L_∞ algebra of Einstein-Cartan-Palatini gravity and its braided noncommutative deformation

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Workshop on Quantum Geometry, Field Theory and Gravity
August 20, 2019

based on joint work with Marija Dimitrijević Ćirić, Voja Radovanović and Richard J. Szabo

Einstein-Cartan-Palatini (ECP) gravity

- ECP gravity is described by the following action functional

$$S = \int_M Tr(e \curlywedge e \curlywedge R) = \int_M (e^a \wedge e^b \wedge R^{cd} \epsilon_{abcd})$$

where $e : TM \to \mathcal{V}$ is a bundle isomorphism onto a fixed (Minkowski) vector bundle $(\mathcal{V}, \eta)$, and $R = d\omega + \frac{1}{2}[\omega, \omega]$ is the curvature of a connection one form on the associated principal $SO(1,3)$-bundle.

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- Locally, or globally for a parallelizable M, $e \in \Omega^1(M, \mathbb{R}^{1,3})$ and $\omega \in \Omega^1(M, \mathfrak{so}(1,3))$.

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- Using the isomorphism $\mathfrak{so}(1,3) \cong \bigwedge^2 \mathbb{R}^{1,3}$, one identifies $\omega = \omega^{ab} E_a \wedge E_b$. The curly wedge combines two different exterior products. The trace is the flat space hodge star.

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corresponding to diffeomorphisms and local lorentz rotations, respectively.

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- The second is then equivalent to the Einstein equation in vacuum.
- To change dimension, modify number of e-fields in action. E.g. in 3d $\mathrm{d}^\omega e = 0$, $R = 0$.

Cyclic L_∞ Algebras and Gauge Theories

- An L_∞ algebra is a graded vector space $X = \bigoplus_{k\in\mathbb{Z}} X^k$ equipped with graded skewsymmetric multilinear maps

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satisfying higher versions of the Jacobi identity.

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- Schematically:

$$\ell_1\ell_1 = 0, \qquad \ell_1\ell_2 - \ell_2\ell_1 = 0, \qquad \ell_1\ell_3 + \ell_2\ell_2 + \ell_3\ell_1 = 0$$

  and infinite more similar higher relations.

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- The first says ℓ_1 is a differential and the second says ℓ_1 is a derivation wrt ℓ_2.
- The third relation represents the breaking of the Jacobi identity in the presence of a non-trivial ℓ_3.

Cyclic L_∞ algebras and Gauge Theories

- Such an algebra is called Cyclic if it is equipped with a non degenerate pairing (of deg -3) $\langle -, - \rangle : X \times X \to \mathbb{R}$. Cyclicity meaning

$$\langle A_1, \ell_n(A_2, \cdots, A_n) \rangle = \langle A_n, \ell_n(A_1, \cdots, A_{n-1}) \rangle$$

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- Complete data of Classical Field theories fit into truncated L_∞ structures with finitely many higher brackets not vanishing. [Hohm, Zwiebach '17]
  Due to duality with BV/BRST. [Jurco, Saemann, Raspollini, Wolf '18]

Cyclic L_∞ algebras and Gauge Theories

The classical data of a gauge field theory is encoded in such an algebra as follows:

- The action S is given by:

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- The Noether identities:

$$\mathcal{I}_\lambda = \ell_1(\mathcal{F}_A) + \ell_2(\mathcal{F}_A, A) + \cdots \equiv 0$$

  <u>offshell</u>.

L_∞ algebra of (3d) ECP gravity

For ECP gravity $X = X_0 \oplus X_1 \oplus X_2 \oplus X_3$, where these are the spaces of

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- $\underline{\ell_1}$

$$\ell_1(\xi, \lambda) = (0, \mathrm{d}\lambda), \quad \ell_1(e, \omega) = (0, 0) \quad \text{and} \quad \ell_1(E, \Omega) = (0, \mathrm{d}\Omega).$$

L_∞ algebra of ECP gravity

- ℓ_2 bracket

$$\ell_2\big((\xi_1,\lambda_1)\,,\,(\xi_2,\lambda_2)\big) = \big([\xi_1,\xi_2]\,,\,-[\lambda_1,\lambda_2] + \mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1\big)$$

$$\ell_2\big((\xi,\lambda)\,,\,(e,\omega)\big) = (-\lambda\cdot e + \mathcal{L}_\xi e, -[\lambda,\omega] + \mathcal{L}_\xi\omega),$$

$$\ell_2\big((\xi,\lambda)\,,\,(E,\Omega)\big) = (-\lambda\cdot E + \mathcal{L}_\xi E, -\lambda\cdot\Omega + \mathcal{L}_\xi\Omega)\ ,$$

$$\ell_2\big((e_1,\omega_1)\,,\,(e_2,\omega_2)\big) = -(2\omega_2\wedge\omega_1, \omega_1\wedge e_2 + \omega_2\wedge e_1)$$

and more related to the Noether identities:

$$\ell_2\big((e,\omega),(E,\Omega)\big) = \\ = \Big(Tr\big(\iota_{\partial_\mu}\mathrm{d}e \curlywedge E + \iota_{\partial_\mu}\mathrm{d}\omega \curlywedge \Omega - \iota_{\partial_\mu}e \curlywedge \mathrm{d}E - \iota_{\partial_\mu}\omega \curlywedge \mathrm{d}\Omega\big) \otimes \mathrm{d}x^\mu\ , \\ E\wedge e - \omega\wedge\Omega \Big)$$

$$\ell_2\big((\xi,\lambda),(\Xi,\Lambda)\big) = \big(Tr(\iota_{\partial_\mu}\mathrm{d}\lambda \curlywedge \Lambda) \otimes \mathrm{d}x^\mu + \mathcal{L}_\xi\Xi\ , -\lambda\cdot\Lambda + \mathcal{L}_\xi\Lambda\big)$$

L_∞ algebra of (3d) ECP gravity

- The gauge transformations are encoded as:

$$\begin{aligned}\delta_{(\xi,\lambda)}(e,\omega) &= \big(-\lambda\cdot e + \mathcal{L}_\xi e\,,\,\mathrm{d}\lambda - [\lambda,\omega] + \mathcal{L}_\xi\omega\big)\\ &= \ell_1(\xi,\lambda) + \ell_2\big((\xi,\lambda)\,,\,(e,\omega)\big)\ ,\end{aligned}$$

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<u>on-shell</u>.

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- The Noether identities:

$$\begin{aligned}\mathcal{I}_{(\xi,\lambda)} &= \big(\mathrm{d}^\omega\mathcal{F}_\omega - \mathcal{F}_e\wedge e, \mathrm{d}x^\mu\otimes Tr(\iota_{\partial_\mu}e\curlywedge \mathrm{d}\mathcal{F}_e - \iota_{\partial_\mu}\mathrm{d}e\curlywedge\mathcal{F}_e) + \cdots\big)\\ &= \ell_1(\mathcal{F}) - \ell_2\big((e,\omega),\mathcal{F}\big) \equiv 0\end{aligned}$$

<u>off-shell</u>.

Cyclic pairing and Action functional

- The action is encoded via the cyclic, non degenerate pairing $\langle -, - \rangle : X_1 \times X_2 \to \mathbb{R}$,

$$\langle (e, \omega), (E, \Omega) \rangle := \int_M Tr(e \curlywedge E + \Omega \curlywedge \omega)$$

with

$$S_{ECP} = \langle (e, \omega), \ell_1(e, \omega) + \ell_2((e, \omega), (e, \omega)) \rangle$$

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- Using gauge invariance and integrating by parts etc, pairing is extended to $\langle -, - \rangle : X_0 \times X_3 \to \mathbb{R}$
- Cyclicity on gauge parameters (X_0) implies formula of Noether identities (Noether's 2nd Theorem).

Braided noncommutative gauge symmetries

- Given a Drinfel'd twist, $\mathcal{F} = f^a \otimes f_a \in U\big(\Gamma(TM)\big) \otimes U\big(\Gamma(TM)\big)$, one defines the braided Lie algebra $\big(\Omega^0_\star(M, \mathfrak{so}(3)), [-,-]_\star\big)$ where

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- The bracket is braided antisymmetric:

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- A braided version of the Jacobi identity is satisfied.

Braided noncommutative gauge symmetries

- Braided $\mathfrak{so}(3)$-connections, $\omega \in \Omega^1_\star(M, \mathfrak{so}(3))$ and coframe fields $e \in \Omega^1_\star(M, \mathbb{R}^3)$ may be defined. Transforming in braided representations:

$$\delta^\star_\lambda e = -\lambda \star e = -\lambda^a{}_b \star e^b E_a$$
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- Note: Gauge transformations satisfy the braided Leibniz rule, e.g.

$$\delta^\star_\lambda (e \otimes \omega) = \delta^\star_\lambda e \otimes \omega + \bar{R}^a e \otimes \delta^\star_{\bar{R}_a \lambda} \omega$$

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- A braided covariant derivative is similarly defined, and thus the braided curvature and torsion:

$$R := \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]_\star$$
$$T := \mathrm{d}^\omega e = \mathrm{d}e + \omega \wedge_\star e$$

Braided noncommutative ECP dynamics

- The Lie algebra of vector fields is twisted to a braided version $\big(\Gamma_\star(TM), [-,-]_\star\big)$, acting via a twisted Lie derivative, e.g.

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- The noncommutative ECP (3d) action is:

$$S^\star_{ECP} := Tr \int_M e \curlywedge_\star R = \int e^a \wedge_\star R^{bc}\,\epsilon_{abc}$$

assuming $\wedge_\star$ is commutative under the integral, this is the only possible deformation.

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- The action is invariant under the braided NC Lie algebra

$$\Gamma_\star(TM) \rtimes \Omega^0_\star\big(M, \mathfrak{so}(3)\big)$$

Braided noncommutative ECP dynamics

- The Equations of motion follow:

$$\mathcal{F}_e = \mathrm{d}\omega + \omega \wedge_\star \omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]_\star =: R$$

$$\mathcal{F}_\omega = \mathrm{d}e + \frac{1}{2}\omega \wedge_\star e + \frac{1}{2}\bar{R}^a\omega \wedge_\star \bar{R}_a e =: T - \frac{1}{2}\omega \wedge_\star e + \frac{1}{2}\bar{R}^a\omega \wedge_\star \bar{R}_a e$$

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$$\mathcal{F}_\omega = \mathrm{d}e + \frac{1}{2}\omega \wedge_\star e + \frac{1}{2}\bar{R}^a\omega \wedge_\star \bar{R}_a e =: T - \frac{1}{2}\omega \wedge_\star e + \frac{1}{2}\bar{R}^a\omega \wedge_\star \bar{R}_a e$$

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Braided noncommutative ECP dynamics

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- Note: Braided non commutativity seems to induce Torsion. Unlike classical case, if connection is kept dynamical, Torsion arises.
- "Problem": Although EOMs are braided gauge covariant, braided gauge symmetries do not produce new solutions, e.g.:

$$\delta^\star_\lambda R[\omega] \neq R[\omega + \delta^\star_\lambda \omega],$$

due to braided Leibniz rule.

Braided symmetries are "gauge"

- Braided Leibniz rule also spoils argument for Noether's 2nd Theorem, since:

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- However, one may modify the argument to a braided version and derive a set of "braided" Noether identities, e.g.

$$\begin{aligned} \mathrm{d}\mathcal{F}_\omega &+ \frac{1}{2}(\omega \wedge_\star \mathcal{F}_\omega + \frac{1}{2}\bar{R}^k\omega \wedge_\star \bar{R}_k\mathcal{F}_\omega - \mathcal{F}_e \wedge_\star e - \bar{R}^k\mathcal{F}_e \wedge e) \\ &+ \frac{1}{4}\omega \wedge_\star \omega \wedge_\star e + \frac{1}{4}\bar{R}^a(\omega \wedge_\star \omega) \wedge_\star \bar{R}_a e - \frac{1}{2}\omega \wedge_\star \bar{R}^k\omega \wedge_\star \bar{R}_k e \\ &\equiv 0 \end{aligned}$$

off-shell. Thus, braided local symmetries do have the right to be called "gauge".

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- The cyclic pairing encoding the action is

$$\langle (e,\omega)\,,\,(E,\Omega)\rangle_\star := \int_M Tr\big(e \curlywedge_\star E + \Omega \curlywedge_\star \omega\big)\ ,$$

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- ℓ_2 bracket

$$\ell_2\big((\xi_1,\lambda_1)\,,\,(\xi_2,\lambda_2)\big)=\big([\xi_1,\xi_2]_\star\,,\,-[\lambda_1,\lambda_2]_\star+\mathcal{L}^\star_{\xi_1}\lambda_2-\mathcal{L}^\star_{\bar{R}^a\xi_2}\bar{R}_a\lambda_1\big)$$

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$$\ell_2\big((e_1,\omega_1)\,,\,(e_2,\omega_2)\big)=-(2\omega_2\wedge_\star\omega_1,\omega_1\wedge_\star e_2+\bar{R}^a\omega_2\wedge\bar{R}_a e_1)$$

and more related to the Noether identities:

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- The gauge transformations are encoded as classically:

$$\begin{aligned}\delta_{(\xi,\lambda)}(e,\omega) &= \big(-\lambda\star e+\mathcal{L}^\star_\xi e\,,\,\mathrm{d}\lambda-[\lambda,\omega]_\star+\mathcal{L}^\star_\xi\omega\big)\\ &= \ell_1(\xi,\lambda)+\ell_2\big((\xi,\lambda)\,,\,(e,\omega)\big)\ ,\end{aligned}$$

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- The equations of motion are encoded, due to cyclicity, as classically:

$$\mathcal{F}_{(e,\omega)} = \big(R, \, T - \frac{1}{2}\omega \wedge_\star e + \frac{1}{2}\bar{R}^a\omega \wedge_\star \bar{R}_a e \big)$$
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- But the Noether identities, due to braided Leibniz rule:

$$\mathcal{I}_{(\xi,\lambda)} = \ell_1(\mathcal{F}) - \frac{1}{2}\big(\ell_2((e,\omega),\mathcal{F}) - \ell_2(\mathcal{F},(e,\omega))\big)$$
$$+ \frac{1}{4}\ell_2\big(\bar{R}^a A, \ell_2^\star(\bar{R}_a(e,\omega),(e,\omega))\big) \equiv 0$$

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- Presented a non commutative deformation of ECP gravity which introduces no extra degrees of freedom. The meaning of braided gauge symmetries was clarified by braided Noether identities.
- Presented the 3d braided deformation of the L_∞ algebra, encoding all the data of the non commutative theory.

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Future plans

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- Investigating a potential dual "Braided BV/BRST" formalism and furthermore quantization of such noncommutative gauge theories.

Thank you for your attention!