Energy cutoff, noncommutativity, fuzzyness: the case of O(D)-covariant fuzzy spheres

Gaetano Fiore, Universitá "Federico II", and INFN, Napoli

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Introduction

Some motivations for noncommutative (NC) space(time) algebras:

- To avoid UV divergences in QFT [Snyder 1947,...].
- As an arena for formulating QG compatible with $\Delta x \gtrsim L_p$ [Mead 1966, Doplicher et al 1994-95,...].
- As an arena for unifying interactions [Connes-Lott '92,...]

Given a quantum theory $\mathcal{T}$ on a commutative space how to find NC candidates $\overline{\mathcal{T}}$ approximating $\mathcal{T}$? One possible mechanism:

Let $\mathcal{H} \equiv$ Hilbert space of the system S, $\mathcal{A} \equiv \mathrm{Lin}(\mathcal{H})$, $\overline{\mathcal{H}} \subset \mathcal{H}$ a subspace, $\overline{P} : \mathcal{H} \mapsto \overline{\mathcal{H}}$ its projection. Then

$$\overline{\mathcal{A}} \equiv \mathrm{Lin}\left(\overline{\mathcal{H}}\right) = \{\overline{A} \equiv \overline{P} A \overline{P} \mid A \in \mathcal{A}\} \neq \mathcal{A}.$$

In particular, if $[x_i, x_j] = 0$, in general $[\overline{x_i}, \overline{x_j}] \neq 0$.

If $\overline{P}H = H\overline{P}$ ($H \equiv$ Hamiltonian of S) then no change in dynamics within $\overline{\mathcal{H}}$. If $\overline{\mathcal{H}} \equiv$ subspace with energies $E \leq \overline{E} \equiv$ cutoff, then $\overline{\mathcal{T}}$ is a low-energy effective approximation of $\mathcal{T}$.

Prototype: Landau model in D=2; $\overline{E} = E_0$ implies $[\overline{x_1}, \overline{x_2}] = \frac{i\hbar c}{ieB}$.

When may this be useful? E. g.:

- If $\overline{\mathcal{H}}^{\perp}$ is practically not accessible in preparing the initial state, nor through the interactions with the environment or the measurement apparatus, then $\overline{\mathcal{T}}$ on $\overline{\mathcal{H}}$ (smaller) is enough.
- If at $E > \overline{E}$ we expect new physics not accountable by $\mathcal{T}$, then $\overline{\mathcal{T}}$ may also help to figure out a new theory $\mathcal{T}'$ valid for all E.

(Of course, the two may co-exist.)

If H is invariant under some group G, then $\overline{\mathcal{H}}, \overline{P}, \overline{\mathcal{T}}$ will be.

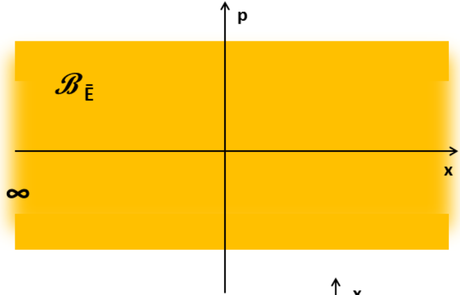
Consider quantum mechanics (QM) on $\mathbb{R}^D$, Hamiltonian $H(x,p)$.

$$\dim(\overline{\mathcal{H}}) \simeq \mathrm{Vol}(\mathcal{B}_{\overline{E}})/h^D,$$

$$\mathcal{B}_{\overline{E}} \equiv \{(x,p) \in \mathbb{R}^{2D} \mid H(x,p) \leq \overline{E}\} = \text{classical phase space below } \overline{E}.$$

$H = p^2 = \Sigma_i\, p_i^2$

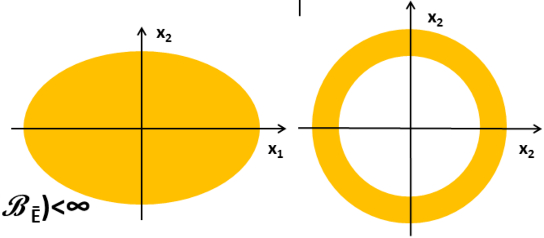
$\dim(\bar{\mathcal{H}}) \sim \mathrm{Vol}(\mathcal{B}_{\bar{E}}) = \infty$



$H = p^2 + V(x)$

confining

$\dim(\bar{\mathcal{H}}) \sim \mathrm{Vol}(\mathcal{B}_{\bar{E}}) < \infty$



Adding a *'dimensional reduction' mechanism* we can obtain a NC, fuzzy approximation of QM on *submanifolds* of $\mathbb{R}^D$.
Here a sphere S^d, $d=D-1$ [GF, F. Pisacane 2017-19].

Consider a quantum particle in $\mathbb{R}^D$ configuration space with Hamiltonian

$$H=-\frac{1}{2}\Delta+V(r); \qquad (1)$$

we fix the minimum $V_0=V(1)$ of the the confining potential $V(r)$ so that the ground state has energy $E_0=0$.

- Choose $V(r)$ and $\overline{E}$ fulfilling

$$V(r)\simeq V_0+2k\,(r-1)^2 \qquad (2)$$

if $V(r)\le\overline{E}$; so that $V(r)$ has a harmonic behavior for $|r-1|\le\sqrt{\frac{\overline{E}-V_0}{2k}}$.

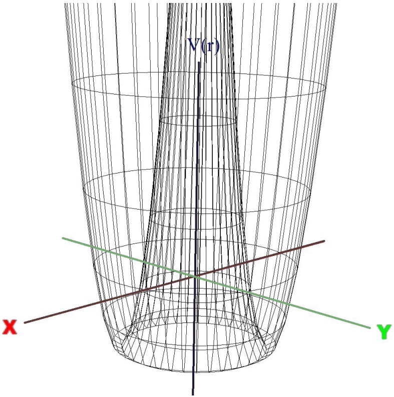


Figure 1 : Three-dimensional plot of $V(r)$

- The minimum on the sphere $r=1$ is sharp if $V''(1)\equiv 4k\gg 0$.
- $\overline{E}$ low enough to *eliminate radial excitations* from Spectrum(H).

Then: $\overline{H}=\overline{L^2}$; the x_i generate all $\overline{\mathcal{A}}$, $[\overline{x_i},\overline{x_j}]\sim\frac{iL_{ij}}{k}$ *à la* Snyder.

- Choose $\overline{E}=\overline{E}(\Lambda)\equiv\Lambda(\Lambda+d-1)$, $k=k(\Lambda)\ge\Lambda^2(\Lambda+1)^2$; diverging with $\Lambda\in\mathbb{N}$. We thus find

$$(\mathcal{H}_\Lambda,\mathcal{A}'_\Lambda)\overset{\Lambda\to\infty}{\longrightarrow}(\mathcal{H},\mathcal{A})\equiv\Big(\mathcal{L}^2(S^d),\,\mathrm{Lin}\big(\mathcal{L}^2(S^d)\big)\Big)$$

This is a $O(D)$-covariant fuzzy sphere $\{S^d_\Lambda\}_{\Lambda\in\mathbb{N}}\equiv\{(\mathcal{H}_\Lambda,\mathcal{A}_\Lambda)\}_{\Lambda\in\mathbb{N}}$, i.e. sequence of finite-dim approximations of ordinary QM on S^d![1]

After briefly reviewing the features of S^1_Λ, S^2_Λ, here I will present various systems of coherent states (SCS) on them and discuss their localization both in configuration and (angular) momentum space.

Finally, I will compare our S^d_Λ with other fuzzy spheres, in particular S^2_Λ with Madore-Hoppe Fuzzy sphere.

[1] A fuzzy space is a sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ of *finite-dimensional* algebras such that $\mathcal{A}_n\overset{n\to\infty}{\longrightarrow}\mathcal{A}\equiv$algebra of regular functions on an ordinary manifold.

Table of contents

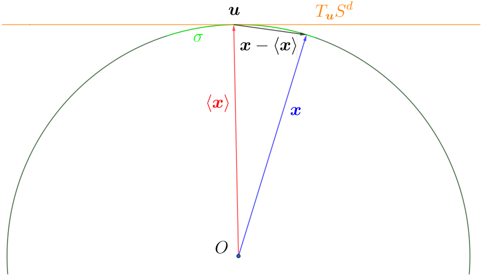
Preliminaries - Localization on $\mathbb{R}^D$, S^d and S^d_Λ

A good measure of the localization of a state ψ in configuration space $\mathbb{R}^D$ is its *spacial dispersion*, i.e. the expectation value on ψ

$$(\Delta\mathbf{x})^2 \equiv \sum_{i=1}^{D}(\Delta x_i)^2 = \langle \mathbf{x}^2\rangle - \langle\mathbf{x}\rangle^2 ; \tag{3}$$

$\mathbf{x} \equiv (x_1, ..., x_n)$, $\langle\mathbf{x}\rangle \equiv (\langle x_1\rangle, ..., \langle x_n\rangle)$ is the average position in $\mathbb{R}^D$;
$\mathbf{x}^2 := \sum_{i=1}^{D} x_i x_i$ measures the square distance from the origin;
$\mathbf{x} - \langle\mathbf{x}\rangle$ measures the displacement from $\langle\mathbf{x}\rangle$;
(3) is the average of the square of the latter, $O(D)$-invariant.

We adopt it also on S^d, S^d_Λ: if ψ is localized in a small region $\sigma \subset S^d$ around a point $\mathbf{u} \equiv \langle\mathbf{x}\rangle \in S^d$ then $(\Delta\mathbf{x})^2$ essentially reduces to the average square displacement in the tangent plane at $\mathbf{u}$, as wished:



Preliminaries - Coherent states

Schrödinger introduced canonical SCS on $\mathbb{R}^D$ $\{\phi_z\}_{z\in\Omega} \subset \mathcal{H}$ $(\Omega\equiv\mathbb{C}^D)$ to saturate Heisenberg uncertainty relation (HUR) $\Delta x_i \Delta p_i \geq \hbar/2$. Heisenberg-Weyl group G maps $\phi_z \mapsto \phi_{z'}$.
Properties:

1. **Strong continuity** of ϕ_z as a function of $z \in \Omega$;
2. **Identity Resolution:** $$I = \int_\Omega dz\, P_z, \quad P_z \equiv |\phi_z\rangle\langle\phi_z|; \qquad (4)$$
3. **Completeness**: $\overline{\mathrm{Span}\,\{\phi_z \mid z \in \Omega\}} = \mathcal{H}. \quad \Leftarrow 2.$

Generic manifold M: 1,2 define *strong* SCS, 1,3 define *weak* SCS; $\Omega \equiv$ topological space, $dz \equiv$ suitable integration measure on Ω.

Assume Lie group G acts on $\mathcal{H}$ via unitary irrep T; fix $\phi_0 \in \mathcal{H}$.
for all $g \in G$ let $\phi_g \equiv T(g)\phi_0$, $H \equiv \{h \in G \mid \phi_h = \exp[i\alpha(h)]\phi_0\}$.
Then $|\phi_g\rangle\langle\phi_g| = |\phi_{gh}\rangle\langle\phi_{gh}| \equiv P_z$, depends only on $z \in \Omega \equiv G/H$.
If $\exists$ left-invariant measure dg on G s.t. $\int_G |\langle\phi_0, T(g)\phi_0\rangle|^2 \, dg < \infty$ ($\Leftarrow G$ compact) then (4) holds with $dz \propto dg$ [Perelomov, Gilmore].

Perelomov: the CS closest states to classical ones are obtained from a ϕ_0 maximizing H; for $G = SO(3)$ it is $H = SO(2)$, and these SCS minimize the dispersion

$$(\Delta \mathbf{L})^2 \equiv \sum_{i=1}^{D} (\Delta L_i)^2 = \left\langle \mathbf{L}^2 \right\rangle - \langle \mathbf{L} \rangle^2 . \qquad (5)$$

Coherent and localized states on S^1_Λ

$\mathcal{B} := \{\psi_\Lambda, \psi_{\Lambda-1}, ..., \psi_{-\Lambda}\} \equiv$ orthonormal basis of $\mathcal{H}_\Lambda$ such that

$$L\psi_n = n\psi_n, \qquad x_\pm\psi_n = \begin{cases} \left[1+\frac{n(n\pm1)}{2k}\right]\psi_{n\pm1} & \text{if } -\Lambda \le \pm n \le \Lambda-1 \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

where $L \equiv L_{12}$, $x_\pm \equiv x_1 \pm ix_2$, $k = k(\Lambda)$ fulfills (**??**). $S^1_\Lambda \xrightarrow{\Lambda\to\infty} S^1$.
L, x_+, x_- and $\mathbf{x}^2 \equiv x_1^2 + x_2^2$ fulfill the $O(2)$-equivariant relations

$$[L, x_\pm] = \pm x_\pm, \quad x_+{}^\dagger = x_-, \qquad L^\dagger = L, \tag{7}$$

$$[x_+, x_-] = -\frac{2L}{k} + \left[1+\frac{\Lambda(\Lambda+1)}{k}\right]\left(\widetilde{P}_\Lambda - \widetilde{P}_{-\Lambda}\right) \equiv L', \tag{8}$$

$$\mathbf{x}^2 = 1+\frac{L^2}{k} - \left[1+\frac{\Lambda(\Lambda+1)}{k}\right]\frac{\widetilde{P}_\Lambda + \widetilde{P}_{-\Lambda}}{2}, \tag{9}$$

where $\widetilde{P}_n : \mathcal{H}_\Lambda \mapsto \mathbb{C}\psi_n$ (projection). We point out that:

- $\mathbf{x}^2 \neq 1$ is a function of L^2; if $n \neq \pm\Lambda$ eigenvectors ψ_n have eigenvalues $\simeq 1$, slightly growing with $|n|$ and $\xrightarrow{\Lambda\to\infty} 1$.
- The ordered monomials $x_+^h L^l x_-^n$ [with degrees h, l, n bounded by (7)-**??**] make up a basis of the $(2\Lambda+1)^2$-dim vector space underlying

$O(2)$-covariant UR and strong SCS systems on S^1_Λ

$\Delta x_1, \Delta x_2$ may vanish separately, but not simultaneously, because

$$(\Delta\mathbf{x})^2 \geq (\Delta\mathbf{x})^2_{min} \sim \frac{1}{\Lambda^2} \tag{11}$$

$$\text{HUR:} \quad \Delta L\,\Delta x_1 \geq \frac{|\langle x_2\rangle|}{2}, \quad \Delta L\,\Delta x_2 \geq \frac{|\langle x_1\rangle|}{2}, \quad \Delta L^2(\Delta\mathbf{x})^2 \geq \frac{\langle\mathbf{x}\rangle^2}{4} \tag{12}$$

for both S^1, S^1_Λ; derived from (7), saturated by the ψ_n ($\Delta L = 0$).

Theorem $\forall\beta \in (\mathbb{R}/2\pi\mathbb{Z})^{2\Lambda+1}$ $\mathcal{S}^\beta \equiv \left\{\boldsymbol{\omega}^\beta_\alpha \equiv \sum_{n=-\Lambda}^{\Lambda} \frac{e^{i(\alpha n+\beta_n)}}{\sqrt{2\Lambda+1}}\psi_n\right\}_{\alpha\in\Omega\equiv S^1}$

is a strong SCS: $$I = \frac{2\Lambda+1}{2\pi}\int_0^{2\pi} d\alpha\, P^\beta_\alpha, \qquad P^\beta_\alpha \equiv \boldsymbol{\omega}^\beta_\alpha\langle\boldsymbol{\omega}^\beta_\alpha, \cdot\rangle. \tag{13}$$

$\mathcal{S}^\beta$ is fully $O(2)$-covariant if $\beta_{-n} = \beta_n$. On all $\boldsymbol{\omega}^\beta_\alpha$ it is $\langle L\rangle = 0$, $(\Delta L)^2 = \frac{\Lambda(\Lambda+1)}{3}$, whereas $(\Delta\mathbf{x})^2$ is minimized by the $\phi_\alpha \equiv \boldsymbol{\omega}^0_\alpha$, with

$$(\Delta\mathbf{x})^2 < \frac{1}{\Lambda+1}\left(\frac{1}{2} + \frac{1}{3\Lambda}\right) \tag{14}$$

$O(2)$-invariant weak SCS minimizing $(\Delta x)^2$

Since $(\Delta\mathbf{x})^2$ is $O(2)$-invariant, so is the set $\mathcal{W}^1$ of states minimizing it.
Hence if $\underline{\chi} \in \mathcal{W}^1$, then $\mathcal{W}^1 = \left\{\underline{\chi}_\alpha \equiv e^{i\alpha L}\underline{\chi}\right\}_{\alpha\in[0,2\pi[}$; is a weak SCS.
We have shown that

$$0 < (\Delta\mathbf{x})^2_{min} = (\Delta\mathbf{x})^2_{\underline{\chi}_\alpha} < \frac{3.5}{(\Lambda+1)^2}. \tag{15}$$

The $\underline{\chi}_\alpha$ are closest to classical states(=points) of S^1, and $S^1 \leftrightarrow \mathcal{W}^1$.

Within the class of strong SCS, ϕ_α are closest to classical points of S^1, and $S^1 \leftrightarrow \mathcal{S}^1 \equiv \{\phi_\alpha\}_{\alpha\in[0,2\pi[}$.

Coherent and localized states on S^2_Λ

Let $\quad L_\pm \equiv L_1 \pm iL_2, \qquad L_0 \equiv L_3, \qquad x_\pm \equiv x_1 \pm ix_2, \qquad x_0 \equiv x_3.$

$\mathcal{B}_\Lambda \equiv \{\psi_l^m\}_{l=0,1,\ldots,\Lambda;\ m=-l,\ldots,l} \equiv$ orthonormal basis of $\mathcal{H}_\Lambda$ such that

$$\mathbf{L}^2\psi_l^m = l(l+1)\psi_l^m, \qquad L_3\psi_l^m = m\psi_l^m. \tag{16}$$

where $\mathbf{L}^2 = L_iL_i$. On the ψ_l^m the L_a, x_a $(a = 0, +, -)$ act as follows:

$$L_0\psi_l^m = m\,\psi_l^m, \quad L_\pm\psi_l^m = \sqrt{(l\mp m)(l\pm m+1)}\psi_l^{m\pm1}, \tag{17}$$

$$x_a\psi_l^m = \begin{cases} c_l A_l^{a,m}\psi_{l-1}^{m+a} + c_{l+1}B_l^{a,m}\psi_{l+1}^{m+a} & \text{if } l<\Lambda, \\ c_l A_l^{a,m}\psi_{\Lambda-1}^{m+a} & \text{if } l=\Lambda, \\ 0 & \text{otherwise}, \end{cases} \tag{18}$$

$$\text{where} \quad A_l^{0,m} \equiv \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}}, \qquad A_l^{\pm,m} \equiv \pm\sqrt{\frac{(l\mp m)(l\mp m-1)}{(2l-1)(2l+1)}},$$
$$B_l^{a,m} \equiv A_{l+1}^{-a,m+a}, \qquad c_0 = c_{\Lambda+1} = 0, \quad c_l \equiv \sqrt{1+\frac{l^2}{k}} \quad 1\le l\le\Lambda, \tag{19}$$

where k is a function of Λ fulfilling (**??**).

The L_i, x_i and $\mathbf{x}^2 \equiv x_i x_i$ fulfill the following $O(3)$-covariant relations:

$$x_i^\dagger = x_i, \quad L_i^\dagger = L_i, \quad [L_i, x_j] = i\varepsilon^{ijh} x_h, \quad [L_i, L_j] = i\varepsilon^{ijh} L_h, \quad x_i L_i = 0,$$

$$[x_i, x_j] = \underbrace{i\varepsilon^{ijh} L_h \left(-\frac{1}{k} + K\widetilde{P}_\Lambda\right)}_{Snyder-like}, \quad \mathbf{x}^2 = 1 + \frac{\mathbf{L}^2+1}{k} - \left[1 + \frac{(\Lambda+1)^2}{k}\right] \frac{\Lambda+1}{2\Lambda+1} \widetilde{P}_\Lambda, \tag{20}$$

$$\prod_{l=0}^{\Lambda} \left[\mathbf{L}^2 - l(l+1)I\right] = 0, \quad \prod_{m=-l}^{l} (L_3 - mI)\widetilde{P}_l = 0, \quad (x_\pm)^{2\Lambda+1} = 0;$$

here $K \equiv \frac{1}{k} + \frac{1+\frac{\Lambda^2}{k}}{2\Lambda+1}$, $\widetilde{P}_l \equiv$ projection on $\mathbf{L}^2 = l(l+1)$ eigenspace. Note:

- $\mathbf{x}^2 \neq 1$ is a function of $\mathbf{L}^2$; if $l < \Lambda$ the eigenvectors ψ_l^m have eigenvalues $\simeq 1$, slightly growing with l and $\xrightarrow{\Lambda\to\infty} 1$.
- Ordered monomials in x_i, L_i [with degrees bounded by $(20)_3$] make up a basis of $\mathcal{A}_\Lambda \equiv End(\mathcal{H}_\Lambda)$: express $\widetilde{P}_l$ as polynomials in $\mathbf{L}^2$.
- The x_i *generate* the $*$-algebra $\mathcal{A}_\Lambda$, because also the L_i can be expressed as non-ordered polynomials in the x_i.
- Actually there are $*$-algebra isomorphisms

$$\mathcal{A}_\Lambda \simeq M_N(\mathbb{C}) \simeq \boldsymbol{\pi}_\Lambda[Uso(4)], \quad N \equiv (\Lambda+1)^2, \tag{21}$$

$\boldsymbol{\pi}_\Lambda \equiv \pi_{\frac{\Lambda}{2}} \otimes \pi_{\frac{\Lambda}{2}}$ unitary representation of $Uso(4) \simeq Usu(2) \otimes Usu(2)$

$O(3)$-covariant UR and strong SCS systems on S^2_Λ

Proposition *The UR* $(\Delta\mathbf{L})^2 \geq |\langle\mathbf{L}\rangle| \;\; \Leftrightarrow \;\; \langle\mathbf{L}^2\rangle \geq |\langle\mathbf{L}\rangle|\,(|\langle\mathbf{L}\rangle|+1)$ (22) *holds on* $\mathcal{H}_\Lambda$*; is saturated by Bloch CS* $\phi_{l,g} \equiv \boldsymbol{\pi}_\Lambda(g)\psi_l^l \in V_l$, $g\in SO(3)$. Holds and new also as $\Lambda\to\infty$, i.e. on $\mathcal{H}=\mathcal{L}^2(S^2)$.

$[L_i,L_j] = i\varepsilon^{ijk}L_k \;\Rightarrow\; \Delta L_1\Delta L_2 \geq \frac{1}{2}|\langle L_3\rangle| +$ permutations $\Rightarrow (\Delta\mathbf{L})^2 \geq \frac{3}{4}|\langle\mathbf{L}\rangle|$
$[L_i,x_j] = i\varepsilon^{ijk}x_k \;\Rightarrow\; \Delta L_1\Delta x_2 \geq \frac{1}{2}|\langle x_3\rangle|$, & permutations. Saturable? Boh
Again, $\Delta x_1, \Delta x_2, \Delta x_3$ may vanish separately, not simultaneously, because

$$(\Delta\mathbf{x})^2 \geq (\Delta\mathbf{x})^2_{min} \sim \frac{1}{\Lambda^2} \tag{23}$$

Set $T=\boldsymbol{\pi}_\Lambda \equiv$ *reducible* unitary repr. of $SO(3)$ on $\mathcal{H}_\Lambda$, $\boldsymbol{\omega} \equiv \sum_{l=0}^{\Lambda}\sum_{h=-l}^{l}\omega_l^h\psi_l^h$

Theorem $\mathcal{S}^\omega \equiv \{\boldsymbol{\omega}_g \equiv \boldsymbol{\pi}_\Lambda(g)\boldsymbol{\omega}\}_{g\in SO(3)}$ *is a strong SCS if*

$\sum_{h=-l}^{l}|\omega_l^h|^2 = \frac{2l+1}{(\Lambda+1)^2}$ $\forall l$*; it is also is fully* $O(3)$*-covariant if* $\omega_l^h = \omega_l^{-h}$.

$$I = \frac{(\Lambda+1)^2}{8\pi^2}\int_{SO(3)} d\mu(g)\,P_g, \qquad P_g := \boldsymbol{\omega}_g\langle\boldsymbol{\omega}_g,\cdot\rangle. \tag{24}$$

Choosing $\boldsymbol{\omega}=\boldsymbol{\phi}^\beta\equiv\sum_{l=0}^{\Lambda}\boldsymbol{\psi}_l^0 e^{i\beta_l}\frac{\sqrt{2l+1}}{\Lambda+1}$, $\beta\in(\mathbb{R}/2\pi\mathbb{Z})^{\Lambda+1}$.

$L_3\boldsymbol{\phi}^\beta=0 \ \Rightarrow$ nontrivial isotropy subgroup $H=\{e^{i\psi L_3}\mid\psi\in\mathbb{R}\}\simeq SO(2)$:
resolution of the identity integrating over $S^2\simeq SO(3)/H\ni g=e^{\varphi l_3}e^{i\theta l_2}$:

$$I=\frac{(\Lambda+1)^2}{4\pi}\int_0^{2\pi}\!d\varphi\int_0^{\pi}\!d\theta\,\sin\theta\,P_g^\beta,\qquad P_g^\beta=\boldsymbol{\phi}_g^\beta\langle\boldsymbol{\phi}_g^\beta,\cdot\rangle,\qquad \boldsymbol{\phi}_g^\beta=\boldsymbol{\pi}_\Lambda(g)\boldsymbol{\phi}^\beta \quad (25)$$

Hence $\mathcal{S}_\beta=\{\boldsymbol{\phi}_g^\beta\}_{g\in S^2}$ is a family of fully $O(3)$-covariant, strong SCSs.
On it $(\Delta\mathbf{L})^2$ is independent of β, while $(\Delta\mathbf{x})^2$ is smallest on the $\boldsymbol{\phi}_g^0$:

$$(\Delta\mathbf{L})^2=\frac{\Lambda(\Lambda+2)}{2},\qquad (\Delta\mathbf{x})^2<\frac{1}{\Lambda+1}. \quad (26)$$

$O(3)$-*invariant weak SCS minimizing* $(\Delta x)^2$

Since $(\Delta\mathbf{x})^2$ is $O(3)$-invariant, so is the set $\mathcal{W}^2$ of states minimizing it. Look for $\underline{\chi} \in \mathcal{W}^2$ s.t. $\langle x_3\rangle = |\langle\mathbf{x}\rangle|$; then $\mathcal{W}^2 = \{\underline{\chi}_g \equiv \boldsymbol{\pi}_\Lambda(g)\underline{\chi}\}_{g\in SO(3)}$.
We have shown that $L_3\underline{\chi} = 0 \;\Rightarrow\; \exists$ nontrivial isotropy subgroup $H = \{e^{i\psi L_3}\,|\,\psi\in\mathbb{R}\} \simeq SO(2)$, whence $\mathcal{W}^2 = \{\underline{\chi}_g \equiv \boldsymbol{\pi}_\Lambda(g)\underline{\chi}\}_{g\in S^2}$.
$\mathcal{W}^2$ is a weak SCS.

The $\underline{\chi}_g$ are closest to classical states(=points) of S^2, and $S^2 \leftrightarrow \mathcal{W}^2$.
At order $O(1/\Lambda^2)$ $\underline{\chi}$ coincides with the eigenvector $\widehat{\chi}$ of x_3 with highest eigenvalue. We have shown that

$$0 < (\Delta\mathbf{x})^2_{min} = (\Delta\mathbf{x})^2_{\underline{\chi}} < \frac{11}{(\Lambda+1)^2}. \tag{27}$$

Within the class of strong SCS, the ϕ_g are closest to classical points of S^2, and $S^2 \leftrightarrow \mathcal{S}^2 \equiv \{\phi_g\}_{g\in S^2}$.

Discussion and conclusions

We have built a sequence $(\mathcal{A}_\Lambda, \mathcal{H}_\Lambda)$ of finite-dim, $O(D)$-covariant $(D = d+1)$ approximations of QM of a spinless particle on the sphere S^d; $\mathbf{x}^2 \gtrsim 1$ collapses to 1 as $\Lambda \to \infty$.

Achieved imposing $E \leq \Lambda(\Lambda+d-1)$ on QM of a particle in $\mathbb{R}^D$ subject to a sharp confining potential $V(r)$ on the sphere $r = 1$.

$\mathcal{A}_\Lambda$ are fuzzy approximations of the *whole algebra of observables* of the particle on S^d (phase space algebra).

$\mathcal{A}_\Lambda \simeq \pi_\Lambda[Uso(D+1)]$, with a suitable irrep π_Λ of $Uso(D+1)$ on $\mathcal{H}_\Lambda$.

$\mathcal{H}_\Lambda$ carries a *reducible* representation of the $Uso(D)$ subalgebra generated by the $\overline{L}_{ij}$: $\mathcal{H}_\Lambda = \bigoplus$irreps fulfilling $L^2 \leq \Lambda(\Lambda+d-1)$.

The same decomposition holds for the subspace $\mathcal{C}_\Lambda \subset \mathcal{A}_\Lambda$ of completely symmetrized polynomials in the $\overline{x}^i$.

As $\Lambda \to \infty$ these resp. become the decompositions (29) of $\mathcal{L}^2(S^d)$ and of $C(S^d)$ acting on $\mathcal{L}^2(S^d)$.

S^2_Λ vs. Madore-Hoppe Fuzzy Sphere S^2_n (seminal fuzzy space):
$\mathcal{A}_n \simeq M_n(\mathbb{C})$, generated by coordinates x^i ($i=1,2,3$) fulfilling

$$[x^i,x^j] = \frac{2i}{\sqrt{n^2-1}}\varepsilon^{ijk}x^k, \quad r^2 := x^ix^i = 1, \qquad n \in \mathbb{N}\setminus\{1\}; \qquad (28)$$

(28) are covariant under $SO(3)$, but not under the whole $O(3)$; in particular not under parity $x^i \mapsto -x^i$.
In fact $L^i = x^i\sqrt{n^2-1}/2$ make up the standard basis of $so(3)$ in the irrep (π_l, V_l) characterized by $L^iL^i = l(l+1)$, $n = 2l+1$.
Does S^2_n approximate the configuration space algebra of a particle on S^2?
Problems: a) parity; b) V_l is irreducible.
Our $[\overline{x_i}, \overline{x_j}] = \frac{iL_{ij}}{k} + \ldots$ are $O(3)$-covariant: a) solved. Moreover,

$$\mathcal{H}_\Lambda \simeq \bigoplus_{l=0}^{\Lambda} V_l, \qquad \mathcal{A}_\Lambda \simeq \bigoplus_{l=0}^{2\Lambda} V_l. \qquad (29)$$

As $\Lambda \to \infty$ we get $\mathcal{L}^2(S^2) \simeq \bigoplus_{l=0}^{\infty} V_l$: b) solved, $C(S^2) \simeq \bigoplus_{l=0}^{\infty} V_l$.

On Madore FS

$$(\Delta \mathbf{x})^2_{min} = \frac{2}{n+1} = \frac{1}{l+1}, \tag{30}$$

($l \equiv$cutoff) whereas on our fuzzy sphere S^2_Λ

$$(\Delta \mathbf{x})^2_{min} < \frac{11}{(\Lambda+1)^2}. \tag{31}$$

The fuzzy spheres of dimension $d=4$ [Grosse, Klimcik, Presnajder 1996], $d\geq 3$ [Ramgoolam 2001, Dolan, O'Connor 2003, ...], are based on $End(V)$ where V carries a particular *irrep* of $SO(d+1)$.
$\mathbf{x}^2$ is central, can be set=1.
Also Snyder-like commutation relations, hence $O(d+1)$-covariant.

In [Steinacker et al. 2016-19] fuzzy 4-spheres S^4_N through reducible repr. of $Uso(5)$ obtained decomposing irreps π of $Uso(6)$ with suitable highest weights (N,n_1,n_2); so $End(V)\simeq\pi[Uso(6)]$, in analogy with our result. The elements X^i of a basis of $so(6)\setminus so(5)$ (as a vector space) play the role of noncommuting cartesian coordinates.
Hence, the $SO(5)$-scalar $\mathbf{x}^2=X^iX^i$ is no longer central, but its spectrum is still very close to 1 *only if* $N\gg n_1,n_2$;
if $n_1=n_2=0$ then $\mathbf{x}^2\equiv 1$ ($\Rightarrow$ irrep), and one recovers the fuzzy 4-sphere [Grosse, Klimcik, Presnajder 1996].

Here $\mathbf{x}^2\simeq 1$ is guaranteed by adopting $x^i=g(L^2)X^ig(L^2)$ rather than X^i as noncommutative cartesian coordinates, and $\mathbf{x}^2=x^ix^i$.

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