The Steinmann Cluster Bootstrap for $\mathcal{N} = 4$ super Yang-Mills Amplitudes

Georgios Papathanasiou

Workshop on Connecting Insights in Fundamental Physics
Corfu, September 1, 2019

1903.10890, 1906.07116 w/ Caron-Huot,Dixon,Dulat,McLeod,Hippel
1812.04640 w/ Drummond,Foster,Gürdogan

Outline

Motivation: Why Amplitudes in $\mathcal{N} = 4$ Super Yang-Mills?

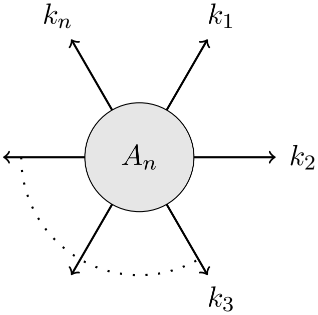
Improving Perturbation Theory: The Amplitude Bootstrap

- Extended Steinmann relations/cluster adjacency
- Coaction principle
- 6 gluons though 7 loops/7 gluons through 4 loops

Conclusions & Outlook

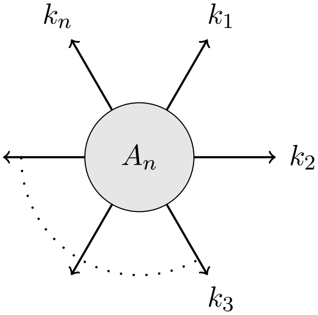
S-matrix of gauge theory beyond Feynman diagrams?

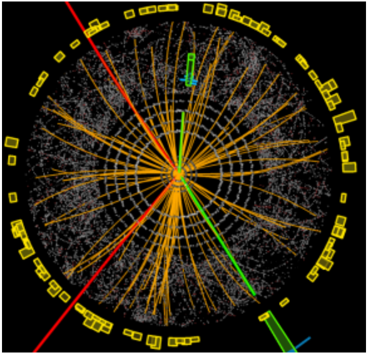
S-matrix of gauge theory beyond Feynman diagrams?



Scattering amplitudes $A = \langle \text{IN}|S|\text{OUT}\rangle$: $d\sigma \propto |A|^2$

S-matrix of gauge theory beyond Feynman diagrams?

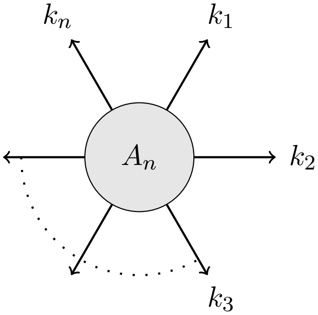


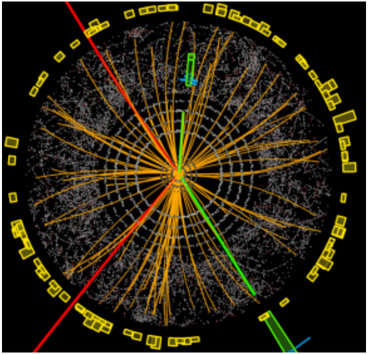


Scattering amplitudes $A = \langle \mathrm{IN}|S|\mathrm{OUT}\rangle$: $d\sigma \propto |A|^2$

- Computing efficiently necessary in practice

S-matrix of gauge theory beyond Feynman diagrams?





Scattering amplitudes $A = \langle \text{IN}|S|\text{OUT}\rangle$: $d\sigma \propto |A|^2$

- Computing efficiently necessary in practice
- Understanding beyond Feynman diagrams mathematically important [Millenium Prize]

Strategy

Focus on the simplest interacting 4D gauge theory

Strategy

Focus on the simplest interacting 4D gauge theory

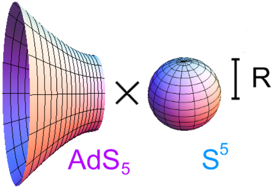
$SU(N)$ maximally supersymmetric Yang-Mills (MSYM) theory

$$\mathcal{L} = -\frac{1}{2g_{YM}^2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \mathrm{fermions} + \mathrm{scalars}$$

Strategy

Focus on the simplest interacting 4D gauge theory

$SU(N)$ maximally supersymmetric Yang-Mills (MSYM) theory in planar limit, $N \to \infty$ with $\lambda = g_{YM}^2 N$ fixed:

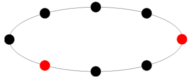


- Planar MSYM $\Leftrightarrow$ Free superstrings
  strongly coupled $\Leftrightarrow$ weakly coupled [Maldacena]

Strategy

Focus on the simplest interacting 4D gauge theory

$SU(N)$ maximally supersymmetric Yang-Mills (MSYM) theory in planar limit, $N \to \infty$ with $\lambda = g_{YM}^2 N$ fixed:

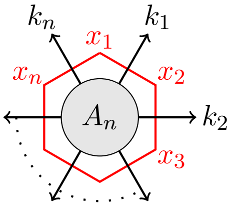
$\mathcal{O} = \mathrm{Tr}[Z^4 W Z^2 W] \quad \Leftrightarrow$ 

- Planar MSYM $\Leftrightarrow$ Free superstrings
  strongly coupled $\Leftrightarrow$ weakly coupled [Maldacena]
- Integrable structures $\Rightarrow$ Exact physical quantities! [Arutyunov,. . . ,Zarembo]

Strategy

Focus on the simplest interacting 4D gauge theory

$SU(N)$ maximally supersymmetric Yang-Mills (MSYM) theory in planar limit, $N \to \infty$ with $\lambda = g^2_{YM} N$ fixed:



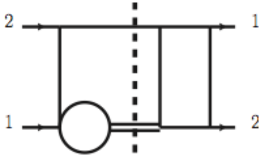
$$k_i = x_{i+1} - x_i = x_{i+1,i}$$
$$s_{i\ldots j} \equiv (k_i + \ldots + k_j)^2 = x^2_{ij+1}$$

- Planar MSYM $\Leftrightarrow$ Free superstrings
  strongly coupled $\Leftrightarrow$ weakly coupled [Maldacena]
- Integrable structures $\Rightarrow$ Exact physical quantities! [Arutyunov,...,Zarembo]
- Amplitudes $\Leftrightarrow$ Wilson Loops; Remarkable symmetries [Alday,Maldacena]

Strategy

Focus on the simplest interacting 4D gauge theory

$SU(N)$ maximally supersymmetric Yang-Mills (MSYM) theory in planar limit, $N \to \infty$ with $\lambda = g_{YM}^2 N$ fixed:



- Planar MSYM $\Leftrightarrow$ Free superstrings
  strongly coupled $\Leftrightarrow$ weakly coupled [Maldacena]
- Integrable structures $\Rightarrow$ Exact physical quantities! [Arutyunov,. . . ,Zarembo]
- Amplitudes $\Leftrightarrow$ Wilson Loops; Remarkable symmetries [Alday,Maldacena]

Ideal theoretical laboratory for developing new computational tools for QCD. E.g. method of symbols: [Goncharov,Spradlin,Vergu,Volovich]

Apply to $|gg \to Hg|^2$ for N^{3}LO Higgs cross-section! [Anastasiou,Duhr et. al.]

Present and future of solving the simplest gauge theory

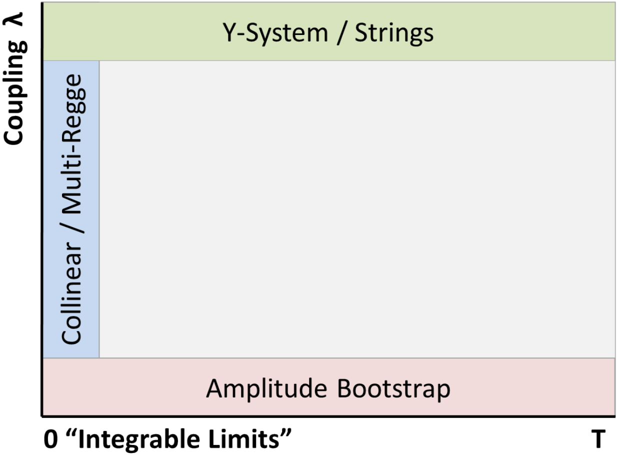
Present and future of solving the simplest gauge theory

Amplitudes A_n with $n = 4, 5$ particles already known to all loops!

Present and future of solving the simplest gauge theory

Amplitudes A_n with $n = 4, 5$ particles already known to all loops!

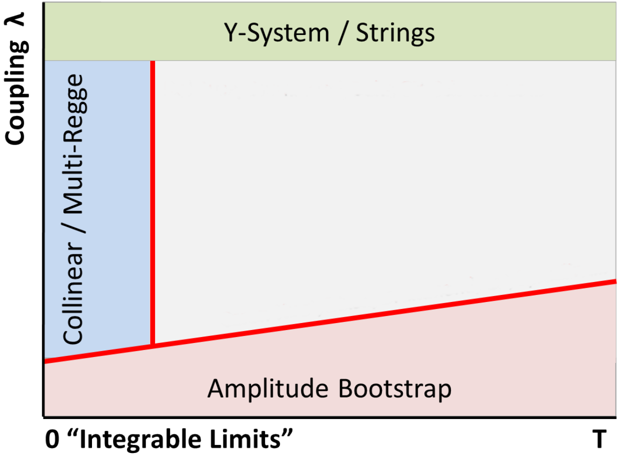
More generally,



Present and future of solving the simplest gauge theory

Amplitudes A_n with $n = 4, 5$ particles already known to all loops!

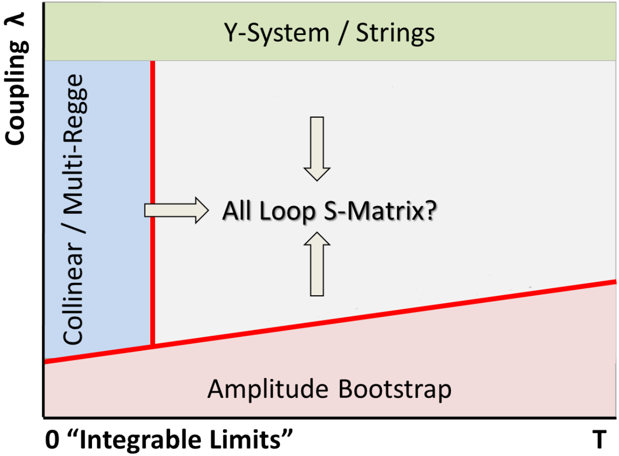
More generally, in this talk,



Present and future of solving the simplest gauge theory

Amplitudes A_n with $n = 4, 5$ particles already known to all loops!

More generally,



The Amplitude Bootstrap

The Amplitude Bootstrap

The most efficient method for computing planar $\mathcal{N} = 4$ amplitudes in general kinematics, at fixed order in the coupling.

The Amplitude Bootstrap

The most efficient method for computing planar $\mathcal{N} = 4$ amplitudes in general kinematics, at fixed order in the coupling.

A. Construct an ansatz for the amplitude assuming

The Amplitude Bootstrap

The most efficient method for computing planar $\mathcal{N} = 4$ amplitudes in general kinematics, at fixed order in the coupling.

A. Construct an ansatz for the amplitude assuming

1. What the general class of *functions* that suffices to express it is

The Amplitude Bootstrap

The most efficient method for computing planar $\mathcal{N} = 4$ amplitudes in general kinematics, at fixed order in the coupling.

A. Construct an ansatz for the amplitude assuming

1. What the general class of *functions* that suffices to express it is
2. What the function *arguments* (encoding the kinematics) are

The Amplitude Bootstrap

The most efficient method for computing planar $\mathcal{N} = 4$ amplitudes in general kinematics, at fixed order in the coupling.

A. Construct an ansatz for the amplitude assuming

1. What the general class of *functions* that suffices to express it is
2. What the function *arguments* (encoding the kinematics) are

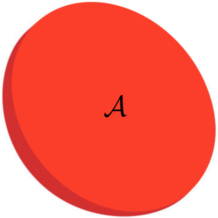
B. Fix the coefficients of the ansatz by imposing consistency conditions (e.g. known near-collinear or multi-Regge limiting behavior)

The Amplitude Bootstrap Evolution

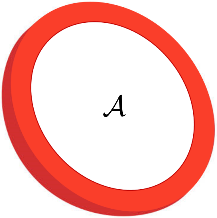
QFT Property	Computation

The Amplitude Bootstrap Evolution

QFT Property	Computation
Physical Branch Cuts	$\mathcal{A}_6^{(L)}, L = 3, 4$
[Gaiotto,Maldacena, Sever,Vieira]	[Dixon,Drummond, (Henn,) Duhr/Hippel,Pennington]



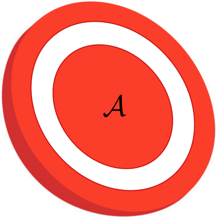
The Amplitude Bootstrap Evolution



QFT Property	Computation
Physical Branch Cuts [Gaiotto,Maldacena, Sever,Vieira]	$\mathcal{A}_6^{(L)}, L = 3, 4$ [Dixon,Drummond, (Henn,) Duhr/Hippel,Pennington]
Cluster Algebras [Golden,Goncharov, Spradlin,Vergu,Volovich]	$\mathcal{A}^{(3)}_{7,\text{MHV}}$ [Drummond, GP, Spradlin]

$$\mathcal{A}_{\text{MHV}} = \mathcal{A}(- - + \ldots +)\,, \quad \mathcal{A}_{\text{NMHV}} = \mathcal{A}(- - - + \ldots +)$$

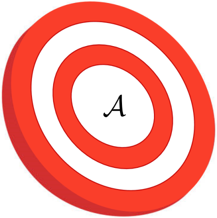
The Amplitude Bootstrap Evolution



QFT Property	Computation
Physical Branch Cuts [Gaiotto,Maldacena, Sever,Vieira]	$\mathcal{A}_6^{(L)}, L = 3, 4$ [Dixon,Drummond, (Henn,) Duhr/Hippel,Pennington]
Cluster Algebras [Golden,Goncharov, Spradlin,Vergu,Volovich]	$\mathcal{A}^{(3)}_{7,\text{MHV}}$ [Drummond, GP, Spradlin]
Steinmann Relation [Steinmann]	$\mathcal{A}_6^{(5)}, \mathcal{A}^{(3)}_{7,\text{NMHV}}, \mathcal{A}^{(4)}_{7,\text{MHV}}$ [Caron-Huot,Dixon,. . .] [Dixon,. . . , GP,Spradlin]

$$\mathcal{A}_{\text{MHV}} = \mathcal{A}(- - + \ldots +)\,, \quad \mathcal{A}_{\text{NMHV}} = \mathcal{A}(- - - + \ldots +)$$

The Amplitude Bootstrap Evolution



QFT Property	Computation
Physical Branch Cuts [Gaiotto,Maldacena, Sever,Vieira]	$\mathcal{A}_6^{(L)}, L = 3, 4$ [Dixon,Drummond, (Henn,) Duhr/Hippel,Pennington]
Cluster Algebras [Golden,Goncharov, Spradlin,Vergu,Volovich]	$\mathcal{A}_{7,\text{MHV}}^{(3)}$ [Drummond, GP, Spradlin]
Steinmann Relation [Steinmann]	$\mathcal{A}_6^{(5)}, \mathcal{A}_{7,\text{NMHV}}^{(3)}, \mathcal{A}_{7,\text{MHV}}^{(4)}$ [Caron-Huot,Dixon,...] [Dixon,..., GP,Spradlin]
Cluster Adjacency [Drummond,Foster, Gurdogan]	$\mathcal{A}_{7,\text{NMHV}}^{(4)}$ [Drummond,Foster, Gurdogan, GP]
Extended Steinmann $\Leftrightarrow$ Coaction Principle	$\mathcal{A}_6^{(6)}, \mathcal{A}_{6,\text{MHV}}^{(7)}$ [Caron-Huot,Dixon,Dulat, McLeod,Hippel,GP]

$$\mathcal{A}_{\text{MHV}} = \mathcal{A}(--+\ldots+)\,, \quad \mathcal{A}_{\text{NMHV}} = \mathcal{A}(---+\ldots+)$$

What are the right functions?

Multiple polylogarithms (MPLs)

What are the right functions?

Multiple polylogarithms (MPLs)

f_k is a MPL of weight k if its differential obeys

$$df_k = \sum_\alpha f_{k-1}^{(\alpha)} \, d\log\phi_\alpha$$

over some set of ϕ_α, with $f_{k-1}^{(\alpha)}$ functions of weight $k-1$.

What are the right functions?
Multiple polylogarithms (MPLs)

f_k is a MPL of weight k if its differential obeys

$$df_k = \sum_\alpha f_{k-1}^{(\alpha)}\, d\log\phi_\alpha \qquad \text{e.g. } d\mathrm{Li}_2(x) = -\log(1-x)\,d\log x\,,$$

over some set of ϕ_α, with $f_{k-1}^{(\alpha)}$ functions of weight $k-1$.

What are the right functions?

Multiple polylogarithms (MPLs)

f_k is a MPL of weight k if its differential obeys

$$df_k = \sum_\alpha f_{k-1}^{(\alpha)} \, d\log\phi_\alpha \qquad \text{e.g. } d\mathrm{Li}_2(x) = -\log(1-x) d\log x\,,$$

over some set of ϕ_α, with $f_{k-1}^{(\alpha)}$ functions of weight $k-1$.

Convenient tool for describing them: The **symbol** $\mathcal{S}(f_k)$ encapsulating recursive application of above definition (on $f_{k-1}^{(\alpha)}$ etc)

$$\mathcal{S}(f_k) = \sum_{\alpha_1,\ldots,\alpha_k} f_0^{(\alpha_1,\alpha_2,\ldots,\alpha_k)} \left(\phi_{\alpha_1} \otimes \cdots \otimes \phi_{\alpha_k}\right).$$

What are the right functions?
Multiple polylogarithms (MPLs)

f_k is a MPL of weight k if its differential obeys

$$df_k = \sum_\alpha f_{k-1}^{(\alpha)}\, d\log\phi_\alpha \qquad \text{e.g. } d\mathrm{Li}_2(x) = -\log(1-x)\, d\log x\,,$$

over some set of ϕ_α, with $f_{k-1}^{(\alpha)}$ functions of weight $k-1$.

Convenient tool for describing them: The **symbol** $\mathcal{S}(f_k)$ encapsulating recursive application of above definition (on $f_{k-1}^{(\alpha)}$ etc)

$$\mathcal{S}(f_k) = \sum_{\alpha_1,\ldots,\alpha_k} f_0^{(\alpha_1,\alpha_2,\ldots,\alpha_k)} \left(\phi_{\alpha_1}\otimes\cdots\otimes\phi_{\alpha_k}\right).$$

Collection of ϕ_α : **symbol alphabet** | $f_0^{(\alpha_1,\ldots,\alpha_k)}$ rational

What are the right functions?
Multiple polylogarithms (MPLs)

f_k is a MPL of weight k if its differential obeys

$$df_k = \sum_\alpha f_{k-1}^{(\alpha)}\, d\log\phi_\alpha \qquad \text{e.g. } d\mathrm{Li}_2(x) = -\log(1-x)d\log x\,,$$

over some set of ϕ_α, with $f_{k-1}^{(\alpha)}$ functions of weight $k-1$.

Convenient tool for describing them: The **symbol** $\mathcal{S}(f_k)$ encapsulating recursive application of above definition (on $f_{k-1}^{(\alpha)}$ etc)

$$\mathcal{S}(f_k) = \sum_{\alpha_1,\ldots,\alpha_k} f_0^{(\alpha_1,\alpha_2,\ldots,\alpha_k)}\,(\phi_{\alpha_1}\otimes\cdots\otimes\phi_{\alpha_k})\,.$$

Collection of ϕ_α : **symbol alphabet** | $f_0^{(\alpha_1,\ldots,\alpha_k)}$ rational

Empeirical evidence: L-loop amplitudes=MPLs of weight $k=2L$
[Duhr,Del Duca,Smirnov] [Arkani-Hamed,Bourjaily,Cachazo,Goncharov,Postnikov,Trnka] [GP]

What are the right variables?

What are the right variables?

More precisely, what is the symbol alphabet?

What are the right variables?

More precisely, what is the symbol alphabet? For $n = 6$, from explicit 2-loop computation [Duhr,Del Duca,Smirnov]

$$\phi_\alpha \in \Phi = \{a, b, c, m_u, m_v, m_w, y_u, y_v, y_w\},$$

$$a = \frac{u}{vw}, \quad m_u = \frac{1-u}{u}, \quad u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \ \text{\& cyclic } u \to v \to w.$$

$$y_u = \frac{u-z_+}{u-z_-}, \quad y_v = \frac{v-z_+}{v-z_-}, \quad y_w = \frac{w-z_+}{w-z_-},$$

$$z_\pm = \frac{1}{2}\left[-1 + u + v + w \pm \sqrt{\Delta}\right], \qquad \Delta = (1-u-v-w)^2 - 4uvw.$$

What are the right variables?

More precisely, what is the symbol alphabet? For $n = 6$, from explicit 2-loop computation [Duhr,Del Duca,Smirnov]

$$\phi_\alpha \in \Phi = \{a, b, c, m_u, m_v, m_w, y_u, y_v, y_w\},$$

$$a = \frac{u}{vw}, \quad m_u = \frac{1-u}{u}, \quad u = \frac{x_{13}^2\, x_{46}^2}{x_{14}^2\, x_{36}^2} \ \text{ \& cyclic } u \to v \to w\,.$$

$$y_u = \frac{u-z_+}{u-z_-}, \quad y_v = \frac{v-z_+}{v-z_-}, \quad y_w = \frac{w-z_+}{w-z_-},$$

$$z_\pm = \frac{1}{2}\Big[-1 + u + v + w \pm \sqrt{\Delta}\Big], \qquad \Delta = (1-u-v-w)^2 - 4uvw.$$

Not every sequence of ϕ_α corresponds to a candidate amplitude:

What are the right variables?

More precisely, what is the symbol alphabet? For $n = 6$, from explicit 2-loop computation [Duhr,Del Duca,Smirnov]

$$\phi_\alpha \in \Phi = \{a, b, c, m_u, m_v, m_w, y_u, y_v, y_w\},$$

$$a = \frac{u}{vw}, \quad m_u = \frac{1-u}{u}, \quad u = \frac{x_{13}^2\, x_{46}^2}{x_{14}^2\, x_{36}^2} \ \text{\& cyclic } u \to v \to w\,.$$

$$y_u = \frac{u-z_+}{u-z_-}, \quad y_v = \frac{v-z_+}{v-z_-}, \quad y_w = \frac{w-z_+}{w-z_-},$$

$$z_\pm = \frac{1}{2}\Big[-1 + u + v + w \pm \sqrt{\Delta}\Big], \qquad \Delta = (1-u-v-w)^2 - 4uvw.$$

Not every sequence of ϕ_α corresponds to a candidate amplitude:

- Locality: Amplitude singularities only when intermediate particles go on-shell ⇒ constrains first symbol entry to a, b, c.

What are the right variables?

More precisely, what is the symbol alphabet? For $n = 6$, from explicit 2-loop computation [Duhr,Del Duca,Smirnov]

$$\phi_\alpha \in \Phi = \{a, b, c, m_u, m_v, m_w, y_u, y_v, y_w\},$$

$$a = \frac{u}{vw}, \quad m_u = \frac{1-u}{u}, \quad u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \text{ \& cyclic } u \to v \to w.$$

$$y_u = \frac{u-z_+}{u-z_-}, \quad y_v = \frac{v-z_+}{v-z_-}, \quad y_w = \frac{w-z_+}{w-z_-},$$

$$z_\pm = \frac{1}{2}\left[-1 + u + v + w \pm \sqrt{\Delta}\right], \qquad \Delta = (1-u-v-w)^2 - 4uvw.$$

Not every sequence of ϕ_α corresponds to a candidate amplitude:

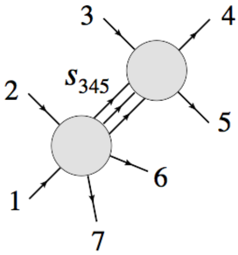
- Locality: Amplitude singularities only when intermediate particles go on-shell ⇒ constrains first symbol entry to a, b, c.
- Integrability: For given $\mathcal{S}$, ensures ∃ function f with this symbol,

$$\partial_{u_i}\partial_{u_j} f = \partial_{u_j}\partial_{u_i} f \quad \Rightarrow \text{linear relations between weights } k, k+1.$$

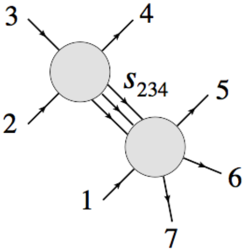
Steinmann Relations [Steinmann] [Cahill,Stapp] [Bartels,Lipatov,Sabio Vera]

Steinmann Relations [Steinmann] [Cahill,Stapp] [Bartels,Lipatov,Sabio Vera]

Double discontinuities vanish for any set of overlapping channels

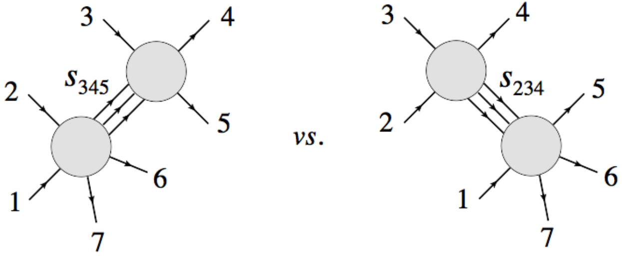


vs.



Steinmann Relations [Steinmann] [Cahill,Stapp] [Bartels,Lipatov,Sabio Vera]

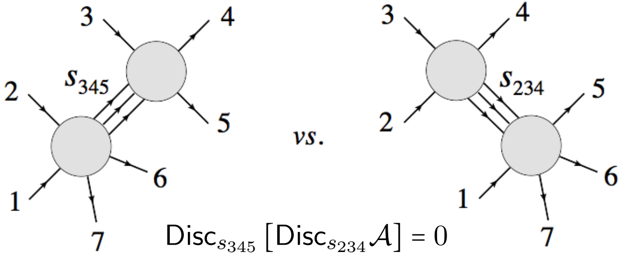
Double discontinuities vanish for any set of overlapping channels



- Channel labelled by Mandelstam invariant we analytically continue

Steinmann Relations [Steinmann] [Cahill,Stapp] [Bartels,Lipatov,Sabio Vera]

Double discontinuities vanish for any set of overlapping channels

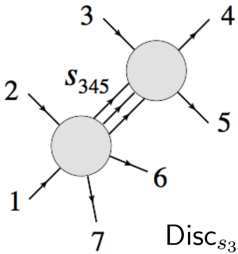


$$\mathrm{Disc}_{s_{345}}\left[\mathrm{Disc}_{s_{234}}\mathcal{A}\right] = 0$$

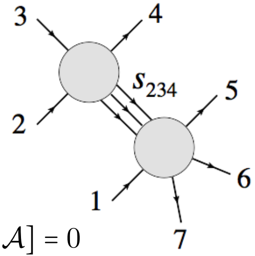
- Channel labelled by Mandelstam invariant we analytically continue
- Channels overlap if they divide particles in 4 nonempty sets.
  Here: $\{2\}$, $\{3,4\}$, $\{5\}$, and $\{6,7,1\}$

Steinmann Relations [Steinmann] [Cahill,Stapp] [Bartels,Lipatov,Sabio Vera]

Double discontinuities vanish for any set of overlapping channels



vs.



$$\text{Disc}_{s_{345}}\left[\text{Disc}_{s_{234}}\mathcal{A}\right] = 0$$

- Channel labelled by Mandelstam invariant we analytically continue
- Channels overlap if they divide particles in 4 nonempty sets. Here: $\{2\}$, $\{3,4\}$, $\{5\}$, and $\{6,7,1\}$
- Focus on $s_{123} \propto \sqrt{a}$ & cyclic (s_{i-1i} more subtle)

[Caron-Huot,Dixon,McLeod,Hippel] [Dixon,Drummond,Harrington,McLeod,GP,Spradlin]

No b, c can appear after a in second symbol entry & cyclic

Steinmann relations

No b, c can appear after a in 2^{nd} symbol entry & cyclic

Extended Steinmann relations

By inspecting known amplitude through five loops:

[Caron-Huot,Dixon,(Dulat,)McLeod,Hippel,GP]

No b, c can appear after a in *any* symbol entry & cyclic

Extended Steinmann relations

By inspecting known amplitude through five loops:

[Caron-Huot,Dixon,(Dulat,)McLeod,Hippel,GP]

> No b, c can appear after a in *any* symbol entry & cyclic

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
First entry	1	3	9	26	75	218	643	1929	5897	?	?	?	?	?
Steinmann	1	3	6	13	29	63	134	277	562	1117	2192	4263	8240	?
Ext. Stein.	1	3	6	13	26	51	98	184	340	613	1085	1887	3224	5431

Figure: Dimensions of the hexagon, Steinmann hexagon, and extended Steinmann hexagon spaces at symbol level.

- Drastically reduces number of candidate amplitudes

Extended Steinmann relations

By inspecting known amplitude through five loops:

[Caron-Huot,Dixon,(Dulat,)McLeod,Hippel,GP]

No b, c can appear after a in *any* symbol entry & cyclic

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
First entry	1	3	9	26	75	218	643	1929	5897	?	?	?	?	?
Steinmann	1	3	6	13	29	63	134	277	562	1117	2192	4263	8240	?
Ext. Stein.	1	3	6	13	26	51	98	184	340	613	1085	1887	3224	5431

Figure: Dimensions of the hexagon, Steinmann hexagon, and extended Steinmann hexagon spaces at symbol level.

- Drastically reduces number of candidate amplitudes
- Must be consequence of original Steinmann holding on any Riemann sheet.

Extended Steinmann relations

By inspecting known amplitude through five loops:

[Caron-Huot,Dixon,(Dulat,)McLeod,Hippel,GP]

No b, c can appear after a in *any* symbol entry & cyclic

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
First entry	1	3	9	26	75	218	643	1929	5897	?	?	?	?	?
Steinmann	1	3	6	13	29	63	134	277	562	1117	2192	4263	8240	?
Ext. Stein.	1	3	6	13	26	51	98	184	340	613	1085	1887	3224	5431

Figure: Dimensions of the hexagon, Steinmann hexagon, and extended Steinmann hexagon spaces at symbol level.

- Drastically reduces number of candidate amplitudes
- Must be consequence of original Steinmann holding on any Riemann sheet.
- Potentially universal: Valid for *individual integrals*!
  [Drummond,Foster,Gürdogan] [Caron-Huot,Dixon,Hippel,McLeod,GP,]

The Coaction on MPLs

Space of MPLs of weight n, $\mathcal{G}_n$, endowed with coaction Δ that "decomposes" it into a tensor product [Goncharov] [Brown]

$$\Delta\mathcal{G}_n \equiv \sum_{k=0}^{n} \Delta_{n-k,k}\mathcal{G}_n = \sum_{k=0}^{n} \mathcal{G}_{n-k} \otimes [\mathcal{G}_k \operatorname{mod}(i\pi)] \,.$$

The Coaction on MPLs

Space of MPLs of weight n, $\mathcal{G}_n$, endowed with coaction Δ that "decomposes" it into a tensor product [Goncharov] [Brown]

$$\Delta\mathcal{G}_n \equiv \sum_{k=0}^{n} \Delta_{n-k,k}\mathcal{G}_n = \sum_{k=0}^{n} \mathcal{G}_{n-k} \otimes [\mathcal{G}_k \, \mathrm{mod}(i\pi)] \, .$$

Also applies to transcendental numbers, e.g.

$$\Delta(i\pi) = (i\pi) \otimes 1 \, , \quad \Delta(\zeta_3^2) = (\zeta_3^2) \otimes 1 + 2\zeta_3 \otimes \zeta_3 + 1 \otimes (\zeta_3^2) \, . \tag{1}$$

The Coaction on MPLs

Space of MPLs of weight n, $\mathcal{G}_n$, endowed with coaction Δ that "decomposes" it into a tensor product [Goncharov] [Brown]

$$\Delta \mathcal{G}_n \equiv \sum_{k=0}^{n} \Delta_{n-k,k} \mathcal{G}_n = \sum_{k=0}^{n} \mathcal{G}_{n-k} \otimes [\mathcal{G}_k \operatorname{mod}(i\pi)] \, .$$

Also applies to transcendental numbers, e.g.

$$\Delta(i\pi) = (i\pi) \otimes 1 \, , \quad \Delta(\zeta_3^2) = (\zeta_3^2) \otimes 1 + 2\zeta_3 \otimes \zeta_3 + 1 \otimes (\zeta_3^2) \, . \tag{1}$$

Valid by construction on $\mathcal{G}$, upgrades symbols to functions.

The Coaction on MPLs

Space of MPLs of weight n, $\mathcal{G}_n$, endowed with coaction Δ that "decomposes" it into a tensor product [Goncharov] [Brown]

$$\Delta\mathcal{G}_n \equiv \sum_{k=0}^{n} \Delta_{n-k,k}\mathcal{G}_n = \sum_{k=0}^{n} \mathcal{G}_{n-k} \otimes [\mathcal{G}_k \, \mathrm{mod}(i\pi)] \, .$$

Also applies to transcendental numbers, e.g.

$$\Delta(i\pi) = (i\pi) \otimes 1 \, , \quad \Delta(\zeta_3^2) = (\zeta_3^2) \otimes 1 + 2\zeta_3 \otimes \zeta_3 + 1 \otimes (\zeta_3^2) \, . \tag{1}$$

Valid by construction on $\mathcal{G}$, upgrades symbols to functions. However, quite nontrivial closure, or "coaction principle" on certain subspaces $\mathcal{H} \subset \mathcal{G}$:

$$\Delta\mathcal{H} \subset \mathcal{H} \otimes \mathcal{G} \, . \tag{2}$$

The Coaction on MPLs

Space of MPLs of weight n, $\mathcal{G}_n$, endowed with coaction Δ that "decomposes" it into a tensor product [Goncharov][Brown]

$$\Delta\mathcal{G}_n \equiv \sum_{k=0}^{n} \Delta_{n-k,k}\mathcal{G}_n = \sum_{k=0}^{n} \mathcal{G}_{n-k} \otimes [\mathcal{G}_k \,\mathrm{mod}(i\pi)] \,.$$

Also applies to transcendental numbers, e.g.

$$\Delta(i\pi) = (i\pi)\otimes 1\,, \quad \Delta(\zeta_3^2) = (\zeta_3^2)\otimes 1 + 2\zeta_3\otimes\zeta_3 + 1\otimes(\zeta_3^2)\,. \tag{1}$$

Valid by construction on $\mathcal{G}$, upgrades symbols to functions. However, quite nontrivial closure, or "coaction principle" on certain subspaces $\mathcal{H}\subset\mathcal{G}$:

$$\Delta\mathcal{H}\subset\mathcal{H}\otimes\mathcal{G}\,. \tag{2}$$

Example: If $\zeta_3\notin\mathcal{H}$, then (1)-(2) imply $\zeta_3^2\notin\mathcal{H}$. "Exclusion principle"!

The Coaction on MPLs

Space of MPLs of weight n, $\mathcal{G}_n$, endowed with coaction Δ that "decomposes" it into a tensor product [Goncharov] [Brown]

$$\Delta \mathcal{G}_n \equiv \sum_{k=0}^{n} \Delta_{n-k,k} \mathcal{G}_n = \sum_{k=0}^{n} \mathcal{G}_{n-k} \otimes [\mathcal{G}_k \bmod(i\pi)] \, .$$

Also applies to transcendental numbers, e.g.

$$\Delta(i\pi) = (i\pi) \otimes 1 \, , \quad \Delta(\zeta_3^2) = (\zeta_3^2) \otimes 1 + 2\zeta_3 \otimes \zeta_3 + 1 \otimes (\zeta_3^2) \, . \tag{1}$$

Valid by construction on $\mathcal{G}$, upgrades symbols to functions. However, quite nontrivial closure, or "coaction principle" on certain subspaces $\mathcal{H} \subset \mathcal{G}$:

$$\Delta \mathcal{H} \subset \mathcal{H} \otimes \mathcal{G} \, . \tag{2}$$

Example: If $\zeta_3 \notin \mathcal{H}$, then (1)-(2) imply $\zeta_3^2 \notin \mathcal{H}$. "Exclusion principle"!

Previously observed in other settings. [Schlotterer, Stieberger] [Panzer, Schnetz] [Schnetz]

The Coaction Principle for the Six-particle Amplitude

The Coaction Principle for the Six-particle Amplitude

Apply to extended Steinmann hexagon space $\mathcal{H}^{\text{hex}}$ of amplitude and its iterated derivatives, at point $a = b = c = 1$, or $u = v = w = 1$.

The Coaction Principle for the Six-particle Amplitude

Apply to extended Steinmann hexagon space $\mathcal{H}^{\text{hex}}$ of amplitude and its iterated derivatives, at point $a = b = c = 1$, or $u = v = w = 1$.

Weight	Multiple Zeta Values	Appear in $\mathcal{H}^{\text{hex}}\big\vert_{u,v,w\to 1}$
0	1	1
1	–	–
2	ζ_2	ζ_2
3	ζ_3	–
4	ζ_4	ζ_4
5	$\zeta_5,\ \zeta_3\zeta_2$	$5\zeta_5 - 2\zeta_3\zeta_2$

The Coaction Principle for the Six-particle Amplitude

Apply to extended Steinmann hexagon space $\mathcal{H}^{\text{hex}}$ of amplitude and its iterated derivatives, at point $a = b = c = 1$, or $u = v = w = 1$.

Weight	Multiple Zeta Values	Appear in $\mathcal{H}^{\text{hex}}\big|_{u,v,w\to 1}$
0	1	1
1	–	–
2	ζ_2	ζ_2
3	ζ_3	–
4	ζ_4	ζ_4
5	$\zeta_5,\ \zeta_3\zeta_2$	$5\zeta_5 - 2\zeta_3\zeta_2$

$$\mathcal{A}_{\text{MHV}}^{\text{fin,old}\,(3)}(1,1,1) = \tfrac{413}{3}\,\zeta_6 + 8(\zeta_3)^2\,, \quad \mathcal{A}_{\text{NMHV}}^{\text{fin,old}\,(3)}(1,1,1) = -\tfrac{940}{3}\zeta_6 + 8(\zeta_3)^2$$

Shift in *common* normalization factor containing known IR divergences,

$$\mathcal{A} = \mathcal{A}^{\text{IR,old}}\mathcal{A}^{\text{fin,old}} = (\rho\mathcal{A}^{\text{IR,old}})(\mathcal{A}^{\text{fin,old}}/\rho)\,, \quad \rho = 1 + 8(\zeta_3)^2 g^6 + \mathcal{O}(g^8)\,,$$

The Coaction Principle for the Six-particle Amplitude

Apply to extended Steinmann hexagon space $\mathcal{H}^{\text{hex}}$ of amplitude and its iterated derivatives, at point $a = b = c = 1$, or $u = v = w = 1$.

Weight	Multiple Zeta Values	Appear in $\mathcal{H}^{\text{hex}}\big|_{u,v,w\to 1}$
0	1	1
1	–	–
2	ζ_2	ζ_2
3	ζ_3	–
4	ζ_4	ζ_4
5	$\zeta_5,\ \zeta_3\zeta_2$	$5\zeta_5 - 2\zeta_3\zeta_2$

$$\mathcal{A}_{\text{MHV}}^{\text{fin,old}\,(3)}(1,1,1) = \frac{413}{3}\,\zeta_6 + 8(\zeta_3)^2\,, \quad \mathcal{A}_{\text{NMHV}}^{\text{fin,old}\,(3)}(1,1,1) = -\frac{940}{3}\zeta_6 + 8(\zeta_3)^2$$

Shift in *common* normalization factor containing known IR divergences,

$$\mathcal{A} = \mathcal{A}^{\text{IR,old}}\mathcal{A}^{\text{fin,old}} = (\rho\mathcal{A}^{\text{IR,old}})(\mathcal{A}^{\text{fin,old}}/\rho)\,, \quad \rho = 1 + 8(\zeta_3)^2 g^6 + \mathcal{O}(g^8)\,,$$

- *Reduces* size of $\mathcal{H}^{\text{hex}}$ $\Rightarrow$ Simpler to bootstrap at higher weight!

Six-gluon amplitude: Results I [Caron-Huot,Dixon,Dulat,McLeod,Hippel,GP'19A]

Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$
1. $\mathcal{H}^{\text{hex}}$	6	27	105	372	1214	3692?
2. Symmetry	(2,4)	(7,16)	(22,56)	(66,190)	(197,602)	(567,1795?)
3. Final-entry	(1,1)	(4,3)	(11,6)	(30,16)	(85,39)	(236,102)
4. Collinear	(0,0)	(0,0)	$(0^*,0^*)$	$(0^*,2^*)$	$(1^{*3},5^{*3})$	$(6^{*2},17^{*2})$
5. LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^{*2},2^{*2})$
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	$(0^*,0^*)$	$(1^*,0^{*2})$
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	$(1,0^*)$
8. N^3LL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
9. Full MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
10. T^1 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(1,0)
11. T^2 OPE	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

Table 1. Remaining parameters in the ansätze for the (MHV, NMHV) amplitude after each constraint is applied, at each loop order. The superscript "$*$" ("$*n$") denotes an additional ambiguity (n ambiguities) which arises only due to lack of knowledge of the cosmic normalization constant ρ at the given stage. The "?" indicates an ambiguity about the number of weight 12 odd functions that are "dropouts"; they are allowed at symbol level but not function level. The seven-loop MHV amplitude was constrained in a somewhat different order. As the parameter counts are not directly comparable it is omitted from the table.

Six-gluon amplitude: Results II

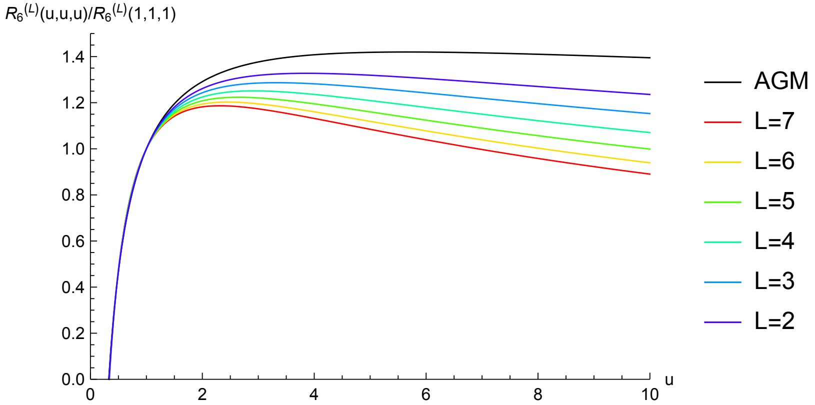


Figure: Normalized perturbative coefficients of the MHV amplitude (remainder), $R_6^{(L)}(u,u,u)/R_6^{(L)}(1,1,1)$, for $L=2$ to 7, plotted along with the strong-coupling result of AGM. The curves all have a remarkably similar shape for $u \lesssim 1$.

Beyond six particles

Beyond six particles

What is the symbol alphabet for n-particle scattering?

Beyond six particles

What is the symbol alphabet for n-particle scattering?

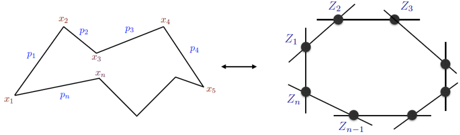
- *Variables a_m of a Grassmannian $Gr(4,n)$ cluster algebra!*
  [Golden,Goncharov,Spradlin,Vergu,Volovich]

Beyond six particles

What is the symbol alphabet for n-particle scattering?

- *Variables a_m of a Grassmannian $Gr(4,n)$ cluster algebra!* [Golden,Goncharov,Spradlin,Vergu,Volovich]

Emerge when parametrizing planar n-particle massless kinematics in terms of n *momentum twistors* Z_i on $\mathbb{CP}^3$ ($Z_i \sim \lambda Z_i$),



$$x_i \sim Z_{i-1} \wedge Z_i$$

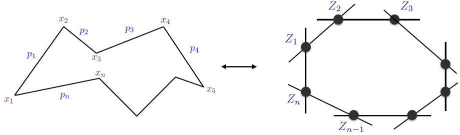
$$(x_i - x_j)^2 \sim \epsilon_{IJKL} Z^I_{i-1} Z^J_i Z^K_{j-1} Z^L_j = \det(Z_{i-1} Z_i Z_{j-1} Z_j) \equiv \langle i-1 i j-1 j \rangle$$

Beyond six particles

What is the symbol alphabet for n-particle scattering?

- *Variables a_m of a Grassmannian $Gr(4, n)$ cluster algebra!* [Golden,Goncharov,Spradlin,Vergu,Volovich]

Emerge when parametrizing planar n-particle massless kinematics in terms of n *momentum twistors* Z_i on $\mathbb{CP}^3$ ($Z_i \sim \lambda Z_i$),



$$x_i \sim Z_{i-1} \wedge Z_i$$

$$(x_i - x_j)^2 \sim \epsilon_{IJKL} Z^I_{i-1} Z^J_i Z^K_{j-1} Z^L_j = \det(Z_{i-1} Z_i Z_{j-1} Z_j) \equiv \langle i-1 i j-1 j \rangle$$

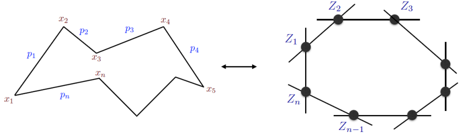
Cluster "$\mathcal{A}$-coordinates" a_m: Certain homogeneous polynomials of $\langle ijkl \rangle$,

Beyond six particles

What is the symbol alphabet for n-particle scattering?

- *Variables a_m of a Grassmannian $Gr(4,n)$ cluster algebra!* [Golden,Goncharov,Spradlin,Vergu,Volovich]

Emerge when parametrizing planar n-particle massless kinematics in terms of n *momentum twistors* Z_i on $\mathbb{CP}^3$ ($Z_i \sim \lambda Z_i$),



$$x_i \sim Z_{i-1} \wedge Z_i$$

$$(x_i - x_j)^2 \sim \epsilon_{IJKL} Z^I_{i-1} Z^J_i Z^K_{j-1} Z^L_j = \det(Z_{i-1} Z_i Z_{j-1} Z_j) \equiv \langle i-1 i j-1 j \rangle$$

Cluster "$\mathcal{A}$-coordinates" a_m: Certain homogeneous polynomials of $\langle ijkl \rangle$,

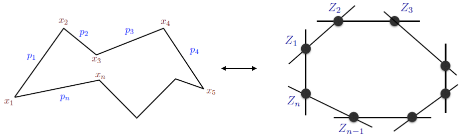
- Grouped into overlapping subsets $\{a_1, \ldots, a_n\}$ of *rank* n, the *clusters*

Beyond six particles

What is the symbol alphabet for n-particle scattering?

- *Variables a_m of a Grassmannian $Gr(4,n)$ cluster algebra!* [Golden,Goncharov,Spradlin,Vergu,Volovich]

Emerge when parametrizing planar n-particle massless kinematics in terms of n *momentum twistors* Z_i on $\mathbb{CP}^3$ ($Z_i \sim \lambda Z_i$),



$$x_i \sim Z_{i-1} \wedge Z_i$$

$$(x_i - x_j)^2 \sim \epsilon_{IJKL} Z^I_{i-1} Z^J_i Z^K_{j-1} Z^L_j = \det(Z_{i-1} Z_i Z_{j-1} Z_j) \equiv \langle i-1 i j-1 j \rangle$$

Cluster "$\mathcal{A}$-coordinates" a_m: Certain homogeneous polynomials of $\langle ijkl \rangle$,

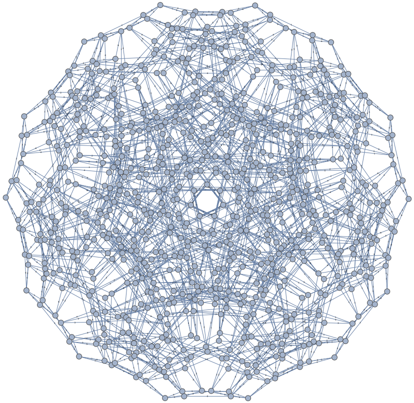
- Grouped into overlapping subsets $\{a_1, \ldots, a_n\}$ of *rank* n, the *clusters*
- Constructed recursively from initial cluster via *mutations*

Cluster adjacency

Cluster adjacency

Remarkably, extended Steinmann relations are equivalent to rule:
[Drummond,Foster,Gürdogan'17]

Two symbol letters are adjacent only if they belong to the same cluster

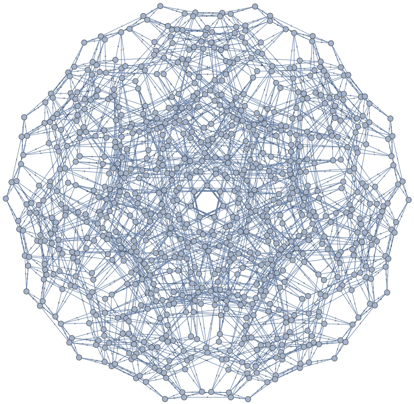


Cluster adjacency

Remarkably, extended Steinmann relations are equivalent to rule: [Drummond,Foster,Gürdogan'17]

Two symbol letters are adjacent only if they belong to the same cluster

In fact, for NMHV, *cluster adjacency* somewhat stronger: Relates transcendental to rational part of the amplitude [Drummond,Foster,Gürdogan'18]

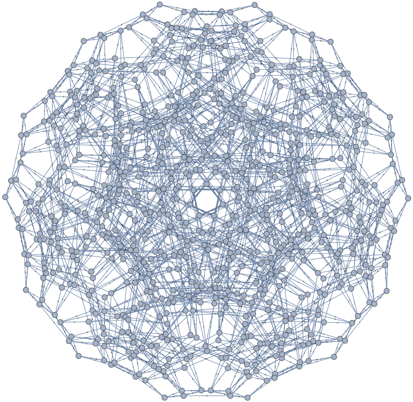


Cluster adjacency

Remarkably, extended Steinmann relations are equivalent to rule: [Drummond,Foster,Gürdogan'17]

Two symbol letters are adjacent only if they belong to the same cluster

In fact, for NMHV, *cluster adjacency* somewhat stronger: Relates transcendental to rational part of the amplitude [Drummond,Foster,Gürdogan'18]



Application: the 4-loop NMHV 7-particle amplitude, and its multi-Regge limit

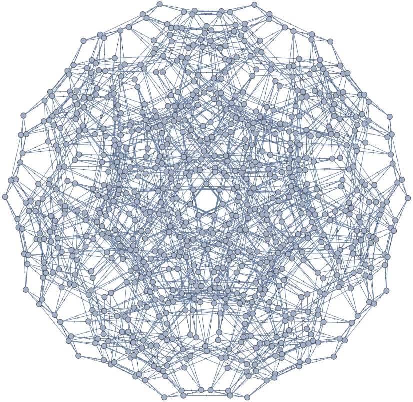
[Drummond,Foster,Gürdogan,Papathanasiou]

Cluster adjacency

Remarkably, extended Steinmann relations are equivalent to rule:
[Drummond,Foster,Gürdogan'17]

Two symbol letters are adjacent only if they belong to the same cluster

In fact, for NMHV, *cluster adjacency* somewhat stronger: Relates transcendental to rational part of the amplitude
[Drummond,Foster,Gürdogan'18]



Application: the 4-loop NMHV 7-particle amplitude, and its multi-Regge limit

[Drummond,Foster,Gürdogan,Papathanasiou]

Bootstrap application to $\mathcal{A}_5$ in QCD
[Gehrmann,Henn,Lo Presti][Abreu,Dormans,Febres Cordero,Ita,Page,Sotnikov]

Higher Multiplicity n

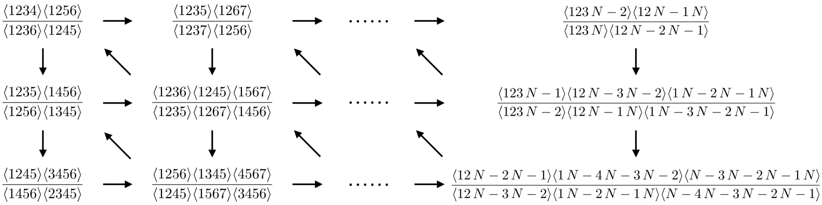
Higher Multiplicity n

For $n \geq 8$, $Gr(4,n)$ cluster algebra becomes infinite

$$
\begin{array}{ccccccc}
\frac{\langle 1234\rangle\langle 1256\rangle}{\langle 1236\rangle\langle 1245\rangle} & \longrightarrow & \frac{\langle 1235\rangle\langle 1267\rangle}{\langle 1237\rangle\langle 1256\rangle} & \longrightarrow & \cdots\cdots & \longrightarrow & \frac{\langle 123\,N-2\rangle\langle 12\,N-1\,N\rangle}{\langle 123\,N\rangle\langle 12\,N-2\,N-1\rangle} \\
\downarrow & \nwarrow & \downarrow & \nwarrow & & \nwarrow & \downarrow \\
\frac{\langle 1235\rangle\langle 1456\rangle}{\langle 1256\rangle\langle 1345\rangle} & \longrightarrow & \frac{\langle 1236\rangle\langle 1245\rangle\langle 1567\rangle}{\langle 1235\rangle\langle 1267\rangle\langle 1456\rangle} & \longrightarrow & \cdots\cdots & \longrightarrow & \frac{\langle 123\,N-1\rangle\langle 12\,N-3\,N-2\rangle\langle 1\,N-2\,N-1\,N\rangle}{\langle 123\,N-2\rangle\langle 12\,N-1\,N\rangle\langle 1\,N-3\,N-2\,N-1\rangle} \\
\downarrow & \nwarrow & \downarrow & \nwarrow & & \nwarrow & \downarrow \\
\frac{\langle 1245\rangle\langle 3456\rangle}{\langle 1456\rangle\langle 2345\rangle} & \longrightarrow & \frac{\langle 1256\rangle\langle 1345\rangle\langle 4567\rangle}{\langle 1245\rangle\langle 1567\rangle\langle 3456\rangle} & \longrightarrow & \cdots\cdots & \longrightarrow & \frac{\langle 12\,N-2\,N-1\rangle\langle 1\,N-4\,N-3\,N-2\rangle\langle N-3\,N-2\,N-1\,N\rangle}{\langle 12\,N-3\,N-2\rangle\langle 1\,N-2\,N-1\,N\rangle\langle N-4\,N-3\,N-2\,N-1\rangle}
\end{array}
$$

Higher Multiplicity n

For $n \geq 8$, $Gr(4,n)$ cluster algebra becomes infinite



- However, in multi-Regge limit, $Gr(4,N) \to A_{N-5} \times A_{N-5}$: finite!
  [Del Duca,Druc,Drummond,Duhr,Dulat,Marzucca,GP,Verbeek]

Higher Multiplicity n

For $n \geq 8$, $Gr(4,n)$ cluster algebra becomes infinite

$$
\begin{array}{ccccccc}
\frac{\langle 1234\rangle\langle 1256\rangle}{\langle 1236\rangle\langle 1245\rangle} & \longrightarrow & \frac{\langle 1235\rangle\langle 1267\rangle}{\langle 1237\rangle\langle 1256\rangle} & \longrightarrow & \cdots\cdots & \longrightarrow & \frac{\langle 123\,N-2\rangle\langle 12\,N-1\,N\rangle}{\langle 123\,N\rangle\langle 12\,N-2\,N-1\rangle} \\
\downarrow & \nwarrow & \downarrow & \nwarrow & & \nwarrow & \downarrow \\
\frac{\langle 1235\rangle\langle 1456\rangle}{\langle 1256\rangle\langle 1345\rangle} & \longrightarrow & \frac{\langle 1236\rangle\langle 1245\rangle\langle 1567\rangle}{\langle 1235\rangle\langle 1267\rangle\langle 1456\rangle} & \longrightarrow & \cdots\cdots & \longrightarrow & \frac{\langle 123\,N-1\rangle\langle 12\,N-3\,N-2\rangle\langle 1\,N-2\,N-1\,N\rangle}{\langle 123\,N-2\rangle\langle 12\,N-1\,N\rangle\langle 1\,N-3\,N-2\,N-1\rangle} \\
\downarrow & \nwarrow & \downarrow & \nwarrow & & \nwarrow & \downarrow \\
\frac{\langle 1245\rangle\langle 3456\rangle}{\langle 1456\rangle\langle 2345\rangle} & \longrightarrow & \frac{\langle 1256\rangle\langle 1345\rangle\langle 4567\rangle}{\langle 1245\rangle\langle 1567\rangle\langle 3456\rangle} & \longrightarrow & \cdots\cdots & \longrightarrow & \frac{\langle 12\,N-2\,N-1\rangle\langle 1\,N-4\,N-3\,N-2\rangle\langle N-3\,N-2\,N-1\,N\rangle}{\langle 12\,N-3\,N-2\rangle\langle 1\,N-2\,N-1\,N\rangle\langle N-4\,N-3\,N-2\,N-1\rangle}
\end{array}
$$

- However, in multi-Regge limit, $Gr(4,N) \to A_{N-5} \times A_{N-5}$: finite!
  [Del Duca,Druc,Drummond,Duhr,Dulat,Marzucca,GP,Verbeek]
- Recently, "tropicalization" proposed to tame this infinity in general kinematics [Arkani-Hamed,Lam,Spradlin;to appear] [Drummond,Foster,Gurdogan,Kalousios]
- Can correspond to amplitude for MHV case at most. [Henke,GP;in progress] Verification/Refinement?

Conclusions

In this presentation, I talked about the beauty, simplicity and utility of amplitudes in maximally supersymmetric Yang-Mills theory.

Conclusions

In this presentation, I talked about the beauty, simplicity and utility of amplitudes in maximally supersymmetric Yang-Mills theory.

I demonstrated that exploiting expected analytic structure of amplitudes tremendously improves perturbation theory:

Conclusions

In this presentation, I talked about the beauty, simplicity and utility of amplitudes in maximally supersymmetric Yang-Mills theory.

I demonstrated that exploiting expected analytic structure of amplitudes tremendously improves perturbation theory:

- Extended Steinmann relation/cluster adjacency: Potential universality

Conclusions

In this presentation, I talked about the beauty, simplicity and utility of amplitudes in maximally supersymmetric Yang-Mills theory.

I demonstrated that exploiting expected analytic structure of amplitudes tremendously improves perturbation theory:

- Extended Steinmann relation/cluster adjacency: Potential universality
- Coaction principle: $\Delta\mathcal{A} = \mathcal{A} \otimes \ldots$, "superselection rule"

Conclusions

In this presentation, I talked about the beauty, simplicity and utility of amplitudes in maximally supersymmetric Yang-Mills theory.

I demonstrated that exploiting expected analytic structure of amplitudes tremendously improves perturbation theory:

- Extended Steinmann relation/cluster adjacency: Potential universality
- Coaction principle: $\Delta\mathcal{A} = \mathcal{A} \otimes \ldots$, "superselection rule"
- Applied to determine 6 gluons/7 loops, 7 gluons/4 loops

Conclusions

In this presentation, I talked about the beauty, simplicity and utility of amplitudes in maximally supersymmetric Yang-Mills theory.

I demonstrated that exploiting expected analytic structure of amplitudes tremendously improves perturbation theory:

- Extended Steinmann relation/cluster adjacency: Potential universality
- Coaction principle: $\Delta\mathcal{A} = \mathcal{A} \otimes \ldots$, "superselection rule"
- Applied to determine 6 gluons/7 loops, 7 gluons/4 loops

Ultimately, can the integrability of planar SYM theory, together with a thorough knowledge of the analytic structure of its amplitudes, lead us to the theory's exact S-matrix?

Conclusions

In this presentation, I talked about the beauty, simplicity and utility of amplitudes in maximally supersymmetric Yang-Mills theory.

I demonstrated that exploiting expected analytic structure of amplitudes tremendously improves perturbation theory:

- Extended Steinmann relation/cluster adjacency: Potential universality
- Coaction principle: $\Delta\mathcal{A} = \mathcal{A} \otimes \ldots$, "superselection rule"
- Applied to determine 6 gluons/7 loops, 7 gluons/4 loops

> Ultimately, can the integrability of planar SYM theory, together with a thorough knowledge of the analytic structure of its amplitudes, lead us to the theory's exact S-matrix?

Higher-point bootstrap? All-order resummation? QCD applications?

Amplitude Normalization

Factor out normalization factor containing known IR divergences $\mathcal{A}^{\mathrm{IR}}$,

$$\mathcal{A} = \mathcal{A}^{\mathrm{IR}} \mathcal{A}_{\mathrm{fin}}.$$

Amplitude Normalization

Factor out normalization factor containing known IR divergences $\mathcal{A}^{\text{IR}}$,

$$\mathcal{A} = \mathcal{A}^{\text{IR}} \mathcal{A}_{\text{fin}}.$$

Meaningful to tune $\mathcal{A}^{\text{IR}}$, so that the finite $\mathcal{A}_{\text{fin}}$ we compute, becomes *simpler/minimal*. Possible choices:

Amplitude Normalization

Factor out normalization factor containing known IR divergences $\mathcal{A}^{\mathsf{IR}}$,

$$\mathcal{A} = \mathcal{A}^{\mathsf{IR}}\mathcal{A}_{\mathsf{fin}}.$$

Meaningful to tune $\mathcal{A}^{\mathsf{IR}}$, so that the finite $\mathcal{A}_{\mathsf{fin}}$ we compute, becomes *simpler/minimal*. Possible choices:

- Originally $\mathcal{A}^{\mathsf{IR}} = \mathcal{A}^{\mathsf{BDS}}$, essentially the exponentiated $\mathcal{A}^{(1)}$.

Amplitude Normalization

Factor out normalization factor containing known IR divergences $\mathcal{A}^{\text{IR}}$,

$$\mathcal{A} = \mathcal{A}^{\text{IR}} \mathcal{A}_{\text{fin}}.$$

Meaningful to tune $\mathcal{A}^{\text{IR}}$, so that the finite $\mathcal{A}_{\text{fin}}$ we compute, becomes *simpler/minimal*. Possible choices:

- Originally $\mathcal{A}^{\text{IR}} = \mathcal{A}^{\text{BDS}}$, essentially the exponentiated $\mathcal{A}^{(1)}$.
- Next, $\mathcal{A}^{\text{IR}} = \mathcal{A}^{\text{BDS-like}}$, all finite $s_{i-1ii+1}$ dependence removed.
  $\Rightarrow$ Only then $\mathcal{A}_{\text{fin}}$ satisfies (extended) Steinmann relations!
- Most recently, "cosmic normalization" $\mathcal{A}^{\text{IR}} = \mathcal{A}^{\text{BDS-like}} \rho$,

[Caron-Huot,Dixon,Dulat,McLeod,Hippel,GP'19A+B]

Amplitude Normalization

Factor out normalization factor containing known IR divergences $\mathcal{A}^{\text{IR}}$,

$$\mathcal{A} = \mathcal{A}^{\text{IR}} \mathcal{A}_{\text{fin}}.$$

Meaningful to tune $\mathcal{A}^{\text{IR}}$, so that the finite $\mathcal{A}_{\text{fin}}$ we compute, becomes *simpler/minimal*. Possible choices:

- Originally $\mathcal{A}^{\text{IR}} = \mathcal{A}^{\text{BDS}}$, essentially the exponentiated $\mathcal{A}^{(1)}$.
- Next, $\mathcal{A}^{\text{IR}} = \mathcal{A}^{\text{BDS-like}}$, all finite $s_{i-1ii+1}$ dependence removed.
  $\Rightarrow$ Only then $\mathcal{A}_{\text{fin}}$ satisfies (extended) Steinmann relations!
- Most recently, "cosmic normalization" $\mathcal{A}^{\text{IR}} = \mathcal{A}^{\text{BDS-like}} \rho$,

[Caron-Huot,Dixon,Dulat,McLeod,Hippel,GP'19A+B]

$$\rho(g^2) = 1 + 8(\zeta_3)^2 g^6 - 160\zeta_3\zeta_5\, g^8 + \Big[1680\zeta_3\zeta_7 + 912(\zeta_5)^2 - 32\zeta_4(\zeta_3)^2\Big] g^{10} - \Big[18816\zeta_3\zeta_9 + 20832\zeta_5\zeta_7 - 448\zeta_4\zeta_3\zeta_5 - 400\zeta_6(\zeta_3)^2\Big] g^{12} + \mathcal{O}(g^{14})$$

so as to satisfy (cosmic Galois) coaction principle.

Amplitude Normalization

Factor out normalization factor containing known IR divergences $\mathcal{A}^{\mathrm{IR}}$,

$$\mathcal{A} = \mathcal{A}^{\mathrm{IR}} \mathcal{A}_{\mathrm{fin}}.$$

Meaningful to tune $\mathcal{A}^{\mathrm{IR}}$, so that the finite $\mathcal{A}_{\mathrm{fin}}$ we compute, becomes *simpler/minimal*. Possible choices:

- Originally $\mathcal{A}^{\mathrm{IR}} = \mathcal{A}^{\mathrm{BDS}}$, essentially the exponentiated $\mathcal{A}^{(1)}$.
- Next, $\mathcal{A}^{\mathrm{IR}} = \mathcal{A}^{\mathrm{BDS\text{-}like}}$, all finite $s_{i-1ii+1}$ dependence removed.
  $\Rightarrow$ Only then $\mathcal{A}_{\mathrm{fin}}$ satisfies (extended) Steinmann relations!
- Most recently, "cosmic normalization" $\mathcal{A}^{\mathrm{IR}} = \mathcal{A}^{\mathrm{BDS\text{-}like}} \rho$,

[Caron-Huot,Dixon,Dulat,McLeod,Hippel,GP'19A+B]

$$\rho(g^2) = 1 + 8(\zeta_3)^2 g^6 - 160\zeta_3\zeta_5\, g^8 + \left[1680\zeta_3\zeta_7 + 912(\zeta_5)^2 - 32\zeta_4(\zeta_3)^2\right] g^{10} - \left[18816\zeta_3\zeta_9 + 20832\zeta_5\zeta_7 - 448\zeta_4\zeta_3\zeta_5 - 400\zeta_6(\zeta_3)^2\right] g^{12} + \mathcal{O}(g^{14})$$

so as to satisfy (cosmic Galois) coaction principle.Call (N)MHV $\mathcal{A}_{\mathrm{fin}}$ (E) $\mathcal{E}$.

B. Identifying the Amplitude within $\mathcal{H}^{\text{hex}}$: Additional Constraints

1. Discrete symmetries, e.g. $p_i \to p_{i+1}$ due to trace cyclicity

B. Identifying the Amplitude within $\mathcal{H}^{\text{hex}}$: Additional Constraints

1. Discrete symmetries, e.g. $p_i \to p_{i+1}$ due to trace cyclicity
2. Dual superconformal symmetry $\Rightarrow$ constrains last symbol entry
   [Caron-Huot,He]

B. Identifying the Amplitude within $\mathcal{H}^{\text{hex}}$: Additional Constraints

1. Discrete symmetries, e.g. $p_i \to p_{i+1}$ due to trace cyclicity
2. Dual superconformal symmetry ⇒ constrains last symbol entry
   [Caron-Huot,He]
3. Collinear limit: After removing universal soft and collinear divergences, finite amplitude remainder obeys $\lim_{i+1\|i} \mathcal{A}_n = \mathcal{A}_{n-1}$

B. Identifying the Amplitude within $\mathcal{H}^{\text{hex}}$: Additional Constraints

1. Discrete symmetries, e.g. $p_i \to p_{i+1}$ due to trace cyclicity
2. Dual superconformal symmetry ⇒ constrains last symbol entry
   [Caron-Huot,He]
3. Collinear limit: After removing universal soft and collinear divergences, finite amplitude remainder obeys $\lim_{i+1\|i} \mathcal{A}_n = \mathcal{A}_{n-1}$
4. Additional limits: multi-Regge and near-collinear (OPE) expansion

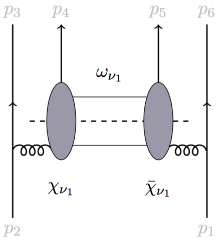
New effective d.o.f.⇒dispersion integrals

B. Identifying the Amplitude within $\mathcal{H}^{\text{hex}}$: Additional Constraints

1. Discrete symmetries, e.g. $p_i \to p_{i+1}$ due to trace cyclicity
2. Dual superconformal symmetry ⇒ constrains last symbol entry [Caron-Huot,He]
3. Collinear limit: After removing universal soft and collinear divergences, finite amplitude remainder obeys $\lim_{i+1\|i} \mathcal{A}_n = \mathcal{A}_{n-1}$
4. Additional limits: multi-Regge and near-collinear (OPE) expansion

New effective d.o.f.⇒dispersion integrals

$$A_6^{\text{MRL}} = \sum_n \left(\frac{z}{z^*}\right)^{\frac{n}{2}} \int \frac{d\nu}{2\pi} \chi_{\nu_1} \bar{\chi}_{\nu_1} |z|^{2i\nu} e^{-L\omega_\nu} =$$



[Bartels,Lipatov] [Duca,Druc,Drummond,Duhr,Dulat,Marzucca,GP,Verbeek]

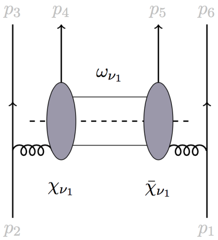
[Basso,Sever,Vieira]

B. Identifying the Amplitude within $\mathcal{H}^{\text{hex}}$: Additional Constraints

1. Discrete symmetries, e.g. $p_i \to p_{i+1}$ due to trace cyclicity
2. Dual superconformal symmetry ⇒ constrains last symbol entry [Caron-Huot,He]
3. Collinear limit: After removing universal soft and collinear divergences, finite amplitude remainder obeys $\lim_{i+1\|i} \mathcal{A}_n = \mathcal{A}_{n-1}$
4. Additional limits: multi-Regge and near-collinear (OPE) expansion

New effective d.o.f.⇒dispersion integrals

$$A_6^{\text{MRL}} = \sum_n \left(\frac{z}{z^*}\right)^{\frac{n}{2}} \int \frac{d\nu}{2\pi} \chi_{\nu_1} \bar{\chi}_{\nu_1} |z|^{2i\nu} e^{-L\omega_\nu} =$$



[Bartels,Lipatov] [Duca,Druc,Drummond,Duhr,Dulat,Marzucca,GP,Verbeek]

[Basso,Sever,Vieira]

Can evaluate in principle at any loop order, [GP] [GP,Drummond] using nested summation algorithms [Moch,Uwer,Weinzierl]

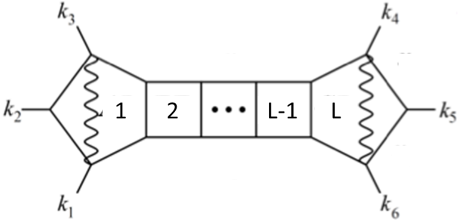
Beyond Perturbation Theory: Coupling Resummation

Beyond Perturbation Theory: Coupling Resummation

For a subspace of $n = 6$ pentagon-ladder integrals, can resum to all loops!

[Caron-Huot,Dixon,McLeod,GP,von Hippel]

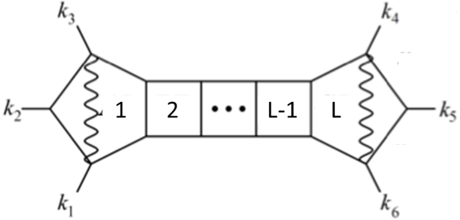
$\Omega^{(L)} =$



Beyond Perturbation Theory: Coupling Resummation

For a subspace of $n = 6$ pentagon-ladder integrals, can resum to all loops!

[Caron-Huot,Dixon,McLeod,GP,von Hippel]

$\Omega^{(L)} =$ 

Obtain extremely simple formula,

$$\Omega \equiv \sum_L g^{2L}\Omega^{(L)} = \int_{-\infty}^{\infty} \frac{d\nu}{2i} z^{i\nu/2} \frac{F^j_{+\nu}(x)F^j_{+\nu}(y) - F^j_{-\nu}(x)F^j_{-\nu}(y)}{\sinh(\pi\nu)},$$

where $g^2 = \frac{\lambda}{16\pi^2}$ and F^j_ν normalized hypergeometric functions:

$$F^j_\nu(x) \equiv \frac{\Gamma(1+\frac{i\nu+j}{2})\Gamma(1+\frac{i\nu-j}{2})}{\Gamma(1+i\nu)} x^{i\nu/2}\, {}_2F_1(\tfrac{i\nu+j}{2}, \tfrac{i\nu-j}{2}, 1+i\nu, x), \quad j \equiv i\sqrt{\nu^2+4g^2}\,.$$

Strong-coupling behavior

From asymptotic analysis of ${}_2F_1$, also obtain expansion of finite-coupling Ω around $g \to \infty$, [Lantos,Papathanasiou;to appear]

Strong-coupling behavior

From asymptotic analysis of ${}_2F_1$, also obtain expansion of finite-coupling Ω around $g \to \infty$, [Lantos,Papathanasiou;to appear]

$$\Omega \simeq -2\pi g \left(\frac{1}{x} - 1\right)^{1/4} \left(\frac{1}{y} - 1\right)^{1/4} \cosh[g(\phi(x) - \phi(y))] e^{-2\pi g}\,,$$

where $\phi(x) = \arccos(2x - 1)$, $u_1 = \frac{1}{1+\sqrt{xy/z}}$, $u_2 = \frac{1}{1+\sqrt{xyz}}$, $\frac{u_3}{(1-x)(1-y)} = u_1 u_2$

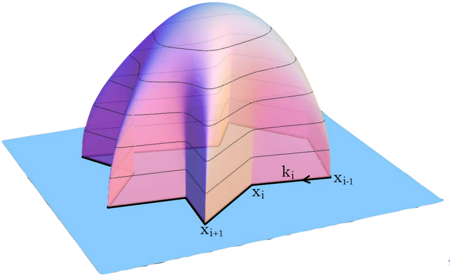
Strong-coupling behavior

From asymptotic analysis of ${}_2F_1$, also obtain expansion of finite-coupling Ω around $g \to \infty$, [Lantos,Papathanasiou;to appear]

$$\Omega \simeq -2\pi g \left(\frac{1}{x}-1\right)^{1/4}\left(\frac{1}{y}-1\right)^{1/4}\cosh[g(\phi(x)-\phi(y))]e^{-2\pi g}\,,$$

where $\phi(x) = \arccos(2x-1)$, $u_1 = \frac{1}{1+\sqrt{xy/z}}$, $u_2 = \frac{1}{1+\sqrt{xyz}}$, $\frac{u_3}{(1-x)(1-y)} = u_1 u_2$

Interpretation as a long string, as expected from gauge/string duality [Alday,Maldacena]



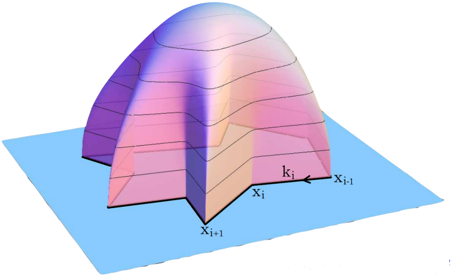
Strong-coupling behavior

From asymptotic analysis of ${}_2F_1$, also obtain expansion of finite-coupling Ω around $g \to \infty$, [Lantos,Papathanasiou;to appear]

$$\Omega \simeq -2\pi g \left(\frac{1}{x} - 1\right)^{1/4} \left(\frac{1}{y} - 1\right)^{1/4} \cosh[g(\phi(x) - \phi(y))] e^{-2\pi g} \,,$$

where $\phi(x) = \arccos(2x-1)$, $u_1 = \frac{1}{1+\sqrt{xy/z}}$, $u_2 = \frac{1}{1+\sqrt{xyz}}$, $\frac{u_3}{(1-x)(1-y)} = u_1 u_2$

Interpretation as a long string, as expected from gauge/string duality [Alday,Maldacena]



Can systematically compute *any* subleading term at strong coupling!

Results: Steinmann Heptagon Symbols

Weight k =	1	2	3	4	5	6	7	8
Heptagon Symbols	7	42	237	1288	6763	?	?	?
Steinmann Relation	7	28	97	322	1030	3192	9570	?
MHV Final Entry	0	1	0	1	1	2	1	4
Well-defined $i\|\|i+1$	-	0	-	0	-	0	-	0

Results: Steinmann Heptagon Symbols

Weight k =	1	2	3	4	5	6	7	8
Heptagon Symbols	7	42	237	1288	6763	?	?	?
Steinmann Relation	7	28	97	322	1030	3192	9570	?
MHV Final Entry	0	1	0	1	1	2	1	4
Well-defined $i\|\|i+1$	-	0	-	0	-	0	-	0

The symbol of the 3-loop 7-particle MHV amplitude is the only weight-6 heptagon symbol which satisfies the last-entry condition and which is finite in the $7 \parallel 6$ collinear limit.[Drummond,GP,Spradlin]

Results: Steinmann Heptagon Symbols

Weight k =	1	2	3	4	5	6	7	8
Heptagon Symbols	7	42	237	1288	6763	?	?	?
Steinmann Relation	7	28	97	322	1030	3192	9570	?
MHV Final Entry	0	1	0	1	1	2	1	4
Well-defined $i \| i+1$	-	0	-	0	-	0	-	0

The symbol of the 3-loop 7-particle MHV amplitude is the only weight-6 heptagon symbol which satisfies the last-entry condition and which is finite in the $7 \parallel 6$ collinear limit.[Drummond,GP,Spradlin]

Weight k =	1	2	3	4	5	6	7	8
Hexagon Symbols	3	9	26	75	218	643	?	?
Steinmann Relation	3	6	13	26	51	98	184	340
MHV Final Entry	0	3	4	11	21	41	76	142
Well-defined $i \| i+1$	-	0	-	2	-	11	-	43

Results: Steinmann Heptagon Symbols

Weight k =	1	2	3	4	5	6	7	8
Heptagon Symbols	7	42	237	1288	6763	?	?	?
Steinmann Relation	7	28	97	322	1030	3192	9570	?
MHV Final Entry	0	1	0	1	1	2	1	4
Well-defined $i\|\|i+1$	-	0	-	0	-	0	-	0

The symbol of the 4-loop 7-particle MHV amplitude is the only weight-8 Steinmann heptagon symbol satisfying the final-entry condition with finite $i \parallel i+1$ limit. [Dixon,Drummond,Harrington,McLeod,GP,Spradlin]

Weight k =	1	2	3	4	5	6	7	8
Hexagon Symbols	3	9	26	75	218	643	?	?
Steinmann Relation	3	6	13	26	51	98	184	340
MHV Final Entry	0	3	4	11	21	41	76	142
Well-defined $i\|\|i+1$	-	0	-	2	-	11	-	43

Cluster algebras [Fomin,Zelevinsky]

Cluster algebras [Fomin,Zelevinsky]

They are commutative algebras with

- Distinguished set of generators a_i, the *cluster variables*

Cluster algebras [Fomin,Zelevinsky]

They are commutative algebras with

- Distinguished set of generators a_i, the *cluster variables*
- Grouped into overlapping subsets $\{a_1, \ldots, a_n\}$ of *rank* n, the *clusters*

Cluster algebras [Fomin,Zelevinsky]

They are commutative algebras with

- Distinguished set of generators a_i, the *cluster variables*
- Grouped into overlapping subsets $\{a_1, \ldots, a_n\}$ of *rank* n, the *clusters*
- Constructed recursively from initial cluster via *mutations*

Cluster algebras [Fomin,Zelevinsky]

They are commutative algebras with

- Distinguished set of generators a_i, the *cluster variables*
- Grouped into overlapping subsets $\{a_1, \ldots, a_n\}$ of *rank* n, the *clusters*
- Constructed recursively from initial cluster via *mutations*

Can be described by quivers.

Cluster algebras [Fomin,Zelevinsky]

They are commutative algebras with

- Distinguished set of generators a_i, the *cluster variables*
- Grouped into overlapping subsets $\{a_1, \ldots, a_n\}$ of *rank* n, the *clusters*
- Constructed recursively from initial cluster via *mutations*

Can be described by quivers. Example: A_3 Cluster algebra

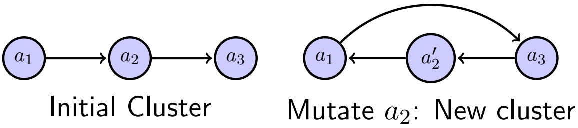
Initial Cluster

Cluster algebras [Fomin,Zelevinsky]

They are commutative algebras with

- Distinguished set of generators a_i, the *cluster variables*
- Grouped into overlapping subsets $\{a_1, \ldots, a_n\}$ of *rank* n, the *clusters*
- Constructed recursively from initial cluster via *mutations*

Can be described by quivers. Example: A_3 Cluster algebra



General rule for mutation at node k:

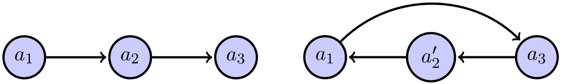
1. $\forall\ i \to k \to j$, add $i \to j$, reverse $i \leftarrow k \leftarrow j$, remove $\rightleftarrows$.

Cluster algebras [Fomin,Zelevinsky]

They are commutative algebras with

- Distinguished set of generators a_i, the *cluster variables*
- Grouped into overlapping subsets $\{a_1, \ldots, a_n\}$ of *rank* n, the *clusters*
- Constructed recursively from initial cluster via *mutations*

Can be described by quivers. Example: A_3 Cluster algebra



Initial Cluster — Mutate a_2: New cluster — $a_2' = (a_1 + a_3)/a_2$ and so on...

General rule for mutation at node k:

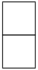
1. $\forall\ i \to k \to j$, add $i \to j$, reverse $i \leftarrow k \leftarrow j$, remove $\rightleftarrows$.
2. In new quiver/cluster, $a_k \to a_k' = \Big(\prod_{\text{arrows } i \to k} a_i + \prod_{\text{arrows } k \to j} a_j \Big) / a_k$

Momentum Twistors Z^I [Hodges]

Momentum Twistors Z^I [Hodges]

- Represent dual space variables $x^\mu \in \mathbb{R}^{1,3}$ as projective null vectors
$$X^M \in \mathbb{R}^{2,4}\,,\; X^2 = 0\,,\; X \sim \lambda X.$$

Momentum Twistors Z^I [Hodges]

- Represent dual space variables $x^\mu \in \mathbb{R}^{1,3}$ as projective null vectors
$$X^M \in \mathbb{R}^{2,4}\,,\; X^2 = 0\,,\; X \sim \lambda X.$$
- Repackage vector X^M of $SO(2,4)$ into antisymmetric representation
$$X^{IJ} = -X^{JI} =$$  of $SU(2,2)$

Momentum Twistors Z^I [Hodges]

- Represent dual space variables $x^\mu \in \mathbb{R}^{1,3}$ as projective null vectors
$$X^M \in \mathbb{R}^{2,4}\,,\; X^2 = 0\,,\; X \sim \lambda X.$$

- Repackage vector X^M of $SO(2,4)$ into antisymmetric representation
$$X^{IJ} = -X^{JI} = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \text{ of } SU(2,2)$$

- Can build latter from two copies of the fundamental $Z^I = \begin{array}{|c|}\hline \\ \hline \end{array}$,
$$X^{IJ} = Z^{[I}\tilde{Z}^{J]} = (Z^I\tilde{Z}^J - Z^J\tilde{Z}^I)/2 \text{ or } X = Z \wedge \tilde{Z}$$

Momentum Twistors Z^I [Hodges]

- Represent dual space variables $x^\mu \in \mathbb{R}^{1,3}$ as projective null vectors
$$X^M \in \mathbb{R}^{2,4}\,,\; X^2 = 0\,,\; X \sim \lambda X.$$

- Repackage vector X^M of $SO(2,4)$ into antisymmetric representation
$$X^{IJ} = -X^{JI} = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \text{ of } SU(2,2)$$

- Can build latter from two copies of the fundamental $Z^I = \begin{array}{|c|}\hline \\ \hline \end{array}$,
$$X^{IJ} = Z^{[I}\tilde{Z}^{J]} = (Z^I\tilde{Z}^J - Z^J\tilde{Z}^I)/2 \text{ or } X = Z \wedge \tilde{Z}$$

- After complexifying, Z^I transform in $SL(4,\mathbb{C})$. Since $Z \sim tZ$, can be viewed as homogeneous coordinates on $\mathbb{P}^3$.

Momentum Twistors Z^I [Hodges]

- Represent dual space variables $x^\mu \in \mathbb{R}^{1,3}$ as projective null vectors
$$X^M \in \mathbb{R}^{2,4}\,,\; X^2 = 0\,,\; X \sim \lambda X.$$

- Repackage vector X^M of $SO(2,4)$ into antisymmetric representation
$$X^{IJ} = -X^{JI} = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \text{ of } SU(2,2)$$

- Can build latter from two copies of the fundamental $Z^I = \begin{array}{|c|}\hline \\ \hline \end{array}$,
$$X^{IJ} = Z^{[I}\tilde{Z}^{J]} = (Z^I\tilde{Z}^J - Z^J\tilde{Z}^I)/2 \text{ or } X = Z \wedge \tilde{Z}$$

- After complexifying, Z^I transform in $SL(4,\mathbb{C})$. Since $Z \sim tZ$, can be viewed as homogeneous coordinates on $\mathbb{P}^3$.

- Can show
$$(x-x')^2 \propto 2X\cdot X' = \epsilon_{IJKL}Z^I\tilde{Z}^J Z'^K\tilde{Z}'^L = \det(Z\tilde{Z}Z'\tilde{Z}') \equiv \langle Z\tilde{Z}Z'\tilde{Z}'\rangle$$

Momentum Twistors Z^I [Hodges]

- Represent dual space variables $x^\mu \in \mathbb{R}^{1,3}$ as projective null vectors
$$X^M \in \mathbb{R}^{2,4}\,,\; X^2 = 0\,,\; X \sim \lambda X.$$

- Repackage vector X^M of $SO(2,4)$ into antisymmetric representation
$$X^{IJ} = -X^{JI} = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \text{ of } SU(2,2)$$

- Can build latter from two copies of the fundamental $Z^I = \begin{array}{|c|}\hline \\ \hline \end{array}$,
$$X^{IJ} = Z^{[I}\tilde{Z}^{J]} = (Z^I\tilde{Z}^J - Z^J\tilde{Z}^I)/2 \text{ or } X = Z \wedge \tilde{Z}$$

- After complexifying, Z^I transform in $SL(4,\mathbb{C})$. Since $Z \sim tZ$, can be viewed as homogeneous coordinates on $\mathbb{P}^3$.

- Can show
$$(x-x')^2 \propto 2X\cdot X' = \epsilon_{IJKL}Z^I\tilde{Z}^J Z'^K\tilde{Z}'^L = \det(Z\tilde{Z}Z'\tilde{Z}') \equiv \langle Z\tilde{Z}Z'\tilde{Z}'\rangle$$

- $(x_{i+i} - x_i)^2 = 0 \quad \Rightarrow X_i = Z_{i-1} \wedge Z_i$

$\mathrm{Conf}_n(\mathbb{P}^3)$ and Graßmannians

Can realize $\mathrm{Conf}_n(\mathbb{P}^3)$ as $4 \times n$ matrix

$$(Z_1|Z_2|\ldots|Z_n)$$

modulo rescalings of the n columns and $SL(4)$ transformations, which resembles a Graßmannian $Gr(4, n)$.

$\mathrm{Conf}_n(\mathbb{P}^3)$ and Graßmannians

Can realize $\mathrm{Conf}_n(\mathbb{P}^3)$ as $4 \times n$ matrix

$$(Z_1|Z_2|\ldots|Z_n)$$

modulo rescalings of the n columns and $SL(4)$ transformations, which resembles a Graßmannian $Gr(4, n)$.

$Gr(k, n)$: The space of k-dimensional planes passing through the origin in an n-dimensional space.

$\mathrm{Conf}_n(\mathbb{P}^3)$ and Graßmannians

Can realize $\mathrm{Conf}_n(\mathbb{P}^3)$ as $4 \times n$ matrix

$$(Z_1|Z_2|\ldots|Z_n)$$

modulo rescalings of the n columns and $SL(4)$ transformations, which resembles a Graßmannian $Gr(4, n)$.

$Gr(k, n)$: The space of k-dimensional planes passing through the origin in an n-dimensional space. Equivalently the space of $k \times n$ matrices modulo $GL(k)$ transformations:

$\mathrm{Conf}_n(\mathbb{P}^3)$ and Graßmannians

Can realize $\mathrm{Conf}_n(\mathbb{P}^3)$ as $4 \times n$ matrix

$$(Z_1|Z_2|\ldots|Z_n)$$

modulo rescalings of the n columns and $SL(4)$ transformations, which resembles a Graßmannian $Gr(4,n)$.

$Gr(k,n)$: The space of k-dimensional planes passing through the origin in an n-dimensional space. Equivalently the space of $k \times n$ matrices modulo $GL(k)$ transformations:

- k-plane specified by k basis vectors that span it $\Rightarrow$ $k \times n$ matrix

$\mathrm{Conf}_n(\mathbb{P}^3)$ and Graßmannians

Can realize $\mathrm{Conf}_n(\mathbb{P}^3)$ as $4 \times n$ matrix

$$(Z_1|Z_2|\dots|Z_n)$$

modulo rescalings of the n columns and $SL(4)$ transformations, which resembles a Graßmannian $Gr(4, n)$.

$Gr(k, n)$: The space of k-dimensional planes passing through the origin in an n-dimensional space. Equivalently the space of $k \times n$ matrices modulo $GL(k)$ transformations:

- k-plane specified by k basis vectors that span it $\Rightarrow$ $k \times n$ matrix
- Under $GL(k)$ transformations, basis vectors change, but still span the same plane.

$\mathrm{Conf}_n(\mathbb{P}^3)$ and Graßmannians

Can realize $\mathrm{Conf}_n(\mathbb{P}^3)$ as $4 \times n$ matrix

$$(Z_1|Z_2|\ldots|Z_n)$$

modulo rescalings of the n columns and $SL(4)$ transformations, which resembles a Graßmannian $Gr(4, n)$.

$Gr(k, n)$: The space of k-dimensional planes passing through the origin in an n-dimensional space. Equivalently the space of $k \times n$ matrices modulo $GL(k)$ transformations:

- k-plane specified by k basis vectors that span it $\Rightarrow$ $k \times n$ matrix
- Under $GL(k)$ transformations, basis vectors change, but still span the same plane.

Comparing the two matrices,

$$\mathrm{Conf}_n(\mathbb{P}^3) = Gr(4, n)/(C^*)^{n-1}$$

Connection with $\mathrm{Conf}_n(\mathbb{P}^3) = Gr(4,n)/(C^*)^{n-1}$

Connection with $\mathrm{Conf}_n(\mathbb{P}^3) = Gr(4,n)/(C^*)^{n-1}$

- Graßmannians $Gr(k,n)$ equipped with cluster algebra structure [Scott]

Connection with $\mathrm{Conf}_n(\mathbb{P}^3) = Gr(4,n)/(C^*)^{n-1}$

- Graßmannians $Gr(k,n)$ equipped with cluster algebra structure [Scott]
- Initial cluster made of a special set of Plücker coordinates $\langle i_1 \dots i_k \rangle$

Connection with $\mathrm{Conf}_n(\mathbb{P}^3) = Gr(4,n)/(C^*)^{n-1}$

- Graßmannians $Gr(k,n)$ equipped with cluster algebra structure [Scott]
- Initial cluster made of a special set of Plücker coordinates $\langle i_1 \ldots i_k \rangle$
- Mutations also yield certain homogeneous polynomials of Plücker coordinates

Connection with $\mathrm{Conf}_n(\mathbb{P}^3) = Gr(4,n)/(C^*)^{n-1}$

- Graßmannians $Gr(k,n)$ equipped with cluster algebra structure [Scott]
- Initial cluster made of a special set of Plücker coordinates $\langle i_1 \dots i_k \rangle$
- Mutations also yield certain homogeneous polynomials of Plücker coordinates
- Crucial observation: For all known cases, symbol alphabet of n-point amplitudes for $n = 6,7$ are $Gr(4,n)$ cluster variables (also known as $\mathcal{A}$-coordinates) [Golden,Goncharov,Spradlin,Vergu,Volovich]

Connection with $\mathrm{Conf}_n(\mathbb{P}^3) = Gr(4,n)/(C^*)^{n-1}$

- Graßmannians $Gr(k,n)$ equipped with cluster algebra structure [Scott]
- Initial cluster made of a special set of Plücker coordinates $\langle i_1 \ldots i_k \rangle$
- Mutations also yield certain homogeneous polynomials of Plücker coordinates
- Crucial observation: For all known cases, symbol alphabet of n-point amplitudes for $n = 6,7$ are $Gr(4,n)$ cluster variables (also known as $\mathcal{A}$-coordinates) [Golden,Goncharov,Spradlin,Vergu,Volovich]

Fundamental assumption of "cluster bootstrap"

Symbol alphabet is made of cluster $\mathcal{A}$-coordinates on $\mathrm{Conf}_n(\mathbb{P}^3)$. For the heptagon, 42 of them.

Imposing Constraints: Physical Singularities

Imposing Constraints: Physical Singularities

Locality: Amplitudes may only have singularities when some intermediate particle goes on-shell.

Imposing Constraints: Physical Singularities

Locality: Amplitudes may only have singularities when some intermediate particle goes on-shell.

Planar colour-ordered amplitudes in massless theories: Only happens when

$$(p_i + p_{i+1} + \cdots + p_{j-1})^2 = (x_j - x_i)^2 \propto \langle i{-}1\, i\, j{-}1\, j \rangle \to 0$$

Imposing Constraints: Physical Singularities

Locality: Amplitudes may only have singularities when some intermediate particle goes on-shell.

Planar colour-ordered amplitudes in massless theories: Only happens when

$$(p_i + p_{i+1} + \cdots + p_{j-1})^2 = (x_j - x_i)^2 \propto \langle i{-}1\, i\, j{-}1\, j \rangle \to 0$$

Singularities of generalised polylogarithm functions are encoded in the first entry of their symbols.

First-entry condition: Only $\langle i{-}1\, i\, j{-}1\, j \rangle$ allowed in the first entry of $\mathcal{S}$

Imposing Constraints: Physical Singularities

Locality: Amplitudes may only have singularities when some intermediate particle goes on-shell.

Planar colour-ordered amplitudes in massless theories: Only happens when

$$(p_i + p_{i+1} + \cdots + p_{j-1})^2 = (x_j - x_i)^2 \propto \langle i{-}1\, i\, j{-}1\, j \rangle \to 0$$

Singularities of generalised polylogarithm functions are encoded in the first entry of their symbols.

> **First-entry condition**: Only $\langle i{-}1\, i\, j{-}1\, j \rangle$ allowed in the first entry of $\mathcal{S}$

Define **(Steinmann) n-gon symbol**: An integrable symbol of the corresponding n-gon alphabet that obeys first-entry condition (and the Steinmann relation).

MHV Constraints: Yangian anomaly equations

MHV Constraints: Yangian anomaly equations

- Tree-level amplitudes exhibit (usual + dual) superconformal symmetry
  [Drummond,Henn,Korchemsky,Sokatchev]

MHV Constraints: Yangian anomaly equations

- Tree-level amplitudes exhibit (usual + dual) superconformal symmetry
  [Drummond,Henn,Korchemsky,Sokatchev]
- Combination of two symmetries gives rise to a Yangian
  [Drummond,Henn,Plefka] [Drummond,Ferro]

MHV Constraints: Yangian anomaly equations

- Tree-level amplitudes exhibit (usual + dual) superconformal symmetry [Drummond,Henn,Korchemsky,Sokatchev]
- Combination of two symmetries gives rise to a Yangian [Drummond,Henn,Plefka] [Drummond,Ferro]
- Although broken at loop level by IR divergences, Yangian anomaly equations governing this breaking have been proposed [Caron-Huot,He]

Consequence for MHV amplitudes: Their differential is a linear combination of $d\log\langle i\, j{-}1\, j\, j{+}1\rangle$, which implies

Last-entry condition: Only $\langle i\, j{-}1\, j\, j{+}1\rangle$ may appear in the last entry of the symbol of any MHV amplitude.

Imposing Constraints: The Collinear Limit

It is baked into the definition of the BDS-subtracted n-particle L-loop MHV remainder function that it should smoothly approach the corresponding $(n-1)$-particle function in any simple collinear limit:

$$\lim_{i+1\parallel i} R_n^{(L)} = R_{n-1}^{(L)}\,.$$

Imposing Constraints: The Collinear Limit

It is baked into the definition of the BDS-subtracted n-particle L-loop MHV remainder function that it should smoothly approach the corresponding $(n-1)$-particle function in any simple collinear limit:

$$\lim_{i+1 \parallel i} R_n^{(L)} = R_{n-1}^{(L)} \,.$$

For $n = 7$, taking this limit in the most general manner reduces the 42-letter heptagon symbol alphabet to 9-letter hexagon symbol alphabet, plus nine additional letters.

Imposing Constraints: The Collinear Limit

It is baked into the definition of the BDS-subtracted n-particle L-loop MHV remainder function that it should smoothly approach the corresponding $(n-1)$-particle function in any simple collinear limit:

$$\lim_{i+1 \parallel i} R_n^{(L)} = R_{n-1}^{(L)} \, .$$

For $n = 7$, taking this limit in the most general manner reduces the 42-letter heptagon symbol alphabet to 9-letter hexagon symbol alphabet, plus nine additional letters.

A function has a well-defined $i+1 \parallel i$ limit only if its symbol is independent of all nine of these letters.