Higher Poisson-Lie T-duality

Fridrich Valach
Charles University

19.9.2019, Corfu

Joint work with Ján Pulmann and Pavol Ševera, arXiv:1909.06151

Summary

Goals

- Put various dualities, such as the Poisson-Lie T-duality and the electric-magnetic duality, on a common footing.
- Obtain a natural higher analogue of the Poisson-Lie T-duality (in a proper BV form).

Tools

- differential graded manifolds, the AKSZ construction

  [Alexandrov, Kontsevich, Schwarz, Zaboronsky 1997]
- (baby version of) the derived intersection of Lagrangian submanifolds

  [Pantev, Toën, Vaquié, Vezzosi 2013]

Differential graded manifolds

graded manifold: space X for which $C^\infty(X)$ is $\mathbb{Z}$-graded

differential graded (dg) manifold: graded manifold with a vector field (differential) Q such that $\deg Q = 1$, $Q^2 = 0$

N-manifold: space X for which $C^\infty(X)$ is $\mathbb{Z}_{\geq 0}$-graded

NQ-manifold: N-manifold with a differential

Example

X an N-manifold $\rightsquigarrow T[1]X$ is an NQ-manifold, with $Q = d$
Note: $C^\infty(T[1]X) \cong \Omega(X), \quad f_{i\dots j}(x)\xi^i \dots \xi^j \mapsto f_{i\dots j}(x)dx^i \dots dx^j$
(NQ manifolds of this type are called **acyclic**)

Differential graded symplectic manifolds

dg symplectic manifold (of degree n): dg manifold with a symplectic form ω such that $\deg \omega = n$, $\mathcal{L}_Q \omega = 0$

Remark: If $n \neq -1$ then there exists a Hamiltonian $H \in C^\infty(X)$ for Q.

$$Q^2 = 0 \quad \leftrightarrow \quad \{H, H\} = 0$$

Example

X an NQ-manifold $\rightsquigarrow T^*[n]X$ is NQ symplectic of deg n, $H_{T^*[n]X} = Q$

classical BV manifold: dg symplectic manifold of degree -1, with H given

A recipe following [Alexandrov, Kontsevich, Schwarz, Zaboronsky 1997] and [Ševera 2016]

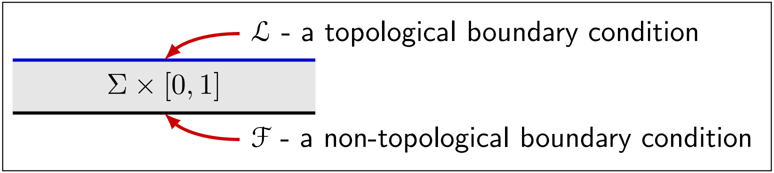
Ingredients: (fixing an oriented n-dimensional manifold Σ)

- NQ symplectic X of deg $n \rightsquigarrow$ define $\mathcal{X} := \mathrm{Maps}(T[1]\Sigma, X)$
- Lagrangian submanifold $L \subset X \rightsquigarrow$ define $\mathcal{L} := \mathrm{Maps}(T[1]\Sigma, L) \subset \mathcal{X}$
- Lagrangian submanifold $\mathcal{F} \subset \mathcal{X}$, using a structure on Σ

BV manifold:

- $\mathcal{M} = \{f \colon T[1](\Sigma \times [0,1]) \to X \;\text{ s.t. }\; f|_0 \in \mathcal{F},\; f|_1 \in \mathcal{L}\}$
- $\omega_{\mathcal{M}}$ induced by ω_X, $Q_{\mathcal{M}}$ given by the difference of Q's

  $\rightsquigarrow$ Hamiltonian $S_{\mathcal{M}}$ = action functional



Duality: $(X, \mathcal{F}, L_1)$ and $(X, \mathcal{F}, L_2)$ correspond to dual theories

Reformulation: derived intersections [Pantev, Toën, Vaquié, Vezzosi 2013]

if X is NQ symplectic and $L \subset X \times \bar{Y}$ is a Lagrangian submanifold, with Y acyclic NQ symplectic

$\rightsquigarrow$ we call the projection $\ell\colon L \to X$ a **Lagrangian map**

if $L, L' \subset X$ are Lagrangian submanifolds

$\rightsquigarrow$ resolve L to a Lagrangian submersion $\ell\colon R \to X$

$\rightsquigarrow$ $\ell^{-1}(L') \subset R$ is called the **derived intersection** of L and L'

Example

The above BV manifold is the derived intersection of the Lagrangian submanifolds $\mathcal{F}, \mathcal{L} \subset \mathcal{X}$.

$$\mathcal{L} \text{ resolved to } \mathcal{R} = \{f\colon T[1](\Sigma \times [0,1]) \to X \text{ s.t. } f|_0 \in \mathcal{L}\}$$

$$\mathcal{R} \to \mathrm{Maps}(T[1]\Sigma, X), \quad f \mapsto f|_1$$

A simplification

Observation:

if $\ell\colon R \to X$ resolves $L \subset X \;\rightsquigarrow\; \mathcal{R} := \mathrm{Maps}(T[1]\Sigma, R)$ resolves $\mathcal{L}$

To obtain a theory:

$$(X, \mathcal{F}, L) \;\rightsquigarrow\; \ell\colon R \to X \;\rightsquigarrow\; \hat{\ell}\colon \mathcal{R} \to \mathcal{X} \;\rightsquigarrow\; \hat{\ell}^{-1}(\mathcal{F}) \subset \mathcal{R}$$

Explicitly:

$$\boxed{S(f) = S_{\mathcal{F}}(\ell \circ f) + \int_{T[1]\Sigma} i_{Q_{T[1]\Sigma}} (r \circ f)^* \vartheta_{\tilde{Y}} + (r \circ f)^* H_{\tilde{Y}} - f^* H_{\mathrm{rel}}}$$

$$f\colon T[1]\Sigma \to R,\ \ell \circ f \in \mathcal{F}$$

$Y = T^*[n]\tilde{Y}$ for $\tilde{Y}$ NQ symplectic of deg $n-1$, $\quad r\colon R \to \tilde{Y}$

$$dS_{\mathcal{F}} = \vartheta_{\mathcal{X}}|_{\mathcal{F}}, \quad dH_{\mathrm{rel}} = \vartheta^{\mathrm{taut}}_{T[1]\tilde{Y}} - \ell^* \vartheta_X, \quad d\vartheta = \omega$$

Example: Poisson-Lie T-duality [Klimčík, Ševera 1995]

Ingredients:

- Σ oriented 2D surface with a pseudoconformal structure
- $X = \mathfrak{g}[1]$, $\omega_X \leftrightarrow$ invariant inner product on $\mathfrak{g}$, $Q_X = \frac{1}{2} c^i_{jk} \xi^k \xi^j \partial_{\xi_i}$
- $L = \mathfrak{h}[1]$, $L' = \mathfrak{h}'[1]$, for $\mathfrak{h}, \mathfrak{h}' \subset \mathfrak{g}$ two complementary Lagrangian Lie subalgebras
- if $E \colon \mathfrak{g} \to \mathfrak{g}$ symmetric, $E^2 = 1$ (generalized metric), then set

$$\mathcal{F} := \{A \in \Omega^1(\Sigma, \mathfrak{g}) \mid *A = EA\} \oplus \Omega^2(\Sigma, \mathfrak{g}) \quad \subset \quad \Omega(\Sigma, \mathfrak{g}) = \mathcal{X}$$

Resolution:

$$L = \mathfrak{h}[1] \subset \mathfrak{g}[1] \quad \rightsquigarrow \quad R = \mathfrak{g}[1] \times H' \to \mathfrak{g}[1]$$

Example: Poisson-Lie T-duality - continuation

BV manifold / Fields:

$$T^*[-1]\left(\mathrm{Maps}(\Sigma, H') \times \Omega^1(\Sigma, \mathfrak{h})\right)$$

Action:

$$S(g,B) = \int_\Sigma \langle B, g^{-1}dg\rangle + \tfrac{1}{2}(g^{-1}\pi_g^{H'})(B,B) + \tfrac{1}{2}\psi(B,B)$$

$$g\colon \Sigma \to H',\ B \in \Omega^1(\Sigma, \mathfrak{h}), \qquad \mathfrak{F} \rightsquigarrow \psi\colon (T^*\Sigma \otimes \mathfrak{h})^{\otimes 2} \to \textstyle\bigwedge^2 T^*\Sigma$$

Integrating out B:

$$S(g) = \tfrac{1}{2}\int_\Sigma e(g^{-1}dg, g^{-1}dg), \quad e_g := (g^{-1}\pi_g^{H'} + \psi)^{-1}$$

Duality: $\mathfrak{h} \leftrightarrow \mathfrak{h}'$

Higher Poisson-Lie T-duality

Ingredients:

- Σ oriented n-dim surface
- $X = \mathfrak{g}[1]$ for $\mathfrak{g}$ a graded Lie algebra concentrated in non-positive degrees, with an invariant pairing of degree $n-2$
- $L = \mathfrak{h}[1]$, $L' = \mathfrak{h}'[1]$, for $\mathfrak{h}, \mathfrak{h}' \subset \mathfrak{g}$ two complementary graded Lagrangian Lie subalgebras
- take $\mathcal{F}$ with $C^\infty(\mathcal{F})$ non-positively graded, with $\mathcal{F} \cap \mathrm{Maps}(T[1]\Sigma, \mathfrak{g}[1])^0$ given by a generating function

$$S_{\mathcal{F}} \in C^\infty(\mathrm{Maps}(T[1]\Sigma, \mathfrak{h}[1])^0)$$

Resolution:

$$L = \mathfrak{h}[1] \subset \mathfrak{g}[1] \quad \rightsquigarrow \quad R = \mathfrak{g}[1] \times H' \to \mathfrak{g}[1]$$

Higher Poisson-Lie T-duality - continuation

Fields:

$$\bar{g} \in \mathrm{Maps}(T[1]\Sigma, H')$$
$$\bar{B} \in \mathrm{Maps}(T[1]\Sigma, \mathfrak{h}[1])^{\geq 0} \cong \Omega(\Sigma, \mathfrak{h}[1])^{\geq 0}$$

Action:

$$\boxed{S(\bar{B}, \bar{g}) = \int_\Sigma \langle \bar{B}, \bar{g}^{-1} d\bar{g} \rangle + \tfrac{1}{2}(\bar{g}^{-1}\pi^{H'}_{\bar{g}})(\bar{B}, \bar{B}) + S_{\mathfrak{F}}(B)}$$

B is the degree 0 part of $\bar{B}$

$$\bar{g}^{-1}\pi^{H'}_{\bar{g}} \in C^\infty(H', \mathfrak{h}' \otimes \mathfrak{h}'), \qquad \bar{g}^{-1} d\bar{g} = i_{d_\Sigma} \bar{g}^* \theta^{H'}_{MC}$$

Duality: $\mathfrak{h} \leftrightarrow \mathfrak{h}'$

Example - Electric-magnetic duality

Need: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$, $S_{\mathfrak{F}} \in C^\infty(\mathrm{Maps}(T[1]\Sigma, \mathfrak{h}[1])^0)$

For electromagnetism:

$$\mathfrak{g} = \overbrace{\mathbb{R}[a]}^{\mathfrak{h}} \oplus \overbrace{\mathbb{R}[b]}^{\mathfrak{h}'} \text{ (abelian)}, \quad S_{\mathfrak{F}} \in C^\infty(\Omega^{a+1}(\Sigma)), \quad S_{\mathfrak{F}}(B) = \tfrac{1}{2} \int_\Sigma B * B$$

$$\bar{x} := \bar{g} \in \mathrm{Maps}(T[1]\Sigma, H') \cong \Omega(\Sigma)$$
$$\bar{B} \in \mathrm{Maps}(T[1]\Sigma, \mathfrak{h}[1])^{\geq 0} \ \cong \ \Omega^{\geq a+1}(\Sigma)$$

$$S(\bar{B}, \bar{x}) = \int_\Sigma \bar{B}\, d\bar{x} + B * B = \int_\Sigma B\, dx + \tfrac{1}{2} B * B + \tfrac{1}{2} \underbrace{B^{(a+2)}}_{x^+}\, d \underbrace{x^{(b)}}_{c} + \dots$$

In 2nd order formalism: $\int_\Sigma dx * dx$, $x \in \Omega^{b+1}(\Sigma)$

Examples of $\mathfrak{h} \subset \mathfrak{g}$

Case $n = 3$:

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^*[1]$ (semi-abelian double)

$\mathfrak{h} = \mathfrak{l} \oplus \mathrm{Ann}\,\mathfrak{l}[1]$ for $\mathfrak{l} \subset \mathfrak{k}$ a Lie subalgebra

Fields: $\mathrm{Maps}(\Sigma, K/L)$, $\Omega^1(\Sigma, \mathfrak{l}^*)$, ghosts for the 1-forms, antifields

Case $n = 4$:

$\mathfrak{g} = \mathfrak{k} \oplus W[1] \oplus \mathfrak{k}^*[2]$

(W symplectic, with a symplectic representation of $\mathfrak{k}$ on W)

$\mathfrak{h} = \mathfrak{l} \oplus U[1] \oplus \mathrm{Ann}\,\mathfrak{l}[2]$

for $\mathfrak{l} \subset \mathfrak{k}$ a Lie subalgebra, preserving $U \subset W$

Fields: $\mathrm{Maps}(\Sigma, K/L)$, $\Omega^1(\Sigma, U^*)$, $\Omega^2(\Sigma, \mathfrak{l}^*)$, $3\times$ ghosts, antifields

Outlook

- Study examples.

- Investigate the global picture $\rightsquigarrow$ replace dg manifolds by higher derived stacks.