University of Burgos

LORENTZIAN AND GALILEAN SNYDER SPACETIMES FROM PROJECTIVE GEOMETRY

Giulia Gubitosi
University of Burgos

based on work with A. Ballesteros and F. Herranz
arXiv:1909.xxxxx

Corfu - September 2019

Snyder's model for noncommutative spacetime (est. 1947)

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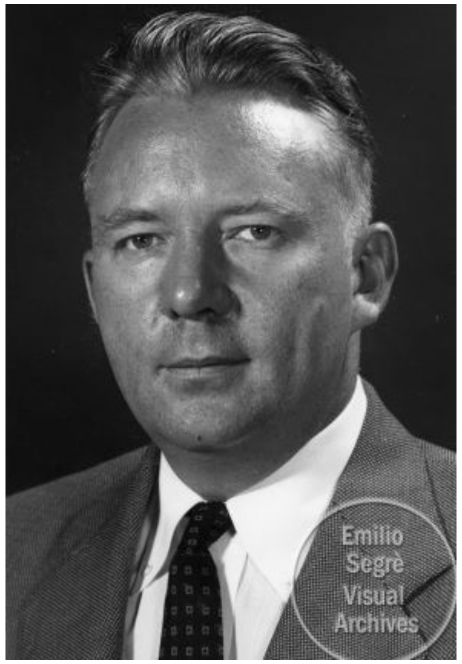

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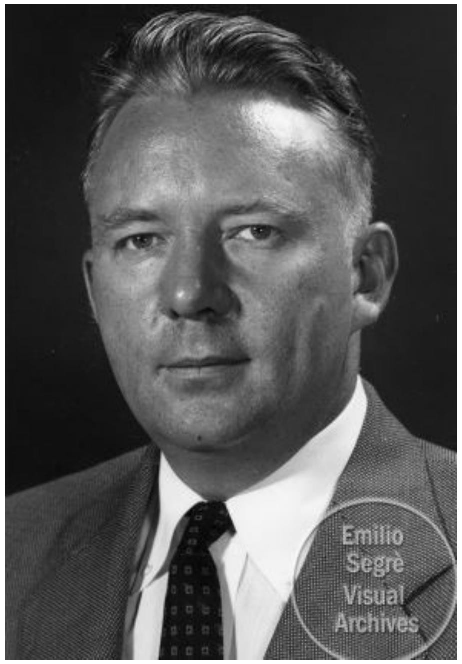

SEPTEMBER 1, 1939 PHYSICAL REVIEW VOLUME 56

On Continued Gravitational Contraction

J. R. Oppenheimer and H. Snyder
University of California, Berkeley, California

(Received July 10, 1939)

When all thermonuclear sources of energy are exhausted a sufficiently heavy star will collapse. Unless fission due to rotation, the radiation of mass, or the blowing off of mass by radiation, reduce the star's mass to the order of that of the sun, this contraction will continue indefinitely. In the present paper we study the solutions of the gravitational field equations which describe this process. In I, general and qualitative arguments are given on the behavior of the metrical tensor as the contraction progresses: the radius of the star approaches asymptotically its gravitational radius; light from the surface of the star is progressively reddened, and can escape over a progressively narrower range of angles. In II, an analytic solution of the field equations confirming these general arguments is obtained for the case that the pressure within the star can be neglected. The total time of collapse for an observer comoving with the stellar matter is finite, and for this idealized case and typical stellar masses, of the order of a day; an external observer sees the star asymptotically shrinking to its gravitational radius.

[a long-forgotten paper, rediscovered in the '60s]

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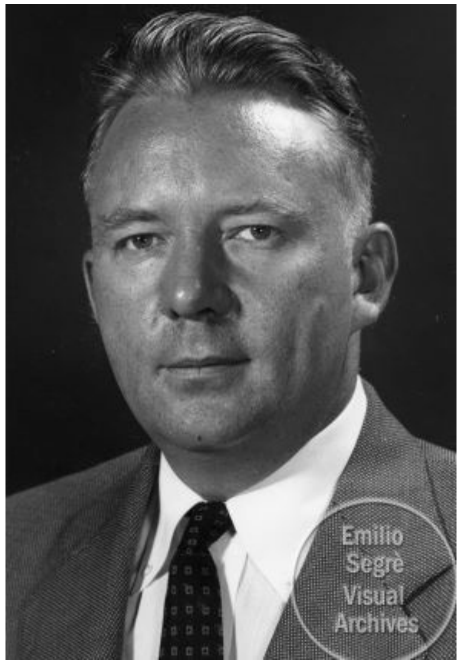

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with a few collaborators, he developed the technique of strong-focussing (1952), which allows to focus particles in accelerators by alternating the gradient of the magnetic field

At the end of last month the very sad news reached us that Dr. Hartland Snyder had died. To us in CERN, Snyder's name is for ever tied to the discovery of the alternating-gradient focusing principle. He shared the honour of this discovery with his colleagues Dr. E.D. Courant and Dr. M.S. Livingston, and with Dr. N.C. Christofilos. Everybody knows what this has meant to CERN. We can only remind ourselves that we were planning a 10-GeV weak-focused synchrotron, a machine at least as expensive and more difficult to make than the PS, when in 1952 we learnt about the elegant new ideas from Brookhaven. This changed the CERN plans entirely, and perhaps also the CERN spirit, as we got a very much more exciting project to concentrate on.

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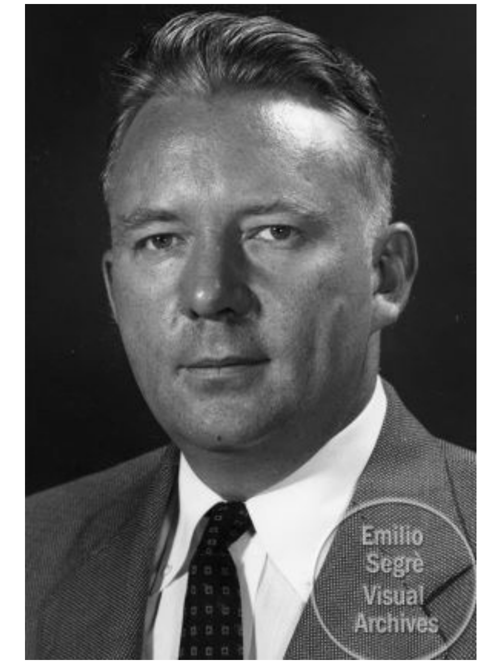

PHYSICAL REVIEW VOLUME 71, NUMBER 1 JANUARY 1, 1947

Quantized Space-Time

HARTLAND S. SNYDER
Department of Physics, Northwestern University, Evanston, Illinois
(Received May 13, 1946)

It is usually assumed that space-time is a continuum. This assumption is not required by Lorentz invariance. In this paper we give an example of a Lorentz invariant discrete space-time.

motivated by the hope that the discreteness induced by spacetime noncommutativity could avoid divergencies in quantum field theory (issue later solved by renormalisation)

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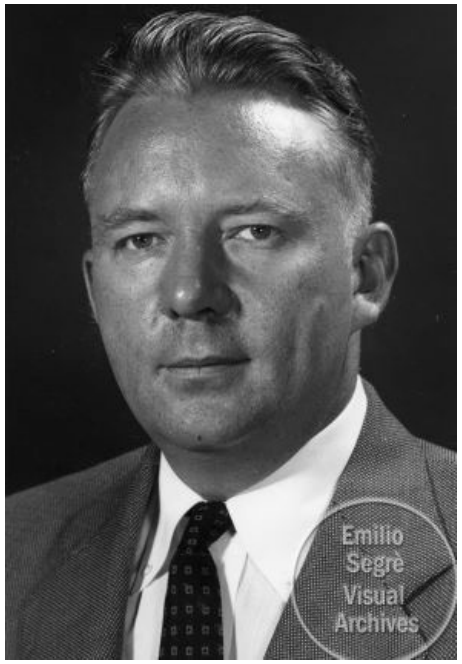

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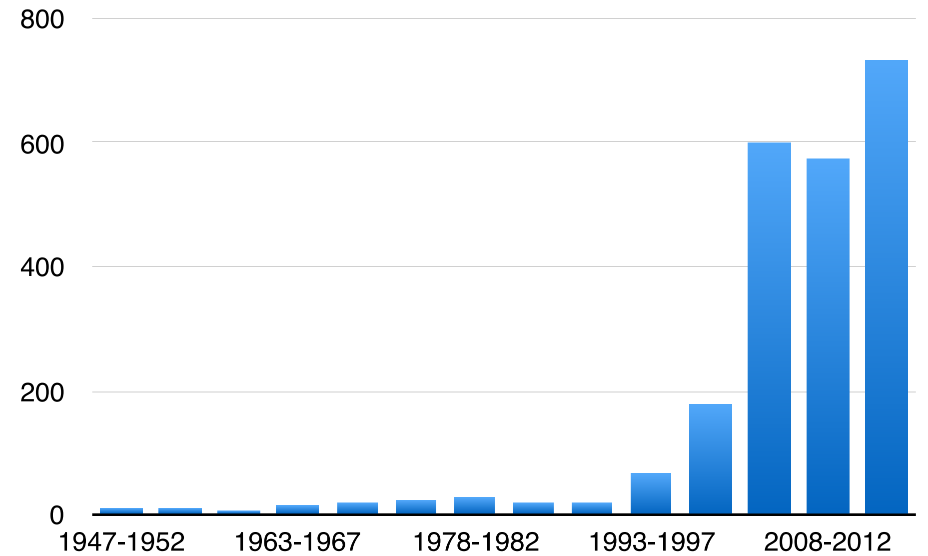

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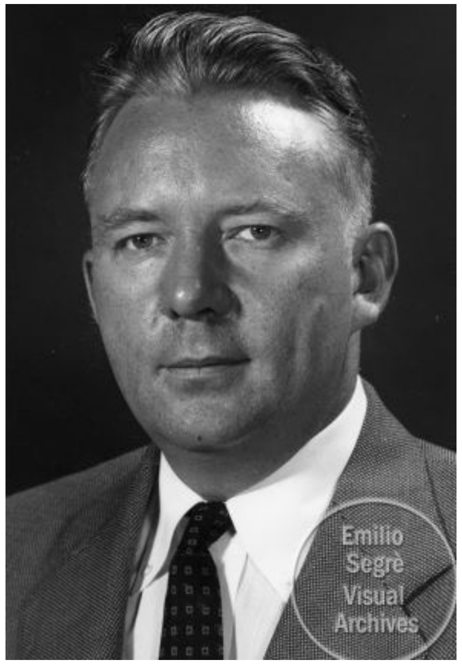

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PHYSICAL REVIEW D VOLUME 55, NUMBER 12 15 JUNE 1997

Minimal length uncertainty relation and ultraviolet regularization

Achim Kempf*
Department of Applied Mathematics & Theoretical Physics and Corpus Christi College, University of Cambridge, Cambridge CB3 9EW, United Kingdom

Gianpiero Mangano†
INFN, Sezione di Napoli, and Dipartimento di Scienze Fisiche, Università di Napoli Federico II, I-80125 Napoli, Italy

(Received 9 December 1996)

Studies in string theory and quantum gravity suggest the existence of a finite lower limit Δx_0 to the possible resolution of distances, at the latest on the scale of the Planck length of 10^{-35} m. Within the framework of the Euclidean path integral we explicitly show ultraviolet regularization in field theory through this short distance structure. Both rotation and translation invariance can be preserved. An example is studied in detail.

JHEP HYPER VERSION RECEIVED: *September 30, 1999,* ACCEPTED: *September 30, 1999*

String theory and noncommutative geometry

Nathan Seiberg and Edward Witten
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Snyder's model for noncommutative spacetime (1947)

✦ The main focus is on not spoiling special-relativistic Lorentz invariance

$$[J_i, J_j] = \epsilon_{ijk} J_k, \qquad [J_i, K_j] = \epsilon_{ijk} K_k, \qquad [K_i, K_j] = -\epsilon_{ijk} J_k$$

→ assume Lorentz-invariant commutators between spacetime coordinates

$$[x_0, x_i] = i a^2 K_i \,, \qquad [x_i, x_j] = i a^2 \epsilon_{ijk} J_k$$

[a is a length scale, $a \sim L_P$, and we use $c = \hbar = 1$]

✦ Momenta are defined as objects that transform as vectors under Lorentz transformations

$$\begin{aligned}
[p_\alpha, p_\beta] &= 0 \\
[p_0, K_i] &= -i p_i \\
[p_i, K_j] &= -i \delta_{ij} p_0 \\
[p_0, J_i] &= 0 \\
[p_j, J_i] &= -i \epsilon^{ijk} p_k \,.
\end{aligned}$$

• *Snyder, Phys. Rev. 1947*

Snyder deformed phase space

✦ Noncommutative coordinates and momenta close a deformed phase space algebra

$$\begin{aligned} [x_i, p_j] &= i(\delta_{ij} + a^2 p_i p_j) \\ [x_0, p_0] &= i(1 - a^2 p_0^2) \\ [x_i, p_0] &= i a^2 p_0 p_i \\ [x_0, p_i] &= -i a^2 p_0 p_i \,. \end{aligned}$$

• *Snyder, Phys. Rev. 1947*

such that the representation of Lorentz generators in undeformed

$$K_i = x_i p_0 + x_0 p_i \,, \qquad J_i = \epsilon_{ijk} x_j p_k$$

✦ This model has been largely studied

noncommutative geometry, generalised uncertainty principle, deformed special relativity, hydrogen atom, harmonic oscillator, path integral, …

Snyder deformed phase space and GUP

✦ One of the most relevant implications is a deformed Heisenberg uncertainty principle and a minimum length uncertainty

$$\Delta x \Delta p \geq \frac{1}{2} |\langle [x, p] \rangle|$$

in 1+1 dimensions:

$$\Delta x \Delta p \geq \frac{1}{2} \left(1 + a^2 \langle p^2 \rangle\right) = \frac{1}{2} \left(1 + a^2 (\Delta p)^2 - a^2 \langle p \rangle^2\right)$$

$$\rightarrow \qquad \Delta x \geq \frac{1}{2} \left(\frac{1 - a^2 \langle p \rangle^2}{\Delta p} + a^2 \Delta p \right)$$

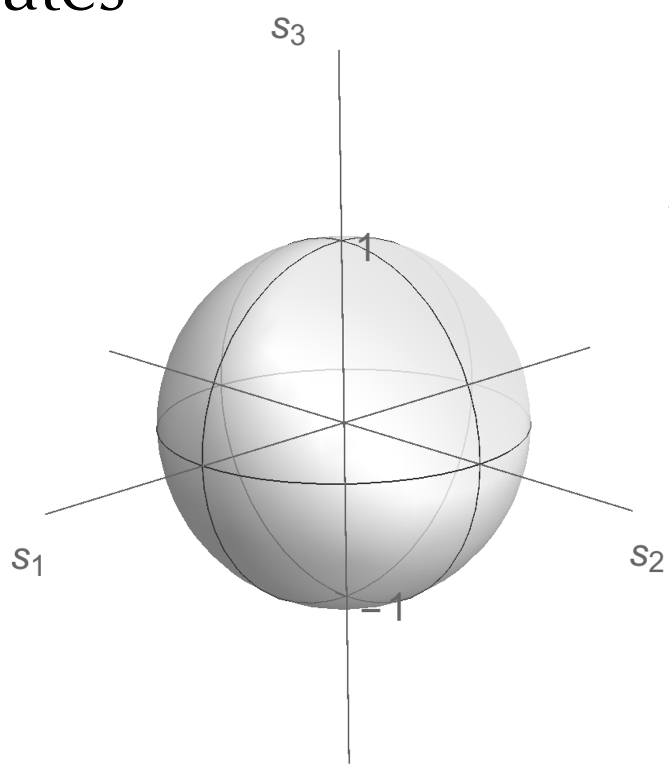
absolute minimum, independent of <*p*>:

$$\Delta x_{\text{min}} = a$$

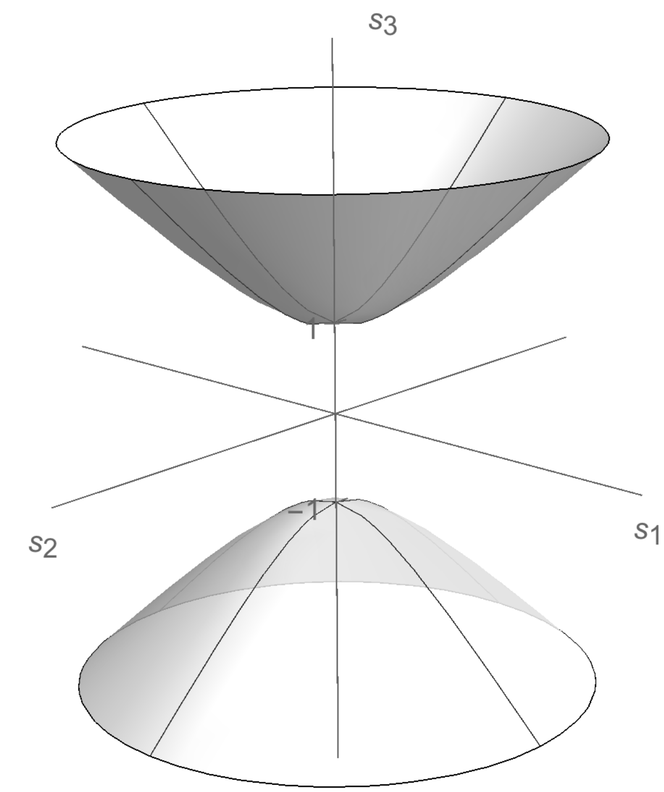
- *Snyder, Phys. Rev. 1947*
- *Kempf, Mangano, PRD 1997*

Euclidean maximally symmetric spaces

- Euclidean sphere ($\omega > 0$) and hyperbolic plane ($\omega < 0$) in embedding coordinates



$$s_3^2 + \omega\left(s_1^2 + s_2^2\right) = 1$$



- Algebra of symmetries $\mathrm{SO}_\omega(3)$

$$[J, P_1] = P_2, \qquad [J, P_2] = -P_1, \qquad [P_1, P_2] = \omega J$$

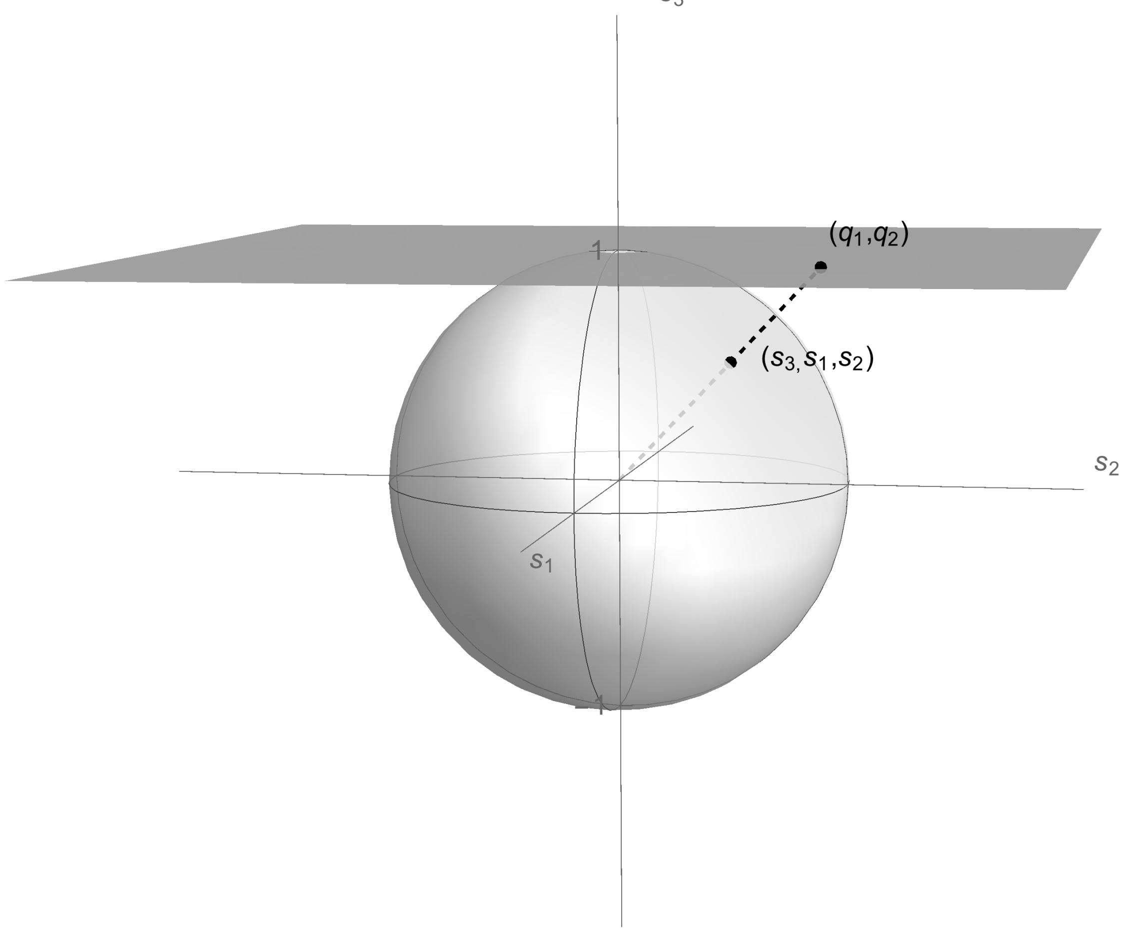
- It generates the spaces via $\mathbf{S}^2_\omega = \mathrm{SO}_\omega(3)/H, \qquad H = \mathrm{SO}(2) = \langle J \rangle.$

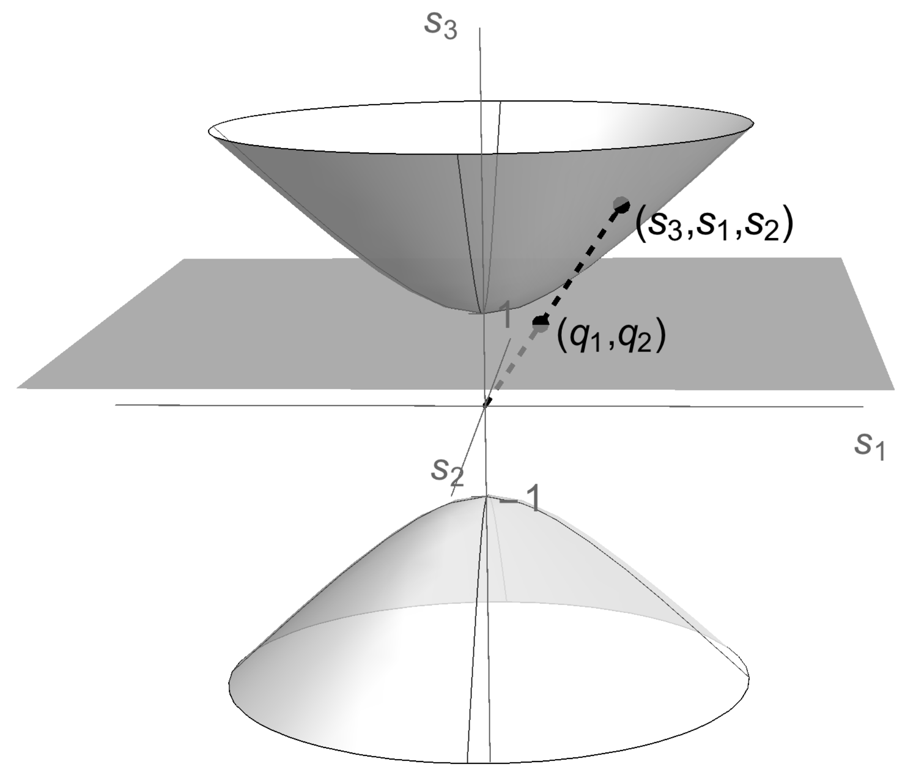
- Action on embedding coordinates

$$\begin{aligned} P_1 &= s_3 \frac{\partial}{\partial s_1} - \omega\, s_1 \frac{\partial}{\partial s_3} \\ P_2 &= s_3 \frac{\partial}{\partial s_2} - \omega\, s_2 \frac{\partial}{\partial s_3} \\ J &= s_1 \frac{\partial}{\partial s_2} - s_2 \frac{\partial}{\partial s_1} \end{aligned}$$

Projective geometry of Euclidean spaces

- Beltrami projective coordinates are obtained via a central stereographic projection with pole (0,0,0) to the plane $s_3=1$





- The relation between Beltrami coordinates and embedding coordinates is

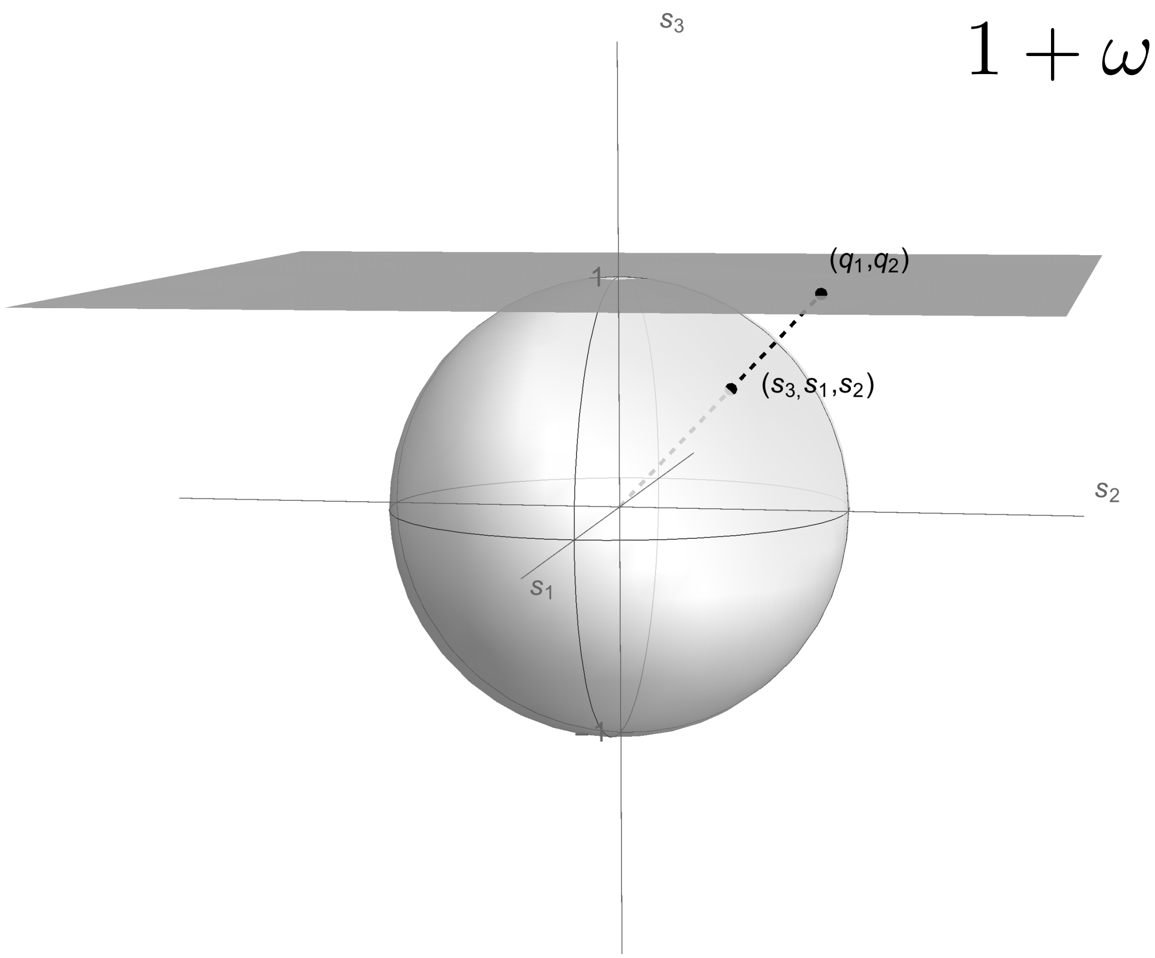
$$q_i = \frac{s_i}{s_3}$$

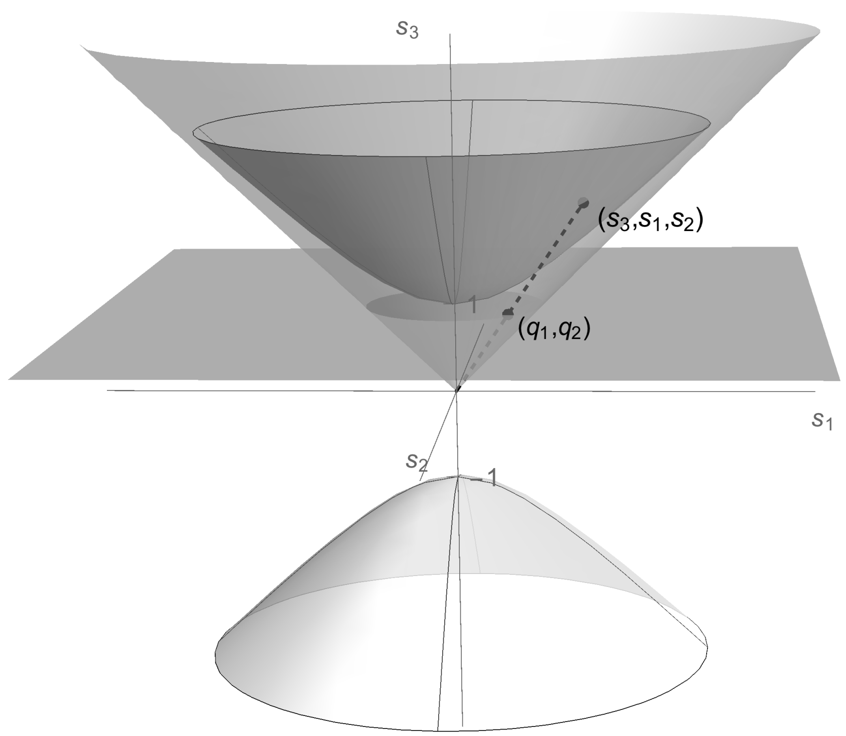
$$s_3 = \frac{1}{\sqrt{1+\omega(q_1^2+q_2^2)}}, \qquad s_i = \frac{q_i}{\sqrt{1+\omega(q_1^2+q_2^2)}}$$

Projective geometry of Euclidean spaces

- The domain of Beltrami coordinates depends on the sign of ω

$$1+\omega(q_1^2+q_2^2)>0$$





$$q_i \in (-\infty,+\infty) \qquad\qquad q_i \in \left(-1/\sqrt{|\omega|},+1/\sqrt{|\omega|}\right)$$

- A curved metric is induced on the Beltrami plane

$$\begin{aligned} d\sigma_\omega^2 &= \frac{1}{\omega}\left(ds_3^2+\omega\left(ds_1^2+ds_2^2\right)\right)\Big|_{\Sigma_\omega} = \frac{\omega(s_1 ds_1+s_2 ds_2)^2}{1-\omega\left(s_1^2+s_2^2\right)}+ds_1^2+ds_2^2 \\ &= \frac{(1+\omega\ q^2)d\ q^2-\omega(\ q\cdot d\ q)^2}{(1+\omega\ q^2)^2} \end{aligned}$$

Euclidean Snyder phase spaces

✦ The goal is to define a non commutative 2-dimensional space with rotational invariance

we can take the sphere/hyperbolic plane as a 'pregeometric' manifold and identify space coordinates with the generators of translations

$$x_1 := P_1 \,, \qquad x_2 := P_2$$

their commutator is naturally consistent with the required symmetry

$$[x_1, x_2] = \omega J$$

✦ Momenta are defined by asking that they transform as vectors under (hyperbolic) rotations — condition satisfied by the Beltrami coordinates!

$$p_1 := q_1 = \frac{s_1}{s_3} \,, \qquad p_2 := q_2 = \frac{s_2}{s_3}$$

Euclidean Snyder phase spaces

✦ The full phase space relations are

$$[x_1, x_2] = \omega J, \qquad [p_x, p_y] = 0,$$
$$[x_1, p_1] = 1 + \omega\ p_1^2, \qquad [x_1, p_2] = \omega\ p_1 p_2,$$
$$[x_2, p_2] = 1 + \omega\ p_2^2, \qquad [x_2, p_1] = \omega\ p_1 p_2,$$

consistent with a standard representation of the (hyperbolic) rotation generator

$$J = x_1 p_2 - x_2 p_1$$

→ the 'pregeometric' manifold (in Beltrami coordinates) is actually the momentum space and space coordinates are translations over the manifold

✦ The phase space is consistent with the spatial restriction of the original Snyder model with $\omega = a^2 > 0$

$$\begin{aligned} [x_i, p_j] &= i(\delta_{ij} + a^2 p_i p_j) & [x_i, p_0] &= i a^2 p_0 p_i \\ [x_0, p_0] &= i(1 - a^2 p_0^2) & [x_0, p_i] &= -i a^2 p_0 p_i\,. \end{aligned}$$

Projective geometry of Lorentzian spaces

- The starting point are 3+1 dimensional maximally symmetric Lorentzian manifolds, (A-)dS

  In embedding coordinates they are defined by the constraint

$$s_4^2 - \Lambda s_0^2 + \frac{\Lambda}{c^2}\left(s_1^2 + s_2^2 + s_3^2\right) = 1$$

  we now keep c explicit to track its effects

- Algebra of symmetries $\mathrm{SO}_\Lambda(4,1)$

$$[J_i, J_j] = \epsilon_{ijk} J_k, \qquad [J_i, P_j] = \epsilon_{ijk} P_k, \qquad [J_i, K_j] = \epsilon_{ijk} K_k,$$

$$[P_0, K_i] = P_i, \qquad [P_j, K_i] = \tfrac{1}{c^2}\,\delta_{ij} P_0, \qquad [K_i, K_j] = -\tfrac{1}{c^2}\,\epsilon_{ijk} J_k,$$

$$[P_0, P_i] = \Lambda K_i, \qquad [P_i, P_j] = \Lambda\,\tfrac{1}{c^2}\,\epsilon_{ijk} J_k, \qquad [P_0, J_i] = 0,$$

- It generates de Sitter ($\Lambda > 0$) and anti-de Sitter ($\Lambda < 0$) space via

$$\mathbf{dS}_\Lambda^{3+1} = \mathrm{SO}_\Lambda(4,1)/\mathrm{SO}(3,1)$$

Projective geometry of Lorentzian spaces

- Beltrami projective coordinates are obtained via a central stereographic projection with pole (0,0,**0**) to the plane s_4=1

$$q_\alpha = \frac{s_\alpha}{s_4}$$

$$s_4 = \frac{1}{\sqrt{1 - \Lambda q_0^2 + \frac{\Lambda}{c^2}\mathbf{q}^2}}, \qquad s_\alpha = \frac{q_\alpha}{\sqrt{1 - \Lambda q_0^2 + \frac{\Lambda}{c^2}\mathbf{q}^2}}$$

- The domain of Beltrami coordinates depends on the sign of Λ:

$$1 - \Lambda q_0^2 + \frac{\Lambda}{c^2}\mathbf{q}^2 > 0 \quad \Rightarrow \begin{cases} q_0^2 - \frac{\mathbf{q}^2}{c^2} < \frac{1}{\Lambda} & \text{if } \Lambda > 0 \\ q_0^2 - \frac{\mathbf{q}^2}{c^2} > \frac{1}{|\Lambda|} & \text{if } \Lambda < 0 \end{cases}$$

- A curved metric is induced on the Beltrami plane

$$\begin{aligned} \mathrm{d}\sigma_\Lambda^2 &= \frac{1}{-\Lambda}\left(\mathrm{d}s_4^2 - \Lambda \mathrm{d}s_0^2 + \frac{\Lambda}{c^2}\,\mathrm{d}\mathbf{s}^2\right)\Big|_{\Sigma_\Lambda} = \frac{-\Lambda\left(s_0\mathrm{d}s_0 - \frac{1}{c^2}\,\mathbf{s}\cdot\mathrm{d}\mathbf{s}\right)^2}{1+\Lambda s_0^2 - \frac{\Lambda}{c^2}\,\mathbf{s}^2} + \mathrm{d}s_0^2 - \frac{1}{c^2}\,\mathrm{d}\mathbf{s}^2 \\ &= \frac{\left(1-\Lambda q_0^2 + \frac{\Lambda}{c^2}\,\mathbf{q}^2\right)\left(\mathrm{d}q_0^2 - \frac{1}{c^2}\,\mathrm{d}\mathbf{q}^2\right) + \Lambda\left(q_0\mathrm{d}q_0 - \frac{1}{c^2}\,\mathbf{q}\cdot\mathrm{d}\mathbf{q}\right)^2}{\left(1-\Lambda q_0^2 + \frac{\Lambda}{c^2}\,\mathbf{q}^2\right)^2} \end{aligned}$$

(anti-)Snyder model from projective geometry

✦ A non commutative Lorentz-invariant spacetime can be constructed by using the (anti-)de Sitter manifold as a 'pregeometric' manifold and identifying the spacetime coordinates with the generators of translations

$$x_0 := \frac{1}{c} P_0, \qquad x_i := c\, P_i$$

their commutator is naturally consistent with the required symmetry

$$[x_0, x_i] = \Lambda\, K_i, \quad [x_i, x_j] = \Lambda\, \epsilon_{ijk} J_k$$

✦ For $\Lambda > 0$ these are the commutators used by Snyder, with $\Lambda \equiv a^2 > 0$ and the Beltrami coordinates correspond to Snyder's choice of momenta

$$p_0 := c\, q_0 = c\, \frac{s_0}{s_4}, \qquad p_i := \frac{1}{c}\, q_i = \frac{1}{c}\, \frac{s_i}{s_4}$$

➡ The Snyder model is a 'curved momentum space' model, where the momentum space is the projection of a de Sitter manifold and spacetime coordinates are translations over the de Sitter manifold. A curved metric is induced on the Beltrami plane:

$$d\mu(p) = \frac{c^4}{(c^2(1 + \Lambda|\mathbf{p}^2|) - \Lambda p_0^2)^{5/2}}\, d^4p\,,$$

(anti-)Snyder phase space

✦ The noncommutative coordinates and the momenta close a deformed phase space algebra

$$[x_0, x_i] = \Lambda\, K_i, \qquad [x_i, x_j] = \Lambda\, \epsilon_{ijk} J_k,$$
$$[x_0, p_\alpha] = \delta_{0\alpha} - \tfrac{\Lambda}{c^2}\, p_0 p_\alpha, \qquad [x_i, p_j] = \delta_{ij} + \Lambda\, p_i p_j,$$
$$[x_i, p_0] = \Lambda\, p_0 p_i, \qquad [p_\alpha, p_\beta] = 0,$$

which reproduces the original Snyder phase space for $\Lambda \equiv a^2 > 0$
The case $\Lambda \equiv -a^2 < 0$ defines the so-called anti-Snyder model

• *Mignemi PRD 2011*

✦ In terms of coordinates and momenta the symmetry generators are

$$K_i = x_0 p_i + \frac{1}{c^2}\, x_i p_0, \qquad J_i = \epsilon_{ijk} x_j p_k$$

✦ The spatial components of the phase space reproduce the Euclidean Snyder model, so it could be natural to suppose that in the corresponding Galilean model, where time is absolute, only these relations survive

The non-relativistic limit of the (anti-)Snyder model

✦ The nonrelativistic limit is usually realised by reducing to the spatial coordinates and momenta

•*Lu, Stern NPB 2012*
•*Mignemi CQG 2012*
•*Ching, Yeo, Ng, IJMPA 2017*

$$\begin{aligned}
[x_0, x_i] &= \Lambda \left(x_0 p_i + \tfrac{1}{c^2}\, x_i p_0 \right) \\
[x_i, x_j] &= \Lambda\, \epsilon_{ijk} J_k \\
[x_0, p_\alpha] &= \delta_{0\alpha} - \tfrac{\Lambda}{c^2}\, p_0 p_\alpha \\
[x_i, p_j] &= \delta_{ij} + \Lambda\, p_i p_j \\
[x_i, p_0] &= \Lambda\, p_0 p_i \\
[p_\alpha, p_\beta] &= 0
\end{aligned}
\quad \longrightarrow \quad
\begin{aligned}
[x_i, x_j] &= \omega \epsilon_{ijk} J_k \\
[p_i, p_j] &= 0, \\
[x_i, p_j] &= \delta_{ij} + \omega\, p_i p_j
\end{aligned}$$

thus one obtains the Euclidean (anti-)Snyder model $\mathbf{dS}^{3+1}_{\Lambda} \longrightarrow \mathbf{S}^3_{\omega}$

✦ However the Euclidean sphere/hyperbolic plane are not the $c \to \infty$ limit of the de Sitter/anti-de Sitter spaces, thus is not compatible with the symmetries of the Galilei algebra

The actual Galilean limit of the Snyder model is best exposed by using the momentum space construction

Galilean limit of (anti-)de Sitter

✦ Take the $c \to \infty$ limit of the (anti-)de Sitter algebra of symmetries

$$
\begin{aligned}
[J_i, J_j] &= \epsilon_{ijk} J_k \\
[J_i, P_j] &= \epsilon_{ijk} P_k \\
[J_i, K_j] &= \epsilon_{ijk} K_k \\
[P_0, K_i] &= P_i \\
[P_j, K_i] &= \tfrac{1}{c^2}\, \delta_{ij} P_0 \\
[K_i, K_j] &= -\tfrac{1}{c^2}\, \epsilon_{ijk} J_k \\
[P_0, P_i] &= \Lambda K_i \\
[P_i, P_j] &= \Lambda \tfrac{1}{c^2}\, \epsilon_{ijk} J_k \\
[P_0, J_i] &= 0
\end{aligned}
\qquad \xrightarrow{c \to \infty} \qquad
\begin{aligned}
[J_i, J_j] &= \epsilon_{ijk} J_k \\
[J_i, P_j] &= \epsilon_{ijk} P_k \\
[J_i, K_j] &= \epsilon_{ijk} K_k \\
[K_i, P_0] &= P_i \\
[K_i, P_j] &= 0 \\
[K_i, K_j] &= 0 \\
[P_0, P_i] &= \Lambda K_i \\
[P_i, P_j] &= 0 \\
[P_0, J_i] &= 0
\end{aligned}
$$

✦ This defines the Newton-Hooke algebra (expanding $\Lambda > 0$, oscillating $\Lambda < 0$) whose associated homogeneous manifold is

$$\mathbf{N}_{\Lambda}^{3+1} = \mathbf{NH}_{\boldsymbol{\Lambda}}(3+1)/\mathrm{ISO}(3).$$

Galilean limit of (anti-)de Sitter

✦ Take the $c \to \infty$ limit of the (anti-)de Sitter algebra of symmetries

$$\begin{aligned}
[J_i, J_j] &= \epsilon_{ijk} J_k \\
[J_i, P_j] &= \epsilon_{ijk} P_k \\
[J_i, K_j] &= \epsilon_{ijk} K_k \\
[P_0, K_i] &= P_i \\
[P_j, K_i] &= \tfrac{1}{c^2}\, \delta_{ij} P_0 \\
[K_i, K_j] &= -\tfrac{1}{c^2}\, \epsilon_{ijk} J_k \\
[P_0, P_i] &= \Lambda K_i \\
[P_i, P_j] &= \Lambda\, \tfrac{1}{c^2}\, \epsilon_{ijk} J_k \\
[P_0, J_i] &= 0
\end{aligned}
\qquad \xrightarrow{c \to \infty} \qquad
\begin{aligned}
[J_i, J_j] &= \epsilon_{ijk} J_k \\
[J_i, P_j] &= \epsilon_{ijk} P_k \\
[J_i, K_j] &= \epsilon_{ijk} K_k \\
[K_i, P_0] &= P_i \\
[K_i, P_j] &= 0 \\
[K_i, K_j] &= 0 \\
[P_0, P_i] &= \Lambda K_i \\
[P_i, P_j] &= 0 \\
[P_0, J_i] &= 0
\end{aligned}$$

✦ This defines the Newton-Hooke algebra (expanding $\Lambda > 0$, oscillating $\Lambda < 0$) whose associated homogeneous manifold is

$$\mathbf{N}_\Lambda^{3+1} = \mathbf{NH}_\Lambda(3+1)/\mathrm{ISO}(3).$$

the constraint defining the manifold in terms of embedding coordinates and the metric on this manifold are degenerate, because of the appearance of an "absolute time", which induces a constant-time foliation (invariant under the action of the NH group)

$$s_4^2 - \Lambda s_0^2 = 1$$

$$\mathrm{d}\sigma_\Lambda^2 = \frac{\mathrm{d}s_0^2}{1 + \Lambda s_0^2}$$

$$\mathrm{d}{\sigma'}_\Lambda^2 = \mathrm{d}\mathbf{s}^2 \quad \text{on} \quad s_0 = \text{constant}$$

Galilean limit of (anti-)de Sitter

✦ Take the $c \to \infty$ limit of the (anti-)de Sitter algebra of symmetries

$$
\begin{array}{rcl}
[J_i, J_j] &=& \epsilon_{ijk} J_k \\
[J_i, P_j] &=& \epsilon_{ijk} P_k \\
[J_i, K_j] &=& \epsilon_{ijk} K_k \\
[P_0, K_i] &=& P_i \\
[P_j, K_i] &=& \frac{1}{c^2}\, \delta_{ij} P_0 \\
[K_i, K_j] &=& -\frac{1}{c^2}\, \epsilon_{ijk} J_k \\
[P_0, P_i] &=& \Lambda K_i \\
[P_i, P_j] &=& \Lambda\, \frac{1}{c^2}\, \epsilon_{ijk} J_k \\
[P_0, J_i] &=& 0
\end{array}
\quad \xrightarrow{c \to \infty} \quad
\begin{array}{rcl}
[J_i, J_j] &=& \epsilon_{ijk} J_k \\
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$$

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Galilean (anti-)Snyder model

✦ The Galilean Snyder phase space can be found as the $c \to \infty$ limit of the Lorentzian Snyder phase space

$$
\begin{aligned}
[x_0, x_i] &= \Lambda\left(x_0 p_i + \tfrac{1}{c^2}\, x_i p_0\right) \\
[x_i, x_j] &= \Lambda\, \epsilon_{ijk} J_k \\
[x_0, p_\alpha] &= \delta_{0\alpha} - \tfrac{\Lambda}{c^2}\, p_0 p_\alpha \\
[x_i, p_j] &= \delta_{ij} + \Lambda\, p_i p_j \\
[x_i, p_0] &= \Lambda\, p_0 p_i \\
[p_\alpha, p_\beta] &= 0
\end{aligned}
\quad \xrightarrow{c \to \infty} \quad
\begin{aligned}
[x_0, x_i] &= \Lambda\, K_i \\
[x_i, x_j] &= \Lambda\, \epsilon_{ijk} J_k \\
[x_0, p_\alpha] &= \delta_{0\alpha} \\
[x_i, p_j] &= \delta_{ij} + \Lambda\, p_i p_j \\
[x_i, p_0] &= \Lambda\, p_0 p_i \\
[p_\alpha, p_\beta] &= 0
\end{aligned}
$$

where the boost K_i reduces to the Galilean boost: $K_i = x_0 p_i$

✦ The commutators between the spatial components of phase space are the same as in the Euclidean Snyder model.

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However there are also nontrivial commutators between space and time components.

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✦ The commutators between the spatial components of phase space are the same as in the Euclidean Snyder model.

However there are also nontrivial commutators between space and time components.

→ There is a remnant spacetime mixing in the Galilean limit of the relativistic model due to spacetime noncommutativity

Conclusions & outlook

- ✦ The Snyder model for noncommutative spacetime introduces a minimum length without violating Lorentz invariance

- ✦ This is achieved by defining spacetime coordinates as translations over a curved maximally symmetric manifold [(A-)dS] and physical momenta as the projective coordinate on a plane of such curved manifold — the momentum space inherits a curved metric

- ✦ The non relativistic limit of the Snyder model is found by performing a similar construction of the manifold that is the Galilean limit of (A-)dS, i.e. the Newton-Hooke manifold.

- ✦ The time foliation of the NH manifold is reminiscent of the "absolute time" of Galilean physics. However, spatial and time translations on these manifold do not commute, so the resulting Galilean Snyder space and time coordinates are not independent

- ✦ This provides an example of QG model where the nonrelativistic limit is qualitatively different from that of standard physics, and time does not completely decouple from space

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QUANTUM GRAVITY PHENOMENOLOGY IN THE MULTI-MESSENGER APPROACH

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