Kac–Moody exceptional field theory

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Outline

- Generalized diffeomorpshims
- Constrained fields and dual graviton
- Twisted self-duality equations

[G. B, M. Cederwall, A. Kleinschmidt, J. Palmkvist, H. Samtleben, 1708.08936]

[G. B, F. Ciceri, G. Inverso, A. Kleinschmidt and H. Samtleben, 1811.04088]

[G. B, A. Kleinschmidt and E. Sezgin, 1907.02080]

[G. B, F. Ciceri, G. Inverso, A. Kleinschmidt and H. Samtleben, 1912.?????]

Motivations

Eleven-dimensional supergravity on $\mathbb{R}^{1,10\text{-}d} \times T^d$

$\hookrightarrow$ Supergravity with $E_{d(d)}$ symmetry [Cremmer–Julia]

Type II string theory on $\mathbb{R}^{1,10\text{-}d} \times T^{d\text{-}1}$

$\hookrightarrow$ U-duality symmetry $E_d(\mathbb{Z})$ [Hull–Townsend]

$E_d(\mathbb{Z}) \supset GL(d,\mathbb{Z})$: global diffeomorphisms of T^d.

$\hookrightarrow$ Global generalised diffeomorphisms $E_d(\mathbb{Z})$? [Hull–Waldram]

- ⋆ U-manifolds compactification: generalised geometry.
- ⋆ Higher dimensional origin of gauged supergravity theories.
- ⋆ Effective theories including $1/2$ BPS states.

Motivations

Exceptional geometries for Kac–Moody groups $E_9 \subset E_{10} \subset E_{11}$.

$\hookrightarrow$ Unique E_{11} exceptional field theory [West]

$\hookrightarrow$ E_{10} cosmological billard [Henneaux–Damour–Nicolai]

E_9 is the first infinite dimensional and nonetheless tractable

$\hookrightarrow$ General gauged supergravity in two dimensions [Samtleben–Weidner]

Integrable system of equations [Breitenlohner–Maison]

$\hookrightarrow$ Integrable structures in eleven dimensions?

$\hookrightarrow$ Integrable structures in gauged supergravity?

String theory effective action on $\mathbb{R}^{1,1} \times T^8$

$\hookrightarrow$ Loop amplitudes — Affine Eisenstein series.

Generalised diffeomorphisms

- ⋆ Diffeomorphisms based on $SL(d)$: $\partial_N Y^M \in GL(d)$

$$\mathcal{L}_\xi V^M = \xi^N \partial_N V^M - \partial_N \xi^M V^N$$

- ⋆ Diffeomorphisms based on $Sp(2d)$: $\partial_N Y^M \in \mathbb{R}_+ \times Sp(2d)$

$$\mathcal{L}_\xi V^M = \xi^N \partial_N V^M - \partial_N \xi^M V^N - \omega_{PN} \omega^{MQ} \partial_Q \xi^P V^N$$

- ⋆ Diffeomorphisms based on $SO(d,d)$: $\partial_N Y^M \in \mathbb{R}_+ \times SO(d,d)$

$$\mathcal{L}_\xi V^M = \xi^N \partial_N V^M - \partial_N \xi^M V^N + \eta_{PN} \eta^{MQ} \partial_Q \xi^P V^N$$

- ⋆ Diffeomorphisms based on G: $\partial_N Y^M \in \mathbb{R}_+ \times G$

$$\mathcal{L}_\xi V^M = \xi^N \partial_N V^M - \Big(\eta_{\alpha\beta} (T^\alpha)^M{}_N (T^\beta)^P{}_Q + (1 - (\lambda,\lambda)) \delta^M_N \delta^P_Q \Big) V^N \partial_P \xi^Q$$

Exceptional diffeomorphism

- ⋆ $SL(d)$ diffeomorphisms close on all functions
- ⋆ $Sp(2d)$ diffeomorphisms close on Poisson commuting functions

$$\omega^{MN}\partial_M F\,\partial_N G = 0 \qquad \Leftrightarrow \qquad [F, G] = i\omega^{MN}\partial_M F\,\partial_N G$$

- ⋆ $SO(d, d)$ diffeomorphisms close on section

$$\eta^{MN}\partial_M F\,\partial_N G = 0 \;\Leftrightarrow\; [F(\sigma), G(\sigma')] = \tfrac{i}{2}\eta^{MN}\partial_M F(\sigma)\,\partial_N G(\sigma')\mathrm{sign}(\sigma-\sigma')$$

- ⋆ G diffeomorphisms (for $\mathfrak{g}_{+\lambda}$ finite dimensional) close on

$$\eta_{\alpha\beta}(T^\alpha)^P{}_M(T^\beta)^Q{}_N\partial_P F\partial_Q G = \partial_N F\partial_M G + ((\lambda,\lambda)-1)\partial_M F\partial_N G$$

Generalised diffeomorphisms

Diffeomorphisms based on E_d, with coordinates in $\overline{R(\Lambda_d)}$

- ⋆ $GL(2)$ on T^2, either type IIB or eleven-dimensional supergravity

...

- ⋆ $Spin(5,5)$ on T^5, coordinates Majorana–Weyl spinors

$$(\Gamma^\Lambda)^{MN}\partial_M F \partial_N G = 0$$

- ⋆ E_6 on T^6, coordinates in the **27**

$$t^{MNP}\partial_M F \partial_N G = 0$$

- ⋆ E_7 on T^7, coordinates in the **56**

$$(t^\alpha)^M{}_P \omega^{PN}\partial_M F \partial_N G = 0 \qquad \omega^{MN}\partial_M F \partial_N G = 0$$

The potential

$\mathcal{M}_{MN}$ symmetric matrix in E_d representation $\overline{R(\Lambda_d)} \cong \left\{ \begin{smallmatrix} \overline{\mathbf{10}} \\ \mathbf{16} \\ \overline{\mathbf{27}} \\ \mathbf{56} \end{smallmatrix} \right\}$.

$\hookrightarrow$ Generalised metric $\sqrt{\text{-}g}^{-1}\mathcal{M}_{MN}$ [Berman–Perry] .

Potential (with normalization: $T^{\alpha M}{}_N T^{\beta N}{}_M = n_d \eta^{\alpha\beta}$) :

$$\begin{aligned} V &= -\frac{1}{4n_d}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_N\mathcal{M}_{KL} + \frac{1}{2}\mathcal{M}^{MN}\partial_M\mathcal{M}^{KL}\partial_L\mathcal{M}_{NK} \\ &= \frac{1}{4}\eta^{\alpha\beta}\mathcal{M}^{MN}\mathcal{J}_{M\alpha}\mathcal{J}_{N\beta} - \frac{1}{2}\mathcal{M}^{PQ}(\mathcal{J}_M)^N{}_P(\mathcal{J}_N)^M{}_Q \end{aligned}$$

with [Hohm–Samtleben]

$$(\mathcal{J}_M)^N{}_P = \mathcal{J}_{M\alpha}T^{\alpha N}{}_P = \mathcal{M}^{NQ}\partial_M\mathcal{M}_{QP}$$

The potential for E_8

$\mathcal{M}_{MN}$ symmetric matrix in E_8 adjoint representation.

$\hookrightarrow$ Generalised metric.

Potential (with normalization: $f^{MP}{}_Q f^{NQ}{}_P = 60\eta^{MN}$) :

$$V = \frac{1}{4}\eta^{PQ}\mathcal{M}^{MN}\mathcal{J}_{M,P}\mathcal{J}_{N,Q} - \frac{1}{2}\mathcal{M}^{PQ}(\mathcal{J}_M)^N{}_P(\mathcal{J}_N)^M{}_Q + \frac{1}{2}\eta^{MQ}\eta^{NP}\mathcal{J}_{M,P}\mathcal{J}_{N,Q}$$

with [Hohm–Samtleben]

$$(\mathcal{J}_M)^N{}_P = -\mathcal{J}_{MQ} f^{QN}{}_P = \mathcal{M}^{NQ}\partial_M \mathcal{M}_{QP}$$

The potential for E_9

$\mathcal{M}_{MN}$ symmetric matrix in the level 1 E_9 basic module $\overline{R(\Lambda_9)}$.

$\hookrightarrow$ Generalised metric.

Potential :

$$V = \frac{1}{4}\eta^{\alpha\beta}\mathcal{M}^{MN}\mathcal{J}_{M\alpha}\mathcal{J}_{N\beta} - \frac{1}{2}\mathcal{M}^{PQ}(\mathcal{J}_M)^N{}_P(\mathcal{J}_N)^M{}_Q + \frac{\rho(\mathcal{M})^2}{2}\mathcal{M}^{PQ}(\mathcal{J}^-_M)^N{}_P(\mathcal{J}^-_N)^M{}_Q - \frac{1}{2\rho(\mathcal{M})^2}\mathcal{M}^{MN}\partial_M\rho(\mathcal{M})\partial_N\rho(\mathcal{M})$$

with

$$(\mathcal{J}_M)^N{}_P = \mathcal{J}_{M\alpha}T^{\alpha N}{}_P = \mathcal{M}^{NQ}\partial_M\mathcal{M}_{QP}$$
$$(\mathcal{J}^-_M)^N{}_P = \mathcal{J}_{M\alpha}\mathcal{S}_{-1}(T^{\alpha N}{}_P) + \chi_M\delta^N_P$$

where $\mathcal{S}_{-1}$ lowers the L_0 weight by 1

χ_M is a new constrained field (like any derivative ∂_M).

The potential for E_9

$\mathcal{M}$ symmetric operator in the level 1 E_9 basic module $\overline{R(\Lambda_9)}$.

$\hookrightarrow$ Generalised metric.

Potential :

$$V = \frac{1}{4}\eta^{\alpha\beta}\langle\mathcal{J}_\alpha|\mathcal{M}^{-1}|\mathcal{J}_\beta\rangle - \frac{1}{2}\langle\mathcal{J}_\alpha|T^\beta\mathcal{M}^{-1}T^{\alpha\dagger}|\mathcal{J}_\beta\rangle + \frac{\rho(\mathcal{M})^2}{2}\langle\mathcal{J}^-_\alpha|T^\beta\mathcal{M}^{-1}T^{\alpha\dagger}|\mathcal{J}^-_\beta\rangle - \frac{1}{8}\langle\mathcal{J}_0|\mathcal{M}^{-1}|\mathcal{J}_0\rangle$$

with

$$\langle\mathcal{J}_\alpha|\otimes\mathcal{M}T^\alpha = \langle\partial_\mathcal{M}|\otimes\mathcal{M}$$
$$\langle\mathcal{J}^-_\alpha|\otimes T^\alpha = \langle\mathcal{J}_\alpha|\otimes\mathcal{S}_{-1}(T^\alpha) + \langle\chi|\otimes\mathbb{1}$$

where $\mathcal{S}_{-1}(T^A_n) = T^A_{n-1}$ and $\mathcal{S}_{-1}(L_n) = L_{n-1}$ and $\mathcal{S}_{-1}(\mathbb{1}) = 0$

$\langle\chi|$ is a new constrained field (like any derivative $\langle\partial|$).

The dual graviton

Einstein equation can be written as the duality equation

$$2g^{mp}\partial_{[n_1}g_{n_2]p} = \frac{1}{9!\sqrt{-g}}g_{n_1q_1}g_{n_2q_2}\varepsilon^{q_1q_2p_1\dots p_9}g^{mq}Y_{p_1\dots p_9;q}$$

which is a tautology using [West–Boulanger–Hohm]

$$\varepsilon^{n_1\dots n_{10}(m}\partial_{n_1}\left(g^{n)p}Y_{n_2\dots n_{10};p} - 10g^{n)p}Y_{[n_2\dots n_{10};p]}\right) = \frac{\partial\mathcal{L}_{\rm B}}{\partial g_{mn}}$$

with

$$\mathcal{L}_{\rm B} = -\frac{1}{2}\sqrt{-g}g^{n_1p_1}\dots g^{n_9p_9}g^{mq}\left(\frac{1}{9!}Y_{n_1\dots n_9;m}Y_{p_1\dots p_9;q} - \frac{1}{8!}Y_{[n_1\dots n_9;m]}Y_{[p_1\dots p_9;q]}\right) - \frac{1}{4}\sqrt{-g}g^{mn}g^{q[p}g^{r]s}\partial_p g_{qm}\partial_r g_{sn}$$

Only in the linearised approximation

$$\varepsilon^{n_1\dots n_{10}(m}\partial_{n_1}\left(g^{n)p}Y_{n_2\dots n_{10};p} - 10g^{n)p}Y_{[n_2\dots n_{10};p]}\right) = 0$$

$$\hookrightarrow \quad Y_{n_1\dots n_9;m} = 9\partial_{[n_1}h_{n_2\dots n_9],m} - \partial_m X_{n_1\dots n_9}$$

The dual graviton

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$$2g^{mp}\partial_{[n_1}g_{n_2]p} = \frac{1}{9!\sqrt{-g}}g_{n_1q_1}g_{n_2q_2}\varepsilon^{q_1q_2p_1\dots p_9}g^{mq}Y_{p_1\dots p_9;q}$$

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with

$$\mathcal{L}_{\rm B} = -\frac{1}{2}\sqrt{-g}g^{n_1p_1}\dots g^{n_9p_9}g^{mq}\left(\frac{1}{9!}Y_{n_1\dots n_9;m}Y_{p_1\dots p_9;q} - \frac{1}{8!}Y_{[n_1\dots n_9;m]}Y_{[p_1\dots p_9;q]}\right) - \frac{1}{4}\sqrt{-g}g^{mn}g^{q[p}g^{r]s}\partial_p g_{qm}\partial_r g_{sn}$$

In general

$$\varepsilon^{n_1\dots n_{10}(m}\partial_{n_1}\left(g^{n)p}Y_{n_2\dots n_{10};p} - 10g^{n)p}Y_{[n_2\dots n_{10};p]}\right) \neq 0$$

$$\hookrightarrow \quad Y_{n_1\dots n_9;m} = 9\partial_{[n_1}h_{n_2\dots n_9],m} - \chi_{m;n_1\dots n_9}$$

The tensor hierarchy

	Weight	E_6	E_7	E_8
$C^{X}_{\mu\nu\sigma\rho}$	$R(\Lambda_3)$	$\mathbf{351}$		
$C^{\Xi}_{\mu\nu\sigma}$	$R(\Lambda_2)$	$\mathbf{78}$	$\mathbf{912}$	
$B^{\Lambda}_{\mu\nu}$	$R(\Lambda_1)$	$\mathbf{27}$	$\mathbf{133}$	$\mathbf{3875} \oplus \mathbf{1}$
A^{M}_{μ}	$R(\Lambda_d)$	$\overline{\mathbf{27}}$	$\mathbf{56}$	$\mathbf{248}$
ϕ_α	$\mathfrak{e}_d$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$
$\Theta_{M\alpha}$	$\mathcal{T}_{-1}$	$\overline{\mathbf{351}}$	$\mathbf{912}$	$\mathbf{3875} \oplus \mathbf{1}$

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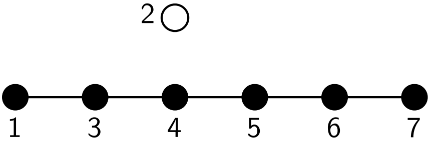
ϕ_α	$\mathfrak{e}_d$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$
$\Theta_{M\alpha}$	$\mathcal{T}_{-1}$	$\overline{\mathbf{351}}$	$\mathbf{912}$	$\mathbf{3875} \oplus \mathbf{1}$

The tensor hierarchy superalgebra

[Palmkvist]

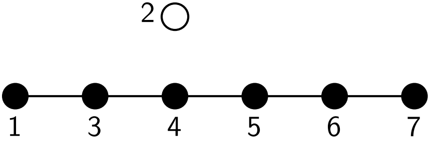
	Weight	E_6	E_7	E_8
P^X	$R(\Lambda_3)$	$\mathbf{351}$	$\mathbf{8645} \oplus \ldots$	$\mathbf{6696000} \oplus \ldots$
P^Ξ	$R(\Lambda_2)$	$\mathbf{78}$	$\mathbf{912}$	$\mathbf{147250} \oplus \ldots$
P^Λ	$R(\Lambda_1)$	$\mathbf{27}$	$\mathbf{133}$	$\mathbf{3875} \oplus \mathbf{1}$
P^M	$R(\Lambda_d)$	$\overline{\mathbf{27}}$	$\mathbf{56}$	$\mathbf{248}$
T^α	$\mathfrak{e}_d$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$
$\bar{P}_{M\alpha}$	$\mathcal{T}_{-1}$	$\overline{\mathbf{351}}$	$\mathbf{912}$	$\mathbf{3875} \oplus \mathbf{1}$

$GL(7,\mathbb{R}) \subset E_{7(7)}$



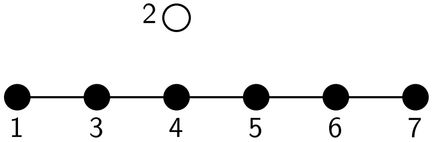
$$
\begin{array}{rcl}
\mathfrak{e}_{7(7)} & \cong & \mathbf{7}^{(-4)} \oplus \overline{\mathbf{56}}^{(-2)} \oplus \left(\mathfrak{gl}_1 \oplus \mathfrak{sl}_7\right)^{(0)} \oplus \mathbf{56}^{(2)} \oplus \overline{\mathbf{7}}^{(4)} \quad , \\
\mathcal{M}_{MN} & & \qquad\qquad\qquad \left(M_{IJ}\right)^{(0)} \oplus \left(C_{IJK}\right)^{(2)} \oplus \left(\tilde{C}^{I}\right)^{(4)} \quad , \\
\mathbf{56} & \cong & \overline{\mathbf{7}}^{(-3)} \oplus \mathbf{21}^{(-1)} \oplus \overline{\mathbf{21}}^{(1)} \oplus \mathbf{7}^{(3)} \quad , \\
A_{\mu}^{M} & & (\tilde{h}_I)^{(3)} \oplus (\tilde{C}^{IJ})^{(1)} \oplus (C_{IJ})^{(-1)} \oplus (h^I)^{(-3)} \quad , \\
\mathbf{133} & \cong & \mathbf{7}^{(-4)} \oplus \overline{\mathbf{56}}^{(-2)} \oplus (\mathbf{1} \oplus \mathbf{48})^{(0)} \oplus \mathbf{56}^{(2)} \oplus \overline{\mathbf{7}}^{(4)} \quad , \\
B_{\mu\nu}^{\alpha} & & \quad \cdots \oplus (C_{IJK}^{+\nabla})^{(2)} \oplus (\tilde{h}_I{}^{J})^{(0)} \oplus (\tilde{C}_{IJKL})^{(-2)} \oplus (C_I)^{(-4)} \quad ,
\end{array}
$$

$GL(7,\mathbb{R}) \subset E_{7(7)}$



$$
\begin{aligned}
\mathfrak{e}_{7(7)} &\cong \mathbf{7}^{(-4)} \oplus \overline{\mathbf{56}}^{(-2)} \oplus \left(\mathfrak{gl}_1 \oplus \mathfrak{sl}_7\right)^{(0)} \oplus \mathbf{56}^{(2)} \oplus \overline{\mathbf{7}}^{(4)} \ , \\
\mathcal{M}_{MN} & \qquad\qquad\qquad\qquad\qquad (M_{IJ})^{(0)} \oplus (C_{IJK})^{(2)} \oplus (\tilde{C}^{I})^{(4)} \ , \\
\mathcal{E} &\cong TM^* \otimes \wedge^7 TM^* \oplus \wedge^5 TM^* \oplus \wedge^2 TM^* \oplus TM \ , \\
A_\mu^M & \quad (\tilde{h}_I)^{(3)} \oplus (\tilde{C}^{IJ})^{(1)} \oplus (C_{IJ})^{(-1)} \oplus (h^I)^{(-3)} \ , \\
\mathbf{133} &\cong \mathbf{7}^{(-4)} \oplus \overline{\mathbf{56}}^{(-2)} \oplus (\mathbf{1} \oplus \mathbf{48})^{(0)} \oplus \mathbf{56}^{(2)} \oplus \overline{\mathbf{7}}^{(4)} \ , \\
B_{\mu\nu}^{\alpha} & \quad \cdots \oplus (C^{+\nabla}_{IJK})^{(2)} \oplus (\tilde{h}_I{}^J)^{(0)} \oplus (\tilde{C}_{IJKL})^{(-2)} \oplus (C_I)^{(-4)} \ ,
\end{aligned}
$$

$GL(7,\mathbb{R}) \subset E_{7(7)}$



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\mathfrak{e}_{7(7)} & \cong & \mathbf{7}^{(-4)} \oplus \overline{\mathbf{56}}^{(-2)} \oplus \left(\mathfrak{gl}_1 \oplus \mathfrak{sl}_7\right)^{(0)} \oplus \mathbf{56}^{(2)} \oplus \overline{\mathbf{7}}^{(4)} \quad , \\
\mathcal{M}_{MN} & & \qquad\qquad\qquad\qquad (M_{IJ})^{(0)} \oplus (C_{IJK})^{(2)} \oplus (\tilde{C}^{I})^{(4)} \ , \\
\mathbf{56} & \cong & \overline{\mathbf{7}}^{(-3)} \oplus \mathbf{21}^{(-1)} \oplus \overline{\mathbf{21}}^{(1)} \oplus \mathbf{7}^{(3)} \ , \\
A_\mu^M & & ({\color{red}\tilde{h}_I})^{(3)} \oplus (\tilde{C}^{IJ})^{(1)} \oplus (C_{IJ})^{(-1)} \oplus (h^I)^{(-3)} \ , \\
\mathbf{133} & \cong & \mathbf{7}^{(-4)} \oplus \overline{\mathbf{56}}^{(-2)} \oplus (\mathbf{1} \oplus \mathbf{48})^{(0)} \oplus \mathbf{56}^{(2)} \oplus \overline{\mathbf{7}}^{(4)} \ , \\
B_{\mu\nu}^{\alpha} & & \quad \cdots \oplus (C_{IJK}^{+\nabla})^{(2)} \oplus ({\color{red}\tilde{h}_I{}^J})^{(0)} \oplus (\tilde{C}_{IJKL})^{(-2)} \oplus (C_I)^{(-4)} \ ,
\end{array}
$$

The tensor hierarchy

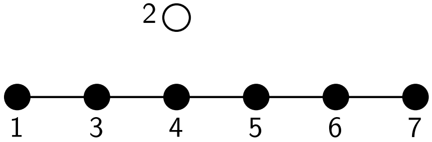
[Hohm–Samtleben]

	Weight	E_6	E_7	E_8
$B_{\mu\nu}^{\Lambda}$	$R(\Lambda_1)$	**27**	$\mathbf{133} \oplus \mathbf{56}$	$\mathbf{3875} \oplus \mathbf{1} \oplus \mathbf{248} \otimes \mathbf{248}$
A_{μ}^{M}	$R(\Lambda_d)$	$\overline{\mathbf{27}}$	**56**	$\mathbf{248} \oplus \mathbf{248}$
ϕ_α	$\mathfrak{e}_d$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$
$\Theta_{M\alpha}$	$\mathcal{T}_{-1}$	$\overline{\mathbf{351}}$	**912**	$\mathbf{3875} \oplus \mathbf{1}$

$$\eta_{\alpha\beta}(T^\alpha)^P{}_M(T^\beta)^Q{}_N\partial_P F\,\chi_Q = \partial_N F\,\chi_M + \tfrac{1}{9-d}\partial_M F\,\chi_N$$

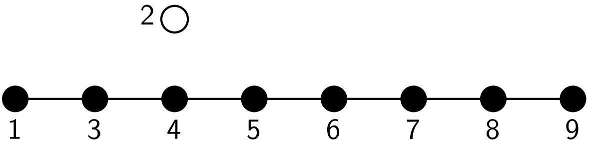
$$\eta_{\alpha\beta}(T^\alpha)^P{}_M(T^\beta)^Q{}_N\,\chi_P\,\chi_Q = \chi_N\,\chi_M + \tfrac{1}{9-d}\,\chi_M\,\chi_N$$

$GL(7,\mathbb{R}) \subset E_{7(7)}$



$$
\begin{array}{rcl}
\mathfrak{e}_{7(7)} & \cong & \mathbf{7}^{(-4)} \oplus \overline{\mathbf{56}}^{(-2)} \oplus \left(\mathfrak{gl}_1 \oplus \mathfrak{sl}_7\right)^{(0)} \oplus \mathbf{56}^{(2)} \oplus \overline{\mathbf{7}}^{(4)} \quad , \\
\mathcal{M}_{MN} & & \left(M_{IJ}\right)^{(0)} \oplus \left(C_{IJK}\right)^{(2)} \oplus \left(\tilde{C}^{I}\right)^{(4)} \quad , \\
\mathbf{56} & \cong & \overline{\mathbf{7}}^{(-3)} \oplus \mathbf{21}^{(-1)} \oplus \overline{\mathbf{21}}^{(1)} \oplus \mathbf{7}^{(3)} \quad , \\
A_\mu^M & & (\tilde{h}_I)^{(3)} \oplus (\tilde{C}^{IJ})^{(1)} \oplus (C_{IJ})^{(-1)} \oplus (h^I)^{(-3)} \quad , \\
B_{\mu\nu M} & & (Y_I)^{(3)} \oplus (0)^{(1)} \oplus (0)^{(-1)} \oplus (0)^{(-3)} \quad , \\
\mathbf{133} & \cong & \mathbf{7}^{(-4)} \oplus \overline{\mathbf{56}}^{(-2)} \oplus (\mathbf{1} \oplus \mathbf{48})^{(0)} \oplus \mathbf{56}^{(2)} \oplus \overline{\mathbf{7}}^{(4)} \quad , \\
B_{\mu\nu}^{\alpha} & & \cdots \oplus (C_{IJK}^{+\nabla})^{(2)} \oplus (\tilde{h}_I{}^{J})^{(0)} \oplus (\tilde{C}_{IJKL})^{(-2)} \oplus (C_I)^{(-4)} \quad , \\
C_{\mu\nu\sigma M}{}^{N} & & \cdots \oplus (Y_{I;JK})^{(2)} \oplus (Y_I{}^{J})^{(0)} \oplus (0)^{(-2)} \oplus (0)^{(-4)} \quad ,
\end{array}
$$

$GL(9,\mathbb{R}) \subset E_9$

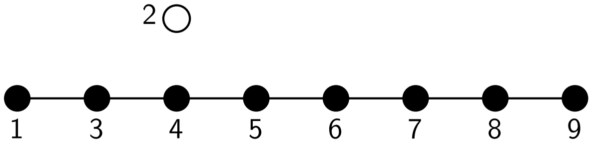


$$
\begin{array}{rcl}
\mathfrak{e}_9 & \cong & \cdots \oplus \left(\mathfrak{gl}_1 \oplus \mathfrak{sl}_9\right)^{(0)} \oplus \mathbf{84}^{(2)} \oplus \overline{\mathbf{84}}^{(4)} \oplus \overline{\mathbf{80}}^{(6)} \oplus \mathbf{84}^{(8)} \oplus \dots \quad , \\
\mathcal{M}_{MN} & & \qquad M_{IJ} \qquad \oplus C_{IJK} \oplus \tilde{C}^{IJK} \oplus \tilde{h}_I{}^J \oplus C^{+\nabla}_{IJK} \oplus \dots \; , \\
R(\Lambda_9) & \cong & \overline{\mathbf{9}}^{(1)} \oplus \mathbf{36}^{(3)} \oplus \mathbf{126}^{(5)} \oplus (\mathbf{9} \otimes \overline{\mathbf{36}})^{(7)} \oplus \dots \; , \\
|A_\mu\rangle & & \cdots \oplus (\tilde{h}_I{}^{JK})^{(-7)} \oplus (\tilde{C}_{IJKLP})^{(-5)} \oplus (C_{IJ})^{(-3)} \oplus (h^I)^{(-1)} \; ,
\end{array}
$$

with gradient duals

$$
\partial_m A_{n_1 n_2 n_3} = \frac{1}{10!\sqrt{-g}} g_{mq} \varepsilon^{q p_1 \dots p_{10}} F_{p_1 \dots p_{10}; n_1 n_2 n_3}
$$

$GL(9,\mathbb{R}) \subset E_9$



$$
\begin{aligned}
\langle \chi | \quad & \quad (Y_I)^{(1)} \oplus (0)^{(3)} \oplus (0)^{(5)} \oplus (0)^{(7)} \oplus \dots \; , \\
\mathfrak{e}_9 \;\cong\; & \cdots \oplus \left(\mathfrak{gl}_1 \oplus \mathfrak{sl}_9\right)^{(0)} \oplus \mathbf{84}^{(2)} \oplus \overline{\mathbf{84}}^{(4)} \oplus \overline{\mathbf{80}}^{(6)} \oplus \mathbf{84}^{(8)} \oplus \dots \; , \\
\mathcal{M}_{MN} \quad & \quad M_{IJ} \oplus C_{IJK} \oplus \tilde{C}^{IJK} \oplus \tilde{h}_I{}^J \oplus C^{+\nabla}_{IJK} \oplus \dots \; , \\
|B_\mu\rangle\langle B| \quad & \cdots \oplus (Y_I{}^{JKLP})^{(-10)} \oplus (Y_{I;JK})^{(-8)} \oplus (Y_I{}^J)^{(-6)} \; , \\
R(\Lambda_9) \;\cong\; & \overline{\mathbf{9}}^{(1)} \oplus \mathbf{36}^{(3)} \oplus \mathbf{126}^{(5)} \oplus (\mathbf{9} \otimes \overline{\mathbf{36}})^{(7)} \oplus \dots \; , \\
|A_\mu\rangle \quad & \cdots \oplus (\tilde{h}_I{}^{JK})^{(-7)} \oplus (\tilde{C}_{IJKLP})^{(-5)} \oplus (C_{IJ})^{(-3)} \oplus (h^I)^{(-1)} \; ,
\end{aligned}
$$

Affine symmetry

$\mathfrak{e}_9$ includes the central extension of the loop algebra over $\mathfrak{e}_8$

$$\begin{aligned} \left[T^A_m, T^B_n\right] &= f^{AB}{}_C\, T^C_{m+n} + \eta^{AB} m\, \delta_{m+n,0}\, \mathbb{1} \ , \\ \left[L_0, T^A_m\right] &= -m T^A_m \ , \end{aligned}$$

with $f^{AC}{}_D f^{BD}{}_C = 60\, \eta^{AB}$.

One defines

$$\eta_{m\alpha\beta} = \sum_{n\in\mathbb{Z}} \eta_{AB}\, T^A_n \otimes T^B_{m-n} - L_m \otimes \mathbb{1} - \mathbb{1} \otimes L_m$$

and

$$\mathcal{S}_m(T^A_n) = T^A_{m+n} \ , \qquad \mathcal{S}_m(L_n) = L_{m+n} \ , \qquad \mathcal{S}(\mathbb{1}) = 0 \ .$$

Duality equations

One can then define the current

$$J = \mathcal{M}^{-1}(d - \langle \partial_{\mathcal{M}} | A \rangle)\mathcal{M} - \eta_{\alpha\beta} \langle \partial_A | T^\alpha | A \rangle (T^\beta + \mathcal{M}^{-1}(T^\beta)^\dagger \mathcal{M}) \\ - \eta_{-1\alpha\beta} \mathrm{tr}\,[T^\alpha B] T^\beta$$

and the duality equation in $\mathfrak{e}_9 \oplus \langle L_{-1}, L_1^* \rangle \cong \widehat{e}_8 \oplus \langle L_0, L_{-1}, L_1^* \rangle$

$$J = \rho^{-1} \star \mathcal{M}^{-1}(\mathcal{S}_1(J) + \chi \mathbb{1})^\dagger \mathcal{M} \ .$$

with the $\mathfrak{e}_9$ invariant bilinear form

$$C_2(J, \chi) = \sum_{n \in \mathbb{Z}} \eta^{AB} J_A^n J_B^{-1-n} - 2 J_{-1} J_K - 2 J_0 \chi$$

The field strength

$$|F\rangle = \varepsilon_{\mu\nu} dx^\mu \wedge dx^\nu \left(\mathcal{S}_{-1}(T^\alpha) \mathcal{M}^{-1} | J_\alpha \rangle + \mathcal{M}^{-1} |\chi\rangle \right) \ .$$

E_{11} tensor hierarchy superalgebra

[B–Kleinschmidt–Palmkvist–Pope–Sezgin]

	E_{11} weight
$\eta_{\alpha\beta}(T^\alpha)^P{}_M(T^\beta)^Q{}_N - \delta_M^Q\delta_N^P + \frac{1}{2}\delta_M^P\delta_N^Q$	$\overline{R(\Lambda_1)} \oplus \overline{R(\Lambda_8)} \oplus \dots$
∂_M	$\overline{R(\Lambda_{11})}$
$B_\alpha,\ B_{[MN]},\ B_{(MN)}$	$\mathfrak{e}_{11} \oplus \overline{R(\Lambda_{10})} \oplus \overline{R(\Lambda_1)}$
F^I	$\mathcal{T}_{-1}$
$\phi^\alpha,\ X^{[MN]},\ Y^{(MN)}$	$\mathfrak{e}_{11} \oplus R(\Lambda_{10}) \oplus R(\Lambda_1)$
ξ^M	$R(\Lambda_{11})$

E_{11} tensor hierarchy superalgebra

[B–Kleinschmidt–Palmkvist–Pope–Sezgin]

	E_{11} weight
$d^2 \propto \eta_{\alpha\beta}(T^\alpha)^P{}_M(T^\beta)^Q{}_N\partial_P\partial_Q - \partial_N\partial_M + \frac{1}{2}\partial_M\partial_N$	$\overline{R(\Lambda_1)} \oplus \overline{R(\Lambda_8)} \oplus \ldots$
$d \uparrow$	
$d = \mathrm{ad}(P^M)\partial_M$	$\overline{R(\Lambda_{11})}$
$B_\alpha,\ B_{[MN]},\ B_{(MN)}$	$\mathfrak{e}_{11} \oplus \overline{R(\Lambda_{10})} \oplus \overline{R(\Lambda_1)}$
$d \uparrow$	
F^I	$\mathcal{T}_{-1}$
$d \uparrow$	
$\phi^\alpha,\ X^{[MN]},\ Y^{(MN)}$	$\mathfrak{e}_{11} \oplus R(\Lambda_{10}) \oplus R(\Lambda_1)$
$d \uparrow$	
ξ^M	$R(\Lambda_{11})$

E_{11} duality equation

One can then define the field strength

$$F^I = C^{IM}{}_\alpha J^\alpha_M + C^{IM}{}_{PQ} \chi_M{}^{PQ}$$

and the duality equation in $\mathcal{T}_{-1}(\mathfrak{e}_{11})$

$$F^I = \mathcal{M}^{IK} \omega_{KJ} F^J \ .$$

Invariant under generalised diffeomorphisms

$$\delta \mathcal{M} = \xi^M \partial_M \mathcal{M} + \eta_{\alpha\beta} T^{\alpha M}{}_N \partial_M \xi^N (\mathcal{M} t^\beta + (t^\beta)^\dagger \mathcal{M})$$

Supersymmetric E_{11} duality equation

One can then define the field strength

$$F^I = C^{IM}{}_\alpha J^\alpha_M + C^{IM}{}_{PQ} \chi_M{}^{PQ}$$

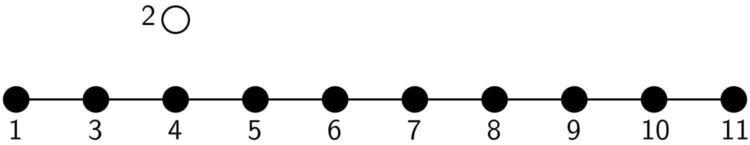
and the duality equation in $\mathcal{T}_{-1}(\mathfrak{e}_{11})$

$$F^I = \mathcal{M}^{IK} \omega_{KJ} F^J + \mathcal{V}^{-1I}{}_{\underline{J}} \Psi \Gamma^{\underline{J}} \Psi \ .$$

Invariant under generalised diffeomorphisms

$$\delta \mathcal{M} = \xi^M \partial_M \mathcal{M} + \eta_{\alpha\beta} T^{\alpha M}{}_N \partial_M \xi^N (\mathcal{M} t^\beta + (t^\beta)^\dagger \mathcal{M})$$

E_{11} duality equation in eleven dimensions

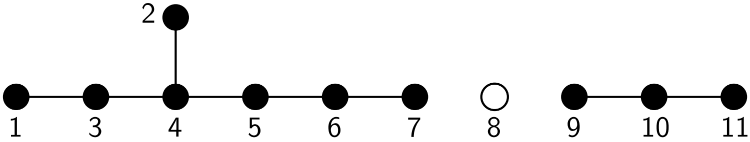


$$F^I = \mathcal{M}^{IK}\omega_{KJ}F^J + \mathcal{V}^{-1I}{}_{\underline{J}}\Psi\Gamma^{\underline{J}}\Psi \ .$$

gives

$$\begin{aligned}
\star\hat{F}_4 &= \hat{F}_7 \ , \\
2g^{mp}\widehat{\partial_{[n_1}g_{n_2]p}} &= \frac{1}{9!\sqrt{-g}} g_{n_1q_1}g_{n_2q_2}\varepsilon^{q_1q_2p_1\ldots p_9}g^{mq}\hat{F}_{p_1\ldots p_9;q} \\
\partial_m\widehat{A_{n_1n_2n_3}} &= \frac{1}{10!\sqrt{-g}} g_{mq}\varepsilon^{qp_1\ldots p_{10}}\hat{F}_{p_1\ldots p_{10};n_1n_2n_3} \\
&\ldots
\end{aligned}$$

E_{11} duality equation in four dimensions



$$F^I = \mathcal{M}^{IK}\omega_{KJ}F^J + \mathcal{V}^{-1I}{}_{\underline{J}}\Psi\Gamma^{\underline{J}}\Psi \ .$$

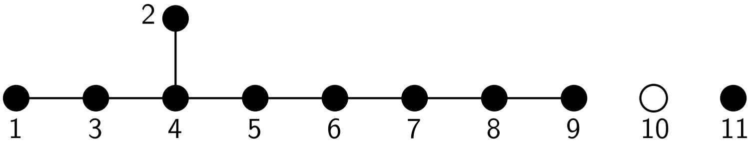
gives [Cremmer–Julia]

$$F_2^I = \mathcal{M}^{IK}\omega_{KJ} \star F_2^J + \mathcal{V}^{-1I}{}_{ij} e^a \wedge e^b O^{ij} + \mathcal{V}^{-1Iij} e^a \wedge e^b O_{ij}$$

with

$$O_{ab}^{ij} = \bar{\psi}_{[a}^{[i}\psi_{b]}^{j]} + \frac{i}{2}\varepsilon_{ab}{}^{cd}\bar{\psi}_c^{[i}\psi_d^{j]} + \frac{1}{2}\bar{\psi}_{ck}\gamma_{ab}\gamma^c\chi^{ijk} + \frac{1}{72}\varepsilon^{ijklpqrs}\bar{\chi}_{klp}\gamma_{ab}\chi_{qrs}$$

E_{11} duality equation in two dimensions

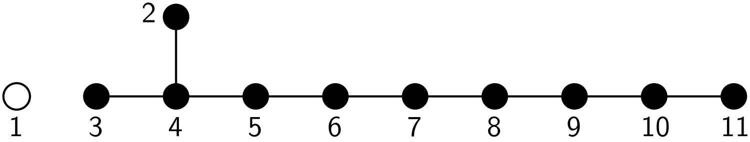


$$F^I = \mathcal{M}^{IK}\omega_{KJ}F^J \ .$$

gives

$$\begin{aligned} J &= \rho^{-1} \star \mathcal{M}^{-1}(\mathcal{S}_1(J) + \chi \mathbb{1})^\dagger \mathcal{M} \\ |F\rangle &= \varepsilon_{\mu\nu} dx^\mu \wedge dx^\nu \left(\mathcal{S}_{-1}(T^\alpha)\mathcal{M}^{-1}|J_\alpha\rangle + \mathcal{M}^{-1}|\chi\rangle\right) \end{aligned}$$

E_{11} duality equation in double field theory



$$F^I = \mathcal{M}^{IK}\omega_{KJ}F^J\ .$$

gives [Hohm–Kwak–Zwiebach]

$$\begin{aligned}(1+\mathcal{H})\partial\!\!\!/ C &= 0\\ 3\mathcal{H}_{[M}{}^{Q}\mathcal{H}_{N|}{}^{R}\partial_R\mathcal{H}_{|P]Q} &= \partial^Q N_{MNPQ} + \frac{1}{16}\bar{C}\Gamma_{MNPQ}\partial^Q C + \chi_{[M;NP]}\end{aligned}$$

Conclusion

- ⋆ E_9 Exceptional field theory
  - ↪ Explicite action to come
  - ↪ Relation to Breitenlohner–Maison linear system
  - ↪ Search for new integrable systems in 11D supergravity
- ⋆ Construction of West E_{11} theory generalising Hohm–Samtleben
  - ↪ Proof of some algebraic identities
  - ↪ Representation theory of $\widetilde{K}(E_{11})$
  - ↪ Action and constrained χ fields equations