_i)J_3^8(1 - E_i), \; i = 1, 2, 3. \tag{44}$$

According to Jacobson any finite (unital) module over $J_3^8$ has the form $J_3^8 \otimes E$ for some finite dimensional real vector space $E$. The above consideration implies that $dim(E)$ should be divisible by three. The analysis of Sect. 4, on the other hand, suggests that E should contain a factor $\mathcal{H}_{16}(\mathbb{C})$. A possible candidate satisfying both conditions is $E = \mathcal{H}_{16}(\mathbb{C}) \otimes \mathcal{H}_3(\mathbb{R})$. Then the rank $3 \times 32$ of the Jordan module $J_2^8 \otimes E \subset J_3^8 \otimes E$, would be equal to the total dimension of the internal space of three generations of fermions.

Using the classification of finite dimensional simple euclidean observable algebras (of 1934) and the quark-lepton symmetry we argue that the observable akgebra of the SM is a multiple of the exceptional Jordan (also called *Albert*) algebra $J_3^8$ that describes the three generations of fundamental fermions. We postulate that the symmetry group of the SM is the subgroup $F_4^\omega$ of $Aut(J_3^8) = F_4$ that respects the quark-lepton splitting. Remarkably, the intersection of $F_4^\omega$ with the automorphism group $Spin(9)$ of the subalgebra $J_2^8 \subset J_3^8$ of a single generation is precisely the gauge group of the SM.
The next big problem we should face is to fix the appropriate $J_3^8$ module and to write down the Lagrangian in terms of fields taking values in this module.