Symmetries and Dualities in sigma models with Wess-Zumino term

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Motivations

- Duality symmetries play a fundamental role in String Theory
- Double Field Theory (DFT) emerges when making explicit T-duality invariance at the low energy effective level
- Poisson-Lie T-duality generalizes Abelian and non-Abelian T-duality
- Dynamics on group manifolds is a natural framework to investigate such issues in a proper geometric setting
- Wess-Zumino-Witten (WZW) model is a string solution

Definition

A **Drinfeld double** is an even-dimensional Lie group D whose Lie algebra $\mathfrak{d}$ can be decomposed into a pair of maximally isotropic subalgebras, $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$, with respect to a non-degenerate (ad)invariant bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{d}$.

$$(\mathfrak{d} = \mathfrak{g} + \tilde{\mathfrak{g}}, \mathfrak{g}, \tilde{\mathfrak{g}}) \longleftarrow \textbf{Manin triple} \longrightarrow D = G \cdot \tilde{G}$$

- Since the bilinear form is non-degenerate, we can use it to identify $\tilde{\mathfrak{g}} = \mathfrak{g}^* \Longleftrightarrow \mathfrak{g}$ Lie bialgebra
- Choosing $T_a \in \mathfrak{g}$, $\tilde{T}^a \in \tilde{\mathfrak{g}}$, such that $(T_a, \tilde{T}^a) \equiv T_A \in \mathfrak{d}$
  $$\langle T_a, T_b \rangle = 0, \quad \langle \tilde{T}^a, \tilde{T}^b \rangle = 0, \quad \langle T_a, \tilde{T}^b \rangle = \delta_a{}^b \longrightarrow O(d,d) \text{ structure}$$
- Lie bracket on $\mathfrak{d}$: $[T_A, T_B] = F_{AB}{}^C T_C$
  $$[T_a, T_b] = f_{ab}{}^c\, T_c, \quad [\tilde{T}^a, \tilde{T}^b] = \tilde{f}^{ab}{}_c\, \tilde{T}^c, \quad [T_a, \tilde{T}^b] = \tilde{f}_a{}^{bc}\, T_c - f^b{}_{ac}\, \tilde{T}^c$$
- Jacobi identity on $\mathfrak{d}$ imposes constraints on the structure constants:
  $$\tilde{f}_a{}^{mc} f^b{}_{dm} - \tilde{f}_a{}^{mb} f^c{}_{dm} - \tilde{f}_d{}^{mc} f^b{}_{am} + \tilde{f}_d{}^{mb} f^c{}_{am} - \tilde{f}_m{}^{bc} f^m{}_{da} = 0$$

Poisson-Lie symmetry

- Let $X^i : \Sigma \to \mathcal{M}$, where (Σ, h) is a 2-dimensional Lorentzian worldsheet and $(\mathcal{M}, g)$ a (pseudo) Riemannian manifold, together with a B-field, which admits a free action of a Lie group G from the right

- Consider the non-linear sigma model

$$S = \int dz d\bar{z}\, E_{ij} \partial X^i \bar{\partial} X^j, \quad \text{with } E_{ij} = g_{ij} + B_{ij}$$

- The infinitesimal generators of the group action are the left-invariant vector fields $\{V_a^i\}$

$$\delta X^i = V_a^i \epsilon^a \longrightarrow \delta S = \int dz d\bar{z}\, \mathcal{L}_{V_a} E_{ij} \partial x^i \bar{\partial} X^j \epsilon^a - \int dJ_a \epsilon^a,$$

$$\text{with } J_a = V_a^i \left(E_{ij} \bar{\partial} X^j d\bar{z} - E_{ji} \partial X^j dz \right).$$

If $\mathcal{L}_{V_a} E_{ij} = 0 \longrightarrow dJ_a = 0$ standard T-duality with isometries

- This can be generalized:

$$dJ_a = \frac{1}{2} \tilde{f}_a{}^{bc} J_b \wedge J_c \longrightarrow \mathcal{L}_{V_a} E_{ij} = -\tilde{f}_a{}^{bc} V_b^k V_c^\ell E_{ik} E_{\ell j} \quad \text{no isometries}$$

$$[V_a, V_b] = f_{ab}{}^c V_c \longrightarrow [\mathcal{L}_{V_a}, \mathcal{L}_{V_b}] E_{ij} = f_{ab}{}^c \mathcal{L}_{V_c} E_{ij} \quad \text{integrability condition}$$

Poisson-Lie symmetry/plurality

$$\left.\begin{array}{l}\mathcal{L}_{V_a}E_{ij} = -\tilde{f}_a{}^{bc}V_b^k V_c^\ell E_{ik}E_{\ell j}\\ [\mathcal{L}_{V_a},\mathcal{L}_{V_b}]E_{ij} = f^c{}_{ab}\mathcal{L}_{V_c}E_{ij}\end{array}\right] \longrightarrow \tilde{f}_a{}^{mc}f^b{}_{dm} - \tilde{f}_a{}^{mb}f^c{}_{dm} - \tilde{f}_d{}^{mc}f^b{}_{am} + \tilde{f}_d{}^{mb}f^c{}_{am} - \tilde{f}_m{}^{bc}f^m{}_{da} = 0$$

Duality

$$(\mathfrak{g},\tilde{\mathfrak{g}}) \longleftrightarrow (\tilde{\mathfrak{g}},\mathfrak{g})$$

$$\left.\begin{array}{l}\mathcal{L}_{\tilde{V}_a}\tilde{E}_{ij} = -f_a{}^{bc}\tilde{V}_b^k \tilde{V}_c^\ell \tilde{E}_{ik}\tilde{E}_{\ell j}\\ [\mathcal{L}_{\tilde{V}_a},\mathcal{L}_{\tilde{V}_b}]\tilde{E}_{ij} = \tilde{f}^c{}_{ab}\mathcal{L}_{\tilde{V}_c}\tilde{E}_{ij}\end{array}\right] \longrightarrow \tilde{f}_a{}^{mc}f^b{}_{dm} - \tilde{f}_a{}^{mb}f^c{}_{dm} - \tilde{f}_d{}^{mc}f^b{}_{am} + \tilde{f}_d{}^{mb}f^c{}_{am} - \tilde{f}_m{}^{bc}f^m{}_{da} = 0$$

- In general, a Drinfeld double has several decompositions $\longrightarrow$ P-L T-plurality

$$T'_A \equiv C_A{}^B T_B : \left[\begin{array}{l}[T'_a,T'_b] = f_{ab}{}^c T'_c, \quad [\tilde{T}'^a,\tilde{T}'^b] = \tilde{f}^{ab}{}_c\tilde{T}'^c, \quad [T'_a,\tilde{T}'^b] = \tilde{f}_a{}^{bc}T'_c - f^b{}_{ac}\tilde{T}'^c\\ \langle T'_A,T'_B\rangle = \langle T_A,T_B\rangle \longrightarrow C_A{}^B \in O(d,d)\end{array}\right.$$

- The algebra of $SL(2,\mathbb{C})$ is spanned by $e_i = \sigma_i/2,\ b_i = ie_i$ with brackets:

  $[e_i, e_j] = i\epsilon_{ij}{}^k e_k, \quad [e_i, b_j] = i\epsilon_{ij}{}^k b_k, \quad [b_i, b_j] = -i\epsilon_{ij}{}^k e_k \longrightarrow e_i\ \mathfrak{su}(2)$ generators

- Two non-degenerate invariant scalar products:

  $\langle v, w\rangle = 2\mathrm{Im}[\mathrm{Tr}(vw)], \quad (v, w) = 2\mathrm{Re}[\mathrm{Tr}(vw)] \quad \forall v, w, \in \mathfrak{sl}(2,\mathbb{C})$
  (Cartan-Killing)

- One can consider the dual vector space $\mathfrak{su}(2)^*$ by introducing a basis $\{\tilde{e}^i\}$ dual to $\{e_i\}$

$$\tilde{e}^i = \delta^{ij}\left(b_j + \epsilon^k{}_{j3}\, e_k\right) : \quad \left\langle \tilde{e}^i, e_j\right\rangle = 2\,\mathrm{Im}\left[\mathrm{Tr}\left(\tilde{e}^i e_j\right)\right] = \delta^i_j$$

These vectors in turn span $\mathfrak{sb}(2,\mathbb{C})$: $\left[\tilde{e}^i, \tilde{e}^j\right] = if^{ij}{}_k \tilde{e}^k$, with $f^{ij}{}_k = \epsilon^{ij\ell}\epsilon_{\ell 3k}$
and each subalgebra acts on the other one non-trivially, by co-adjoint action:

$$\left[\tilde{e}^i, e_j\right] = i\epsilon^i_{jk}\tilde{e}^k + if^{ki}_j e_k$$

- Both subalgebras $\mathfrak{su}(2)$ and $\mathfrak{sb}(2,\mathbb{C})$ are maximally isotropic w. r. t. the scalar product $\langle\cdot,\cdot\rangle$

  $\langle e_i, e_j\rangle = 0, \quad \langle \tilde{e}^i, \tilde{e}^j\rangle = 0 \longrightarrow (\mathfrak{sl}(2,\mathbb{C}), \mathfrak{su}(2), \mathfrak{sb}(2,\mathbb{C}))$ is a Manin triple$\rightarrow$

$$\rightarrow SL(2,\mathbb{C}) = SU(2)\cdot SB(2,\mathbb{C})$$

$SL(2,\mathbb{C})$ as a Drinfeld double: $O(3,3)$ and Riemannian metrics

- In doubled notation $e_I = \begin{pmatrix} e_i \\ \tilde{e}^i \end{pmatrix}$, with $e_i \in \mathfrak{su}(2)$, $\tilde{e}^i \in \mathfrak{sb}(2,\mathbb{C})$

$$\langle e_I, e_J \rangle = \eta_{IJ} = \begin{pmatrix} 0 & \delta_i^j \\ \delta_j^i & 0 \end{pmatrix} \longrightarrow O(3,3) \text{ invariant metric}$$

- Consider the scalar product $(v,w) = 2\mathrm{Re}[\mathrm{Tr}(vw)]$ on $\mathfrak{sl}(2,\mathbb{C})$, we have another splitting w. r. t. this

$$(e_i, e_j) = -(b_i, b_j) = \delta_{ij}, \quad (e_i, b_j) = 0 \longleftarrow \text{not positive-definite}$$

By denoting C_+ and C_- the two subspaces spanned by $\{e_i\}$ and $\{b_i\}$ respectively, this scalar product, with the splitting $C_+ \oplus C_-$, defines a positive-definite Riemannian metric via $\mathcal{H} = (,)_{C_+} - (,)_{C_-} \longrightarrow ((,))$

$$((e_i, e_j)) := (e_i, e_j)\,; \quad ((b_i, b_j)) := -(b_i, b_j)\,; \quad ((e_i, b_j)) := (e_i, b_j) = 0$$

- In doubled notation we have

$$((e_I, e_J)) = \mathcal{H}_{IJ} = \begin{pmatrix} \delta_{ij} & {\epsilon_{3i}}^j \\ -{\epsilon^i}_{j3} & \delta^{ij} + {\epsilon^i}_{\ell 3}{\epsilon^j}_{k3}\delta^{\ell k} \end{pmatrix} \longrightarrow \mathcal{H}^T \eta \mathcal{H} = \eta$$

(pseudo-orthogonal $O(3,3)$)

- Let $\varphi : (t, \sigma) \in \Sigma \to g \in SU(2)$, where Σ denotes the worldsheet with Minkowski $(1, -1)$ signature
- Consider the non-linear sigma model

$$S_0 = \frac{1}{4\lambda^2} \int_\Sigma \mathrm{Tr}\left[\varphi^* \left(g^{-1}dg\right) \wedge *\varphi^* \left(g^{-1}dg\right)\right]$$

with a WZ term

$$\kappa S_{WZ} = \frac{\kappa}{24\pi} \int_{\mathcal{B}} \mathrm{Tr}\left[\tilde{\varphi}^* \left(\tilde{g}^{-1}d\tilde{g} \wedge \tilde{g}^{-1}d\tilde{g} \wedge \tilde{g}^{-1}d\tilde{g}\right)\right],$$

where $\mathcal{B}$ is a 3-manifold whose boundary is the compactification of the original two-dimensional source space, while $\tilde{g}$ is the extension of g on $\mathcal{B}$.

- The action $S = S_0 + \kappa S_{WZ}$ leads to the equations of motion

$$\begin{aligned} \partial_t A - \partial_\sigma J &= -\frac{\kappa\lambda^2}{4\pi}[A, J], \\ \partial_t J - \partial_\sigma A &= -[A, J], \quad \text{(integrability condition)} \\ \lim_{|\sigma|\to\infty} g(\sigma) &= 1 \end{aligned}$$

where $A = \left(g^{-1}\partial_t g\right)^i e_i$ and $J = \left(g^{-1}\partial_\sigma g\right)^i e_i$ are the $\mathfrak{su}(2)(\mathbb{R})$-valued currents

- Introducing canonical momenta I as fiber coordinates of the cotangent bundle $T^*SU(2)$, the model is described the Hamiltonian

$$H = \frac{1}{4\lambda^2}\int d\sigma \, \left(\delta^{ij} I_i I_j + \delta_{ij} J^i J^j\right)$$

  and e.t. Poisson brackets:

$$\begin{aligned}
\{I_i(\sigma), I_j(\sigma')\} &= 2\lambda^2 {\epsilon_{ij}}^k I_k(\sigma)\delta(\sigma-\sigma') + \frac{\kappa\lambda^4}{2\pi}\epsilon_{ijk} J^k(\sigma)\delta(\sigma-\sigma') \\
\{I_i(\sigma), J^j(\sigma')\} &= 2\lambda^2 \left[{\epsilon_{ki}}^j J^k(\sigma)\delta(\sigma-\sigma') - \delta_i^j \delta'(\sigma-\sigma')\right] \\
\{J^i(\sigma), J^j(\sigma')\} &= 0.
\end{aligned}$$

The Poisson algebra is the semi-direct sum of an Abelian algebra and a Kac–Moody algebra associated to $SU(2)$.

- We want to deform this algebra to a semi-simple one in such a way that the resulting brackets, together with the deformed Hamiltonian lead to an equivalent description of the dynamics

- It is possible to give an equivalent description of the dynamics in terms of a new Poisson algebra and a modified Hamiltonian, with the currents on an equal footing ($a = \frac{\kappa\lambda^2}{4\pi}$, $\xi_\tau = 2\lambda^2(1-\tau^2)$):

$$\{I_i(\sigma), I_j(\sigma')\} = \xi_\tau \left[\epsilon_{ij}{}^k I_k(\sigma)\delta(\sigma-\sigma') + a\,\epsilon_{ijk} J^k(\sigma)\delta(\sigma-\sigma')\right]$$
$$\{I_i(\sigma), J^j(\sigma')\} = \xi_\tau \left[(\epsilon_{ki}{}^j J^k(\sigma) + a\tau^2\,\epsilon_i{}^{jk} I_k(\sigma))\delta(\sigma-\sigma') - (1-\tau^2)\delta_i{}^j\delta'(\sigma-\sigma')\right]$$
$$\{J^i(\sigma), J^j(\sigma')\} = \xi_\tau\tau^2 \left[\epsilon^{ijk} I_k(\sigma)\delta(\sigma-\sigma') + a\,\epsilon^{ij}{}_k J^k(\sigma)\delta(\sigma-\sigma')\right],$$

with deformed Hamiltonian

$$H_\tau = \frac{1}{4\lambda^2(1-\tau^2)^2}\int d\sigma \left(\delta^{ij} I_i I_j + \delta_{ij} J^i J^j\right).$$

- However, our goal is to recover the $SL(2,\mathbb{C})$ algebra, but this is not easily understood from these rather complicated brackets.

- Performing the rotation (take τ to be purely imaginary)

$$S_i(\sigma) = \frac{1}{\xi(1-a^2\tau^2)}\left[I_i(\sigma) - a\delta_{ik}J^k(\sigma)\right]$$

$$B^i(\sigma) = \frac{i\tau}{\xi(1-a^2\tau^2)}\left[-ai\tau\delta^{ik}I_k(\sigma) - \frac{1}{i\tau}J^i(\sigma)\right],$$

we have ($C_\tau = \frac{a}{\lambda^2(1-a^2\tau^2)^2}$, $C'_\tau = \frac{(1+a^2\tau^2)}{2\lambda^2(1-a^2\tau^2)^2}$)

$$\{S_i(\sigma), S_j(\sigma')\} = \epsilon_{ij}{}^k S_k(\sigma)\delta(\sigma-\sigma') + C_\tau\delta_{ij}\delta'(\sigma-\sigma')$$

$$\{B^i(\sigma), B^j(\sigma')\} = \tau^2\epsilon^{ijk}S_k(\sigma)\delta(\sigma-\sigma') + \tau^2 C_\tau\delta^{ij}\delta'(\sigma-\sigma')$$

$$\{S_i(\sigma), B^j(\sigma')\} = \epsilon_{ki}{}^j B^k(\sigma)\delta(\sigma-\sigma') + C'_\tau\delta_i{}^j\delta'(\sigma-\sigma').$$

The Hamiltonian is rewritten as

$$H_\tau = \lambda^2\int d\sigma\Big[\left(1+a^2\tau^4\right)\delta^{ij}S_iS_j + \left(1+a^2\right)\delta_{ij}B^iB^j - 2a\left(1+\tau^2\right)\delta^i{}_jS_iB^j\Big].$$

- To make the Drinfeld double structure explicit we can let S_i be unchanged since they already span the $\mathfrak{su}(2)$ algebra, and transform the B^i generators as follows:

$$K^i(\sigma) = B^i(\sigma) - i\tau\epsilon^{i\ell 3}S_\ell(\sigma),$$

leading to

$$\{S_i(\sigma), S_j(\sigma')\} = \epsilon_{ij}{}^k S_k(\sigma)\delta(\sigma-\sigma') + C\delta_{ij}\delta'(\sigma-\sigma')$$
$$\{K^i(\sigma), K^j(\sigma')\} = i\tau f^{ij}{}_k K^k(\sigma)\delta(\sigma-\sigma') + C\tau^2(\delta^{ij} + \epsilon_p{}^{i3}\epsilon^{jp3})\delta'(\sigma-\sigma')$$
$$\{S_i(\sigma), K^j(\sigma')\} = \left[\epsilon_{ki}{}^j K^k(\sigma) + i\tau f^{jk}{}_i S_k(\sigma)\right]\delta(\sigma-\sigma') + \left(C'\delta_i^j + i\tau C\epsilon_i{}^{j3}\right)\delta'(\sigma-\sigma')$$

and Hamiltonian (in doubled notation $S_I \equiv (S_i, K^i)$)

$$H_\tau = \lambda^2 \int d\sigma\, S_I \left(\mathcal{M}_\tau\right)^{IJ} S_J,$$

with

$$\mathcal{M}_\tau = \begin{pmatrix} (1+a^2\tau^4)\delta^{ij} - \tau^2(1+a^2)\epsilon_p{}^{i3}\epsilon^{pj3} & i\tau(1+a^2)\epsilon_j{}^{i3} - a(1+\tau^2)\delta^i{}_j \\ i\tau(1+a^2)\epsilon_i{}^{j3} - a(1+\tau^2)\delta_i^j & (1+a^2)\delta_{ij} \end{pmatrix}.$$

- Introduce another imaginary parameter α in such a way to make the role of the subalgebras $\mathfrak{su}(2)(\mathbb{R})$ and $\mathfrak{sb}(2,\mathbb{C})(\mathbb{R})$ symmetric without modifying the dynamics
- Let us go back to the S and B generators and consider the following pair of Poisson brackets and Hamiltonian:

$$\{S_i(\sigma), S_j(\sigma')\} = \epsilon_{ij}{}^k S_k(\sigma)\delta(\sigma-\sigma') + \frac{a}{\lambda^2\left(1+a^2\tau^2\alpha^2\right)^2}\delta_{ij}\delta'(\sigma-\sigma')$$

$$\{B^i(\sigma), B^j(\sigma')\} = -\tau^2\alpha^2\epsilon^{ijk}S_k(\sigma)\delta(\sigma-\sigma') - \frac{a\tau^2\alpha^2}{\lambda^2\left(1+a^2\tau^2\alpha^2\right)^2}\delta^{ij}\delta'(\sigma-\sigma')$$

$$\{S_i(\sigma), B^j(\sigma')\} = \epsilon_{ki}{}^j B^k(\sigma)\delta(\sigma-\sigma') + \frac{1-a^2\tau^2\alpha^2}{2\lambda^2\left(1+a^2\tau^2\alpha^2\right)^2}\delta_i{}^j\delta'(\sigma-\sigma'),$$

$$H_{\tau,\alpha} = \lambda^2\int d\sigma\Big[\left(1+a^2\tau^4\alpha^4\right)\delta^{ij}S_iS_j + \left(1+a^2\right)\delta_{ij}B^iB^j - 2a\left(1-\tau^2\alpha^2\right)\delta^i{}_jS_iB^j\Big].$$

One can observe this is just the pair in S and B, under the mapping $\tau \to i\alpha\tau$, so the equations of motion following from it do not change.

- Define new generators as

$$U_i = i\alpha S_i, \quad V^i = \frac{1}{i\alpha} B^i - i\tau \epsilon^{i\ell 3} S_\ell .$$

The algebra satisfied by these new generators is given by

$$\{U_i(\sigma), U_j(\sigma')\} = i\alpha \epsilon_{ij}{}^k U_k(\sigma)\delta(\sigma-\sigma') - \frac{a\alpha^2}{\lambda^2\left(1+a^2\tau^2\alpha^2\right)^2}\delta_{ij}\delta'(\sigma-\sigma')$$

$$\{V^i(\sigma), V^j(\sigma')\} = i\tau f^{ij}{}_k V^k(\sigma)\delta(\sigma-\sigma') + \frac{a\tau^2}{\lambda^2\left(1+a^2\tau^2\alpha^2\right)^2}(\delta^{ij} + \epsilon_p{}^{i3}\epsilon^{jp3})\delta'(\sigma-\sigma')$$

$$\{U_i(\sigma), V^j(\sigma')\} = \left[i\alpha\epsilon_{ki}{}^j V^k(\sigma) + i\tau f^{jk}{}_i U_k(\sigma)\right]\delta(\sigma-\sigma') + \frac{1}{2\lambda^2\left(1+a^2\tau^2\alpha^2\right)^2}\left[\left(1-a^2\tau^2\alpha^2\right)\delta_i{}^j + 2a\, i\tau\, i\alpha\, \epsilon_i{}^{j3}\right]\delta'(\sigma-\sigma').$$

- $i\tau \to 0 \longrightarrow \mathfrak{c}_1 = \mathfrak{su}(2)(\mathbb{R})\dot{\oplus}\,\mathfrak{a}$ current algebra (original model)
- $i\alpha \to 0 \longrightarrow \mathfrak{c}_3 = \mathfrak{sb}(2,\mathbb{C})(\mathbb{R})\dot{\oplus}\,\mathfrak{a}$ current algebra.
- For all the other values of α and τ this algebra is isomorphic to $\mathfrak{c}_2 \simeq \mathfrak{sl}(2,\mathbb{C})(\mathbb{R})$ and upon suitable rescaling we obtain a two-parameter family of models all equivalent to the WZW model.

- The Hamiltonian can be rewritten in terms of U and V as

$$H_{\tau,\alpha} = \lambda^2 \int d\sigma\, U_I \left(\mathcal{M}_{\tau,\alpha}\right)^{IJ} U_J,$$

with

$$\mathcal{M}_{\tau,\alpha} = \begin{pmatrix} \frac{1+a^2\tau^4\alpha^4}{(i\alpha)^2}\delta^{ij} - \tau^2(1+a^2)\epsilon_p{}^{i3}\epsilon^{pj3} & i\tau i\alpha(1+a^2)\epsilon_j{}^{i3} - a(1-\tau^2\alpha^2)\delta^i{}_j \\ i\tau i\alpha(1+a^2)\epsilon_i{}^{j3} - a(1-\tau^2\alpha^2)\delta_i{}^j & (i\alpha)^2(1+a^2)\delta_{ij} \end{pmatrix}.$$

- U and V play a symmetric role $\longrightarrow$ we can perform a $O(3,3)$ transformation

$$\tilde{V}(\sigma) = U(\sigma), \quad \tilde{U}(\sigma) = V(\sigma)$$

Explicitly, under such a rotation we obtain the dual Hamiltonians

$$\tilde{H}_{\tau,\alpha} = \lambda^2 \int d\sigma \left[(m_{\tau,\alpha})^{ij} \tilde{V}_i \tilde{V}_j + (m_{\tau,\alpha})_{ij} \tilde{U}^i \tilde{U}^j + \tilde{V}_i \tilde{U}^j (m_{\tau,\alpha})^i{}_j + \tilde{V}_j \tilde{U}^i (m_{\tau,\alpha})_i{}^j \right],$$

and the dual Poisson algebras

$$\{\tilde{V}_i(\sigma), \tilde{V}_j(\sigma')\} = i\alpha \epsilon_{ij}{}^k \tilde{V}_k(\sigma)\delta(\sigma - \sigma') - \frac{a\alpha^2}{\lambda^2 (1 + a^2\tau^2\alpha^2)^2} \delta_{ij} \delta'(\sigma - \sigma')$$

$$\{\tilde{U}^i(\sigma), \tilde{U}^j(\sigma')\} = i\tau f^{ij}{}_k \tilde{U}^k(\sigma)\delta(\sigma - \sigma') + \frac{a\tau^2}{\lambda^2 (1 + a^2\tau^2\alpha^2)^2} (\delta^{ij} + \epsilon_p{}^{i3} \epsilon^{jp3}) \delta'(\sigma - \sigma')$$

$$\{\tilde{V}_i(\sigma), \tilde{U}^j(\sigma')\} = \left[i\alpha \epsilon_{ki}{}^j \tilde{U}^k(\sigma) + i\tau f^{jk}{}_i \tilde{V}_k(\sigma) \right] \delta(\sigma - \sigma')$$
$$+ \frac{1}{2\lambda^2 (1 + a^2\tau^2\alpha^2)^2} \left[\left(1 - a^2\tau^2\alpha^2\right) \delta_i{}^j + 2a\, i\tau\, i\alpha\, \epsilon_i{}^{j3} \right] \delta'(\sigma - \sigma')$$

- The new family of models DWZW, has target configuration space the group manifold of $SB(2,\mathbb{C})$, spanned by the fields $\tilde{V}_i$, while momenta $\tilde{U}^i$ span the fibers of the target phase space.

Conclusions and future perspectives

- Description of T-duality properties of the $SU(2)$ WZW model by means of an equivalent one-parameter deformation reformulation
- Introduction of a two-parameter family of dual models with target configuration space $SB(2,\mathbb{C})$
- Duality is of Poisson-Lie type

- It would be relevant to define a natural dual model on $SB(2,\mathbb{C})$
- Formulation of a doubled theory on the $SL(2,\mathbb{C})$ group manifold
- Quantization of the interpolating model
- New CFTs?