EQUIVARIANT BATALIN-VILKOVISKY FORMALISM

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equivariant methods played a fundamental role in obtaining exact results in supersymmetric quantum field theories

ex. Nekrasov's computation of $N=2$ 4d SUSY in Ω-background

Here we want to address equivariance from a gauge theoretical point of view

PLAN of THE TALK

- Basics of BV and equivariant cohomology
- AKSZ solution of CME
- Equivariant AKSZ and BV
- Example: Donaldson-Witten

EQUIVARIANT COHOMOLOGY

$\mathfrak{g}$ Lie algebra

A $\mathfrak{g}$-differential algebra is a dga (A, d) with $X \to L_X$ and $X \to \iota_X$ $\forall X \in \mathfrak{g}$ s.t.

$$[L_X, d] = 0 \quad [L_X, L_Y] = L_{[X,Y]} \quad [\iota_X, d] = L_X \quad [\iota_X, \iota_Y] = 0 \quad [\iota_X, L_Y] = \iota_{[X,Y]}$$

$$A[u] = A \otimes S\mathfrak{g}^* \qquad d_G = d - u^a \iota_{v_a} \qquad |u| = 2$$

$$d_G^2 = u^a L_{v_a}$$

$$\left(A[u]^{\mathfrak{g}}, d_G\right)$$ Cartan complex for $H_G(A)$

BASIC FACTS of BV QUANTIZATION [Batalin Vilkovisky '83]

$S_0(\varphi)$ classical action

$\delta_c S_0 = 0$ gauge invariance

$\Phi = \{\varphi, c\}$ $\equiv$ fields

physical fields

ghosts

$|\phi^+| = -|\phi| - 1$

$\Phi^\vee = \{\varphi^\vee, c^\vee\}$ $\equiv$ antifields

$\{\phi_i, \phi^{\vee j}\} = \delta_i^{\,j}$ antibracket

$\Delta = \dfrac{\delta}{\delta\phi^{\vee i}} \dfrac{\delta}{\delta\phi_i}$ BV-laplacian

Look for $S(\phi, \phi^\vee)$, $|S| = 0$, s.t.

i) $S(\phi, 0) = S_0(\varphi)$

ii) $\frac{1}{2}\{S, S\} - i\hbar\Delta S = 0$ Quantum master equation

The fundamental fact about BV geometry

$$\frac{d}{dt} \int_{\phi^\vee = \frac{\partial \Psi_t}{\partial \phi}} \left(e^{\frac{i}{\hbar}S} \mathcal{O} \right) = 0$$

$\Psi_t =$ (a family of) gauge fixing fermions

if $\{S, \mathcal{O}\} - i\hbar\Delta\mathcal{O} = 0$

$\mathcal{O} =$ quantum observable

The classical limit $\hbar \to 0$ of QME

$$\{S, S\} = 0 \qquad \text{Classical master equation}$$

$$Q_{BV} = \{S, -\} \qquad Q_{BV}^2 = 0$$

We tipically first solve CME and then look for QME.

The easiest situation is when S solves CME and $\Delta S = 0$

EXAMPLE Chern-Simons action

$V = \mathfrak{g}[1] \ni c \quad |c| = 1 \qquad \mathcal{C} = c + A + A^\vee + c^\vee$

$$S = \int \mathcal{C} d\mathcal{C} + \frac{1}{6}\langle \mathcal{C}, [\mathcal{C}, \mathcal{C}] \rangle \qquad S\big|_{A^\vee = c^\vee = 0} = \int A dA + \frac{1}{6}\langle A, [A, A] \rangle$$

gauge fixing: $c^\vee = 0 \quad A, A^\vee$ coexact

Add the trivial sector $\left\{ \begin{array}{l} \bar c, b \text{ 0-forms.} \\ \bar c^\vee, b^\vee \end{array} \right.$ $\quad S^{tr} = \int b \bar c^\vee$

$$\Psi = \int \bar c \, d * A \Rightarrow \boxed{A^\vee = d^+ * \bar c} \qquad \bar c^\vee = d * A \qquad S^{tr} = \int b \, d * A \xrightarrow[\text{over } b]{\text{int}} d^+ A = 0$$

AKSZ CONSTRUCTION of SOLUTIONS of CME

[Alexandrov, Kontsevich, Schwarz, Zaboronsky '97]

Define a solution of CME in terms of the following geometrical data

SOURCE $\Sigma \equiv d$-dimensional smooth manifold

TARGET $\mathcal{M}$ graded manifold with

i) a symplectic form $\omega \in \Omega^2 \mathcal{M}$ $\quad |\omega| = d-1$

ii) an hamiltonian cohomological vector field, i.e. $\Theta \in C(\mathcal{M})$ $\quad |\Theta| = d$ $\quad \{\Theta, \Theta\} = 0$

$$D_\Theta = \{\Theta, -\} \quad |D_\Theta| = 1 \text{ and } D_\Theta^2 = 0$$

$\mathcal{M} = V$ graded vector space

SPACE of FIELDS (and ANTIFIELDS)

Let $\{X^A\}$ coordinates on V and $\{u^\alpha\}$ on Σ

Let $\theta^\alpha = \dot{u}^\alpha$ denote the (odd) velocities (coordinates on $T[1]\Sigma$)

Define the space of fields and antifields $\mathcal{F}_\Sigma$ space of the superfields

$$\mathbb{X}^A = X^A + X^A_\alpha \theta^\alpha + \dots + \qquad |X^A_{\alpha_1 \dots \alpha_r}| = |X^A| - r$$

$$\mathcal{F}_\Sigma = \Omega\Sigma \otimes V$$

$$\Omega = \int_{T[1]\Sigma} du\, d\theta \; \omega_{AB}(\mathbb{X})\, \delta\mathbb{X}^A \, \delta\mathbb{X}^B$$

$$S = \int_{T[1]\Sigma} \left[\mathbb{X}^A \alpha_{AB} \delta \mathbb{X}^B + \Theta(\mathbb{X}) \right] = S_0 + S_\Theta$$

$$d\alpha = \omega$$

S solves the CME

There is a canonical choice of gauge fixing

Choose a metric on Σ

$$\Omega^k \Sigma = H^k(\Sigma) \oplus \underbrace{d\,\Omega^{k-1}}_{\text{exact}} \oplus \underbrace{d^\dagger \Omega^{k+1}}_{\text{coexact}}$$

$$\mathbb{X} = \mathbb{X}_h + \mathbb{X}_{ex} + \mathbb{X}_{co}$$

Integrate over $\mathcal{L} = \underbrace{\Omega_{co}}_{\text{coexact form}} \otimes V$

EQUIVARIANT AKSZ

Let the Lie algebra $\mathfrak{g}$ act on Σ by $x \in \mathfrak{g} \to V_x \in \mathcal{X}(\Sigma)$

ι_{V_x}, L_{V_x} contraction operator and Lie derivative acting on $\Omega\Sigma$

In the language of graded geometry

$$\iota_{V_x}, L_{V_x} \in \mathrm{Vect}(T[1]\Sigma) \equiv \text{infinitesimal diffeo of } T[1]\Sigma$$

The space of fields $\mathcal{F}_\Sigma$ of the AKSZ construction is a space of maps from $T[1]\Sigma$ to the target V

$$\mathcal{F}_\Sigma =: \mathrm{Map}(T[1]\Sigma, V)$$

Diffeos of the source (and of the target) by composition become diffeos of $\mathcal{F}_\Sigma$

$\hat{\iota}_{V_x}, \hat{L}_{V_x} \in \mathrm{Vect}(\mathcal{F}_\Sigma)$ and they are hamiltonian

$\mathcal{A} = C(\mathcal{F}_\Sigma)$ together with Q_{BV}, $\hat{\iota}_{V_x}$, $\hat{L}_{V_x}$
is a $\mathfrak{g}$-differential algebra

the equivariant extension of the AKSZ action

$$S^c = S - u^a S_{\hat{\iota}_{V_a}} \qquad |u^a| = 2$$

so that

$$Q^c_{BV} = \{ S^c_{BV}, - \} = Q_{BV} - u^a \hat{\iota}_{V_a}$$

$$\frac{1}{2} \{ S^c , S^c \} + u^a S_{\hat{L}_{V_a}} = 0$$ Modified classical master equation

How to modify the BV formalism?

Main idea

let $T := \frac{1}{2}\{S^c, S^c\} - i\hbar \Delta S^c = -u^a S_{\hat{L}V_a} - i\hbar \Delta S^c$

$\int_{\mathcal{L}} e^{\frac{i}{\hbar} S^c} \mathcal{O}$ is invariant under the deformations of $\mathcal{L}$

if

i) $T = 0$ on $\mathcal{L}$

ii) $\Delta_{S,T}\mathcal{O}$ is proportional to T

where $\Delta_{S,T}\mathcal{O} = e^{-\frac{i}{\hbar} S^c} \Delta\left(e^{\frac{i}{\hbar} S^c} \mathcal{O}\right) = \Delta\mathcal{O} + \frac{i}{\hbar} Q^c_{BV} \mathcal{O} + \left(\frac{i}{\hbar}\right)^2 T\mathcal{O}$

We have to work modulo T $\quad I_T$ = ideal generated by T

$N_T = \{ \mathcal{O} \in \mathcal{A} \mid \{T, \mathcal{O}\} \in I_T \}$

$\mathcal{A}_T = N_T / I_T$

REMARK $A_T = N_T / I_T$ can be interpreted as the algebra of functions over $\underline{C_T}$, the symplectic reduction of the zero locus of T.

Since $T|_{\mathcal{L}} = 0 \Rightarrow \mathcal{L} \subset C_T \Rightarrow \{T, -\} \subset T\mathcal{L}$

$\mathcal{L}$ cannot gauge fix completely $\Rightarrow$ G compact

A_T can be worked out more explicitly. Under some natural hypothesis on Δ, one can show that

i) $A[u]^{\mathfrak{g}} \subset N_T$

ii) $A[u]^{\mathfrak{g}}$ is invariant under Q^c_{BV} and $\Delta_{S,T}$

iii) $T \in A[u]^{\mathfrak{g}}$

$\left(A[u]^{\mathfrak{g}}, Q_{BV}^{c}\right)$ is the Cartan model computing the equivariant cohomology of the $\mathfrak{g}$-differential algebra $(A, Q_{BV}, \hat{\iota}_v, \hat{L}_v)$

The quantum pre observables are then

$$\left(A[u]^{\mathfrak{g}}/I'_T, \Delta_{S,T}\right)$$

A quantum observable is then $\mathcal{O} \in A[u]^{\mathfrak{g}}$ s.t.

$$\left(\Delta + \frac{i}{\hbar} Q_{BV}^{c}\right)\mathcal{O} \sim T$$

GAUGE FIXING $\mathcal{L} = (\Omega\Sigma)^{\text{coexact}} \otimes V$

$T\big|_{\mathcal{L}} = 0$ if the ~~metric~~ is $\mathfrak{g}$-invariant

EQUIVARIANT DONALDSON-WITTEN THEORY

$d=4$ $\quad V = \mathbb{k}[1] \oplus \mathbb{k}[2]$ $\quad \mathbb{k}$ - Lie algebra $(\mathbb{k} \neq \mathfrak{g})$

$\quad\quad\quad\quad \overset{\Psi}{c} \quad\quad \overset{\Psi}{\phi}$

with nondegenerate invariant pairing $\langle\, , \rangle$

$$\Theta = \frac{1}{2}\langle \phi, \phi \rangle + \frac{1}{2}\langle \phi, [c,c] \rangle \qquad |\Theta| = 4$$

$$D_\Theta c = \phi + \frac{1}{2}[c,c]$$

$$D_\Theta \phi = [c, \phi]$$

Weil differential

$$\mathbb{C} = c + A + \chi + \psi^\vee + \phi^\vee$$

$$\Phi = \phi + \psi + \chi^\vee + A^\vee + c^\vee$$

$$S = \int \langle \Phi, d_\Sigma \mathbb{C} \rangle + \frac{1}{2}\langle \Phi, \Phi \rangle + \frac{1}{2}\langle \Phi, [\mathbb{C}, \mathbb{C}] \rangle$$

Let us discuss gauge fixing $\pm$ (anti) self dual 2-forms

$$\chi = \chi^{+} + \chi^{-} \qquad \begin{cases} \chi^{-} = 0 \\ \chi^{\vee +} = 0 \end{cases}$$
$$\chi^{\vee} = \chi^{\vee +} + \chi^{\vee -}$$

In other degrees we require they are coexact

$$c^{\vee} = \phi^{\vee} = 0$$

To impose $(A, A^{\vee})$ and $(\psi, \psi^{\vee})$ are coexact we have to introduce

for $A, A^{\vee}$ $\begin{cases} \bar{c}, b & \text{zero forms of degree } -1, 0 \\ \bar{c}^{\vee}, b^{\vee} \end{cases}$

and add $S_1^{tr} = \int \langle \bar{c}^{\vee}, b \rangle$ to the action

for $\psi, \psi^{\vee}$ $\begin{cases} \varphi, \eta & \text{zero forms of degree } -2, -1 \\ \varphi^{\vee}, \eta^{\vee} \end{cases}$

and add $S_2^{tr} = \int \langle \varphi^{\vee}, \eta \rangle$ to the action

Choose the gauge fixing fermion as

$$\Psi = \int_\Sigma \langle \bar{c}, d * A \rangle + \langle \varphi, d * \psi \rangle$$

integration over $\bar{c}$ and φ imposes $d^\dagger A = 0$
$d^\dagger \psi = 0$

the gauge fixing condition imposes $A^+ = d^\dagger * \bar{c}$
$\psi^+ = d^\dagger * \varphi$

$(A, \phi, \psi, \chi^+, \varphi, \eta)$ reproduces the vector multiplet of DW theory and $Q = \delta_{SUSY} + \delta_{BRST}$

(already observed in [Ikeda, '11])

EQUIVARIANT EXTENSION

We simply replace $d_\Sigma \to d_\Sigma - u^a \iota_{v_a}$

$$S^c = \int_{T[1]\Sigma_4} \langle \mathbb{\Phi}, d_G \mathbb{C} \rangle + \frac{1}{2} \langle \mathbb{\Phi}, \mathbb{\Phi} \rangle + \frac{1}{2} \langle \mathbb{\Phi}, [\mathbb{C}, \mathbb{C}] \rangle$$

$$= S - u^a \int \langle \psi, \iota_{v_a} \phi^\vee \rangle + \langle \chi^\vee, \iota_{v_a} \psi^\vee \rangle + \langle A^\vee, \iota_{v_a} \chi \rangle + \langle C^\vee, \iota_{v_a} A \rangle$$

the gauge fixing is done choosing an invariant metric on Σ_4

In the trivial sector we now have to add

$$\int \langle \bar{c}^\vee, b \rangle + \langle b^\vee, L_v \bar{c} \rangle + \langle \varphi^\vee, \eta \rangle + \langle \eta^\vee, L_v \varphi \rangle$$

we have to keep the structure of $\mathfrak{g}$ - differential algebra

By repeating the construction in the non equivariant case one recovers the trasformations of the equivariant DW theory used in [Nekrasov '03]

(A relevant class of) Classical equivariant observables is obtained from any $F \in W(\mathbb{K}) = S\mathfrak{g}^* \otimes \Lambda \mathfrak{g}^*$

$$D_{\oplus} F = 0$$

and any equivariant cycle γ

$$O_{F,\gamma} = \int_\gamma F(c+A, \phi+\psi-F(A))$$

- The equivariant extension of the AKSZ solution is a straightforward procedure

- We have recovered known and meaningful examples like DW and SYM_2, related to SUSY

- We have to study unknown and hopefully meaningful examples, like CS and PSM