κ-deformed special relativity

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COST Action CA18108

Quantum Gravity in the Multi Messenger Approach (QGMM)

Leader: Jose Manuel Carmona (U. Zaragoza)
Vice-leader: Giovanni Amelino-Camelia (Federico II U. Naples)

WG2: Phenomenology

Leader: Flavio Mercati (Federico II U. Naples)
Vice-leader: Stefano Liberati (SISSA)

2+1D QG is a topological theory (no local propagating degrees of freedom).

Coupling point particles (*i.e.* conical defects) leads to deformed relativistic kinematics ($SL(2,\mathbb{R})$-valued momenta, deformed Poisson structure, ...) [Matschull–Welling, CQG 1998]

Certain formulation of QG+scalar [Freidel–Livine PRL 2006]:

$$\int \mathcal{D}[g]\mathcal{D}[\phi]e^{iS[g,\phi]} = \int \mathcal{D}[\phi]e^{iS_{eff}[\phi]}$$

$S_{eff}[\phi]$ = nonlocal theory (infinite derivatives of ϕ),

can be reformulated as ordinary QFT on a *noncommutative spacetime:*

$$[x^\mu, x^\nu] \propto L_p\, x^\rho$$

symmetric under *Hopf algebra* / *Quantum Group* deformation of $ISO(2,1)$:

$$x'^\mu = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1\,,$$

s.t.

$$[x'^\mu, x'^\nu] \propto L_p\, x'^\rho\,, \qquad \Rightarrow \qquad \begin{cases} [\Lambda^\mu{}_\nu, \Lambda^\rho{}_\sigma] \\ [\Lambda^\mu{}_\nu, a^\rho] & \neq 0 \\ [a^\mu, a^\nu] \end{cases}$$

Quantum groups

On top of the ordinary Lie group structures...

$$\begin{cases} \Delta[\Lambda^\mu{}_\nu] = \Lambda^\mu{}_\rho \otimes \Lambda^\rho{}_\nu \,, \\ \Delta[a^\mu] = \Lambda^\mu{}_\rho \otimes a^\rho + a^\mu \otimes 1 \,, \end{cases} \text{(composition)}$$

$$\begin{cases} S[\Lambda^\mu{}_\nu] = (\Lambda^{-1})^\mu{}_\nu \,, \\ S[a^\mu] = -(\Lambda^{-1})^\mu{}_\rho \, a^\rho \,, \end{cases} \text{(inverse)}$$

$$\begin{cases} \epsilon[\Lambda^\mu{}_\nu] = \delta^\mu{}_\rho \,, \\ \epsilon[a^\mu] = 0 \,, \end{cases} \text{(identity)}$$

...we have an additional layer: a noncommutative algebra of functions over the group manifold

$$\begin{cases} [\Lambda^\mu{}_\nu, \Lambda^\rho{}_\sigma] \\ [\Lambda^\mu{}_\nu, a^\rho] \\ [a^\mu, a^\nu] \end{cases} \neq 0 \,.$$

Group axioms: Δ *co*associative and $1 \otimes S \circ \Delta = \epsilon = S \otimes 1 \circ \Delta$.

Δ, S and ϵ must be *homomorphisms* w.r.t. $[\cdot, \cdot]$.

$[\cdot, \cdot]$ can be seen as the quantization of a Poisson bracket $\{\cdot, \cdot\}$.
Quantum groups are quantization of Poisson-Lie groups.

The infinitesimal version of a Poisson-Lie group is a *Lie bialgebra.*

This is a Lie algebra:

$$[X_i, X_j] = c_{ij}{}^k X_k\,, \qquad c_{ij}{}^l c_{lk}{}^m + c_{ki}{}^l c_{lj}{}^m + c_{jk}{}^l c_{li}{}^m = 0\,,$$

whose dual $\langle X_i, x^j \rangle = \delta^i{}_j$ is another Lie algebra:

$$\{x^i, x^j\} = f^{ij}{}_k x^k\,, \qquad f^{jk}{}_i f^{lm}{}_j + f^{jl}{}_i f^{mk}{}_j + f^{jm}{}_i f^{kl}{}_j = 0\,,$$

with compatibility conditions between their structure constants:

$$f^{ab}{}_k c_{ij}{}^k = f^{ak}{}_i c_{kj}{}^b + f^{kb}{}_i c_{kj}{}^a + f^{ak}{}_j c_{ik}{}^b + f^{kb}{}_j c_{ik}{}^a\,,$$

equivalent to coassociativity of Δ. *Cocommutator* (dual map to $\{\cdot,\cdot\}$):

$$\delta(X^i) = f^{ij}{}_k X_j \wedge X_k\,,$$

the compatibility condition states that δ is a 1-cocycle:

$$\delta([X_i, X_j]) = [X_i, X_j \otimes 1 + 1 \otimes X_j] + [X_i \otimes 1 + 1 \otimes X_i, X_j].$$

Which Quantum Group for 3+1D?

[FM–Sergola, NPB 2018]

Dimensionless version of (A)dS/Poincaré algebra [so(4,1)-so(3,2)-iso(3,1)]:

$$[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\sigma\mu} M^{\rho\nu} + \eta^{\sigma\nu} M^{\rho\mu},$$

$$[Q^{\mu}, M^{\rho\sigma}] = \eta^{\mu\rho} Q^{\sigma} - \eta^{\mu\sigma} Q^{\rho}, \qquad [Q^{\mu}, Q^{\nu}] = \lambda M^{\mu\nu}.$$

where $P_\mu = \sqrt{|\Lambda|}\, Q_\mu$ and $\lambda = \mathrm{sgn}(\Lambda)$.

most generic cocommutator:

$$\delta(Q_\mu) = a_\mu{}^{\rho\sigma} Q_\rho \wedge Q_\sigma + b_\mu{}^{\rho\sigma\gamma} Q_\rho \wedge M_{\sigma\gamma} + c_\mu{}^{\rho\sigma\gamma\delta} M_{\rho\sigma} \wedge M_{\gamma\delta},$$
$$\delta(M_{\mu\nu}) = d_{\mu\nu}{}^{\rho\sigma} Q_\rho \wedge Q_\sigma + e_{\mu\nu}{}^{\rho\sigma\gamma} Q_\rho \wedge M_{\sigma\gamma} + f_{\mu\nu}{}^{\rho\sigma\gamma\delta} M_{\rho\sigma} \wedge M_{\gamma\delta},$$

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where $P_\mu = \sqrt{|\Lambda|}\, Q_\mu$ and $\lambda = \text{sign}(\Lambda)$.

most generic cocommutator:

$$\delta(Q) = a\, Q \wedge Q + b\, Q \wedge M + c\, M \wedge M,$$
$$\delta(M) = d\, Q \wedge Q + e\, Q \wedge M + f\, M \wedge M,$$

Constraints imposed by Dimensional Analysis

$$\delta(Q) = a\, Q \wedge Q + b\, Q \wedge M + c\, M \wedge M,$$
$$\delta(M) = d\, Q \wedge Q + e\, Q \wedge M + f\; M \wedge M,$$

$a, b \cdots =$ functions of the two physical scales in the theory:
$L_p \sim 10^{-35} m$ and $\Lambda \sim 10^{-52} m^{-2}$.

$a, b \ldots$ are dimensionless. Can only be functions of $q = L_p \sqrt{|\Lambda|} \sim 10^{-61}$.

Assuming that $\lim_{q \to 0} a(q) = 0, \ldots$ and that they are analytic:

$$a(q) = q\, a^{(1)} + \frac{1}{2} q^2\, a^{(2)} + \mathcal{O}(q^3)\,,$$
$$\vdots$$
$$f(q) = q\, f^{(1)} + \frac{1}{2} q^2\, f_\mu^{(2)} + \mathcal{O}(q^3)\,,$$

Constraints imposed by Dimensional Analysis

reintroducing the dimensional translation generators $P_\mu = \sqrt{|\Lambda|}\, Q_\mu$:

$$\delta(P_\mu) = L_p \left(a^{(1)} P \wedge P + \overbrace{\sqrt{|\Lambda|}\, b^{(1)} P \wedge M + |\Lambda|\, c^{(1)} M \wedge M}^{\text{suppressed by } \Lambda} \right) + \mathcal{O}(q^2),$$

$$\delta(M_{\mu\nu}) = L_p \left(\underbrace{\frac{d^{(1)}}{\sqrt{|\Lambda|}}}_{\xrightarrow[\Lambda\to 0]{}\infty} P \wedge P + e^{(1)} P \wedge M + \underbrace{\sqrt{|\Lambda|}\, f^{(1)} M \wedge M}_{\text{suppressed by } \Lambda} \right) + \mathcal{O}(q^2).$$

Constraints imposed by Dimensional Analysis

[Zakrzewski, CMP 1997]: if algebra is Poincaré there are 23 independent solutions of the bialgebra axioms.

However, if we only keep the blue terms, we are left with 7-parameter family:

$$\delta(P_\mu) = v^\nu \, P_\nu \wedge P_\mu \, ,$$
$$\delta(M_{\mu\nu}) = v_{[\mu} \, M_{\nu]\rho} \wedge P^\rho + v^\rho P_\rho \wedge M[u_1, u_2, u_3] \, .$$

$M[u_1, u_2, u_3]$: generic element of stabilizer of $v^\rho P_\rho$. "Reshetikhin twist" [Daszkiewicz, IJMPA 2008]

v^μ are 4 $\mathcal{O}(L_p)$ parameters. 'vectorial' generalization of κ-Poincaré.

Imposing 'manifest' spatial isotropy: $u^i = 0, \quad v^\mu = \frac{1}{\kappa}\delta^\mu{}_0$.

$$\Downarrow$$

'timelike' κ-Poincaré [Lukierski–Nowicki–Ruegg–Tolstoy, PLB 1991].

κ-Poincaré group

[Lukierski–Ruegg, PLB 1994]

$$\begin{cases} \Delta[\Lambda^\mu{}_\nu] = \Lambda^\mu{}_\rho \otimes \Lambda^\rho{}_\nu \,, \\ \Delta[a^\mu] = \Lambda^\mu{}_\rho \otimes a^\rho + a^\mu \otimes 1 \,, \end{cases}$$

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$$\begin{cases} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma} \\ [\Lambda^\mu{}_\nu, \Lambda^\rho{}_\sigma] = 0 \\ [\Lambda^\mu{}_\nu, a^\rho] = i \Big[(\Lambda^\mu{}_\sigma v^\sigma - v^\mu) \, \Lambda^\rho{}_\nu \\ \qquad\qquad + (\Lambda^\sigma{}_\nu v_\sigma - v_\nu) \, \eta^{\mu\rho} \Big] \\ [a^\mu, a^\nu] = i \, (v^\mu \, a^\nu - v^\nu \, a^\mu) \,, \end{cases}$$

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The translations close a subalgebra $[a,a] = a$ (‘coisotropy’ w.r.t. $SO(3,1)$, [Ballesteros–Meusburger–Naranjo, J. Phys. A, 2017]),

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The translations close a subalgebra $[a, a] = a$ ('coisotropy' w.r.t. $SO(3, 1)$, [Ballesteros–Meusburger–Naranjo, J. Phys. A, 2017]),

Lorentz transformations close a subalgebra, and the translations act on them, $[\Lambda, a] = \Lambda$ ('coreductivity', [Ballesteros–FM–Gutierrez, arXiv:1909.01000]).

κ-Minkowski noncommutative spacetime

Coisotropy+coreductivity $[a, a] = a$, $[\Lambda, a] = \Lambda \to \delta\Lambda\delta a \geq \langle\Lambda\rangle/2$

$\Downarrow$

quantum homogeneous space from translation subalgebra a^μ

[Lizzi–Manfredonia–FM-Poulain, PRD 2019]

$$[x^\mu, x^\nu] = i\,(v^\mu\, x^\nu - v^\nu\, x^\mu)\ , \qquad x^\mu \in \mathcal{A}\,, \quad \text{“}\kappa\text{-Minkowski”}$$

on which the κ-Poincaré group acts covariantly:

$$x'^\mu = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1 \quad \Rightarrow \quad [x'^\mu, x'^\nu] = i\left(v^\mu\, x'^\nu - v^\nu\, x'^\mu\right)\ .$$

Curved momentum spaces associated to κ-Minkowski

The algebra $[x^\mu, x^\nu] = i\,(v^\mu\, x^\nu - v^\nu\, x^\mu)$ admits a faithful 5D representation:

$$\rho(x^0) = i \begin{pmatrix} 0 & 0 & 0 & 0 & v^0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ v^0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \rho(x^1) = i \begin{pmatrix} 0 & v^0 & 0 & 0 & v^1 \\ v^0 & 0 & 0 & 0 & v^0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ v^1 & -v^0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho(x^2) = i \begin{pmatrix} 0 & 0 & v^0 & 0 & v^2 \\ 0 & 0 & 0 & 0 & 0 \\ v^0 & 0 & 0 & 0 & v^0 \\ 0 & 0 & 0 & 0 & 0 \\ v^2 & 0 & -v^0 & 0 & 0 \end{pmatrix} \qquad \rho(x^3) = i \begin{pmatrix} 0 & 0 & 0 & v^0 & v^3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ v^0 & 0 & 0 & 0 & v^0 \\ v^3 & 0 & 0 & -v^0 & 0 \end{pmatrix}$$

Curved momentum spaces associated to κ-Minkowski

...as linear combinations of $so(4,1)$ matrices:

$$\begin{aligned}
\rho(x^0) &= i\,v^0\,M_{04}\,,\\
\rho(x^1) &= iv^0(M_{01}+M_{41})+i\,v^1\,M_{04}\,,\\
\rho(x^2) &= i\,v^0(M_{02}+M_{42})+i\,v^2\,M_{04}\,,\\
\rho(x^3) &= i\,v^0(M_{03}+M_{43})+i\,v^3\,M_{04}\,.
\end{aligned}$$

Exponentialization generates the 4D Lie group $SB(3)$:

$$G(p_\mu) = e^{i\,p_1\,\rho(x^1)}\,e^{i\,p_2\,\rho(x^2)}\,e^{i\,p_3\,\rho(x^3)}\,e^{i\,p_0\,\rho(x^0)}$$

p_μ = coordinates on the group = momenta!

[Kowalski-Glikman, Ballesteros–Herranz, Ballesteros–Gubitosi ...]

Curved momentum spaces associated to κ-Minkowski

Starting from a fiducial point X^A in 5D ambient space, the group action generates a 4D manifold:

$$G(p_\mu)^A{}_B X^B = S^A(p^\mu)\,,$$

$$S^A(p^\mu) = \text{embedding coordinates} : \mathbb{R}^4 \to \mathcal{M} \subset \mathbb{R}^{4+1}$$

since $G(p_\mu)^A{}_B \in SO(4,1)$:

$$\eta_{AB} S^A(p^\mu) S^B(p^\mu) = \textit{const.}\,, \quad \eta_{AB} = \text{diag}(-1,1,1,1,1)\,.$$

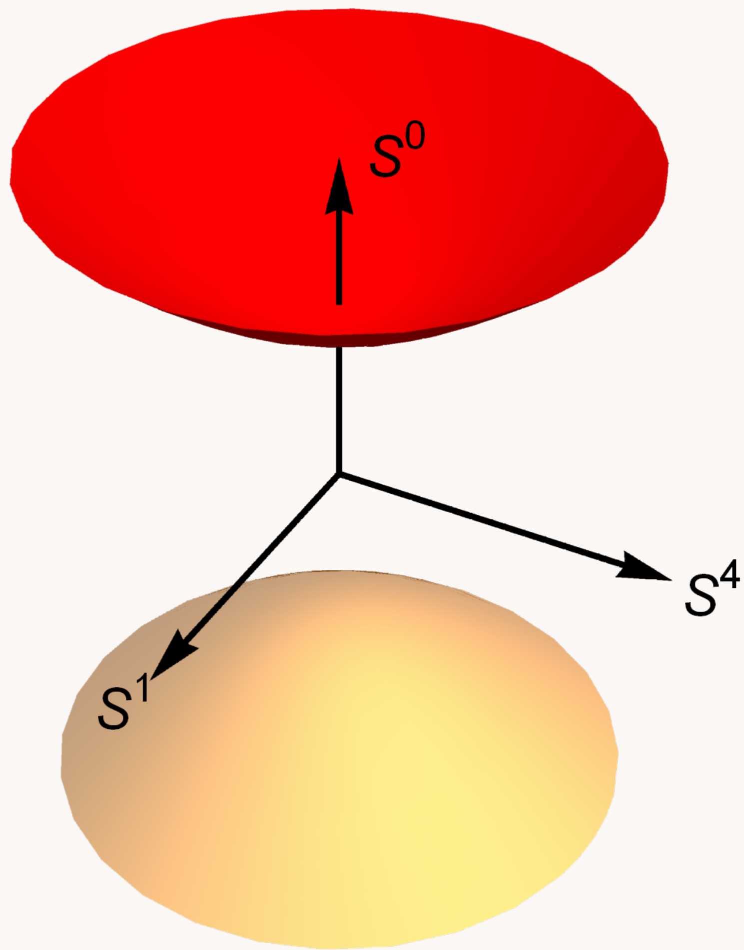
Depending on choice of X^A, we generate different momentum spaces.

[Lizzi–Manfredonia–FM, arXiv:1910.xxxx]

If X^A is timelike, *e.g.* $X^A = (1,0,0,0,0)$

$\Downarrow$

S^A embed (half of) a two-sheeted hyperboloid in 5D Minkowski space:

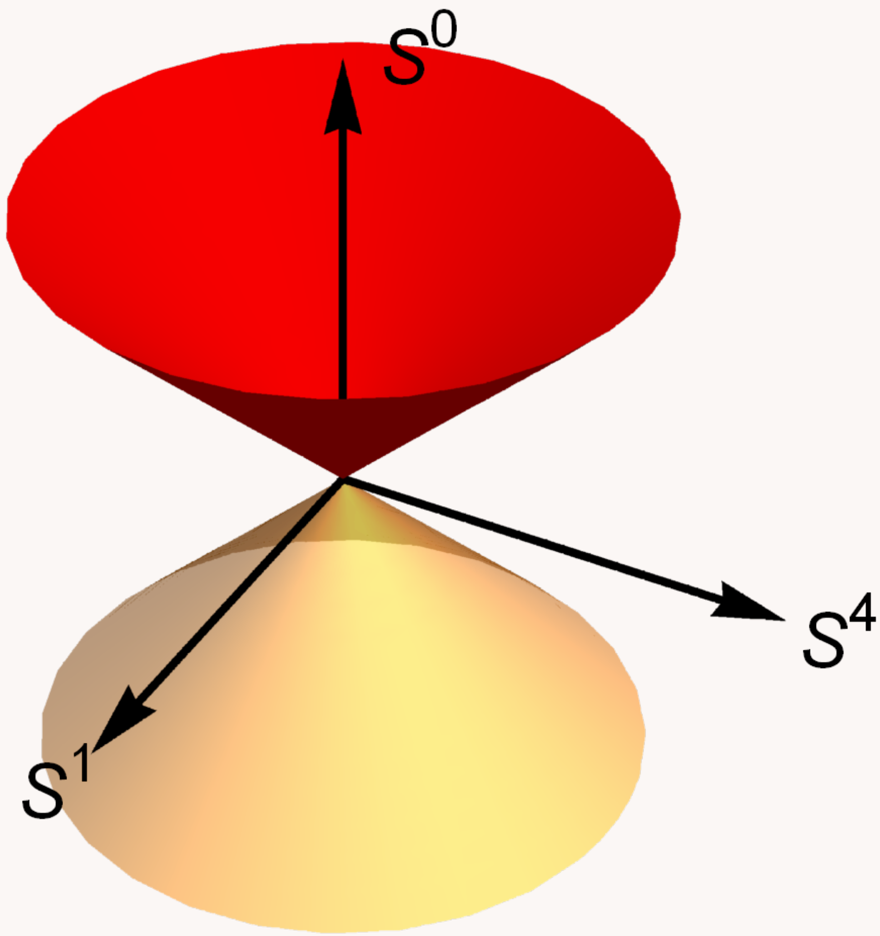


Induced metric is *Euclidean.*

If X^A is lightlike, *e.g.* $X^A = (\pm 1, 0, 0, 0, 1)$

$\Downarrow$

S^A embed a (future or past) lightcone in 5D Minkowski space:

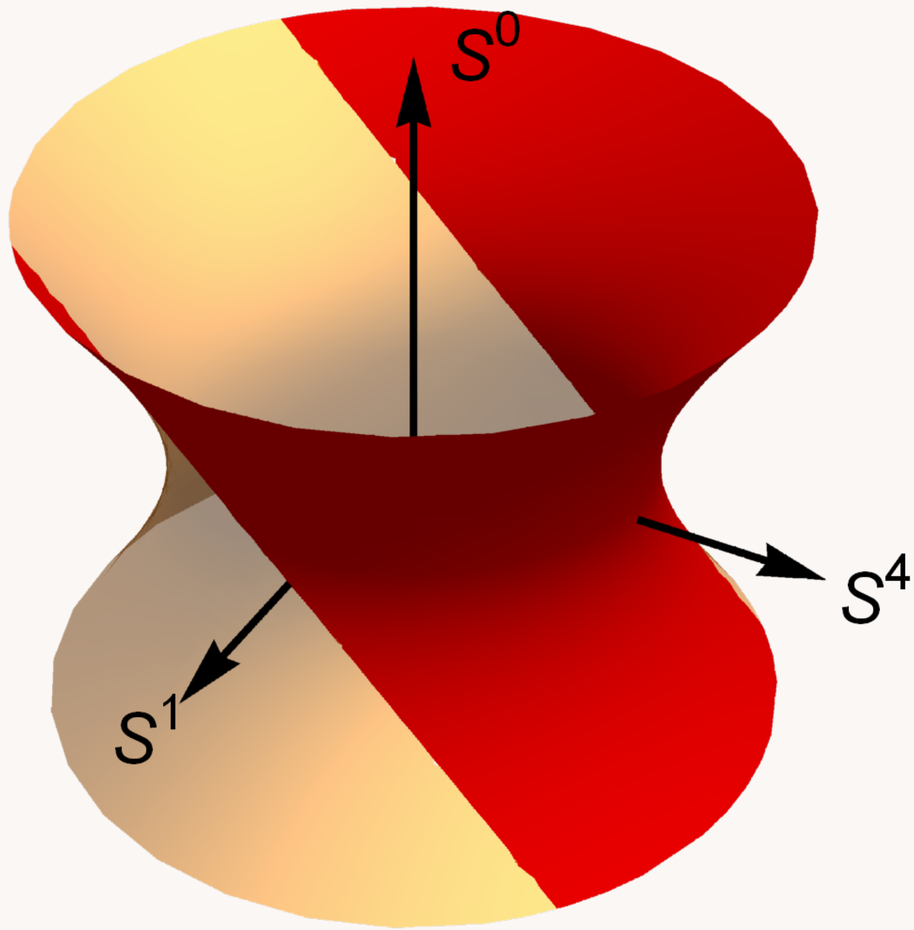


Induced metric is *degenerate*, $\det g = 0$.

If X^A is spacelike, *e.g.* $X^A = (0, 0, 0, 0, 1)$,
we have the additional condition $S^0 + S^4 > 0$

$\Downarrow$

S^A embed half of a one-sheeted hyperboloid in 5D Minkowski space:



Induced metric is *Lorentzian.*

A few special choices of X^A give rise to degenerate ($D < 4$) group orbits.

The *timelike*, *lightlike* and *spacelike* choices of X^A correspond to:

$$\begin{cases} \Delta[\Lambda^\mu{}_\nu] = \Lambda^\mu{}_\rho \otimes \Lambda^\rho{}_\nu \,, \\ \Delta[a^\mu] = \Lambda^\mu{}_\rho \otimes a^\rho + a^\mu \otimes 1 \,, \end{cases}$$

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with, respectively:

$\eta^{\mu\nu} = \text{diag}(1,1,1,1)$, $\begin{cases} \eta^{\mu\nu} = \text{diag}(0,1,1,1) \\ \eta_{\mu\nu} = \text{diag}(1,0,0,0) \end{cases}$, $\eta^{\mu\nu} = \text{diag}(-1,1,1,1)$,

Euclidean group ISO(4) — Carroll group — Poincaré group ISO(3,1)

can be obtained as three Inönü–Wigner contractions of $SO(4,1)$.
([Lizzi–Manfredonia–FM, 1910.xxxx])

de Sitter momentum space

Focusing on the Lorentzian case: $X^A = (0,0,0,0,1)$:

$$G(p_\mu)^A{}_B X^B = \left(\sinh\frac{p_0}{\kappa} + \frac{e^{\frac{p_0}{\kappa}}\|\vec{p}\|^2}{2\kappa^2} \, , \, \frac{e^{\frac{p_0}{\kappa}}\vec{p}}{\kappa} \, , \, \cosh\frac{p_0}{\kappa} - \frac{e^{\frac{p_0}{\kappa}}\|\vec{p}\|^2}{2\kappa^2} \right) ,$$

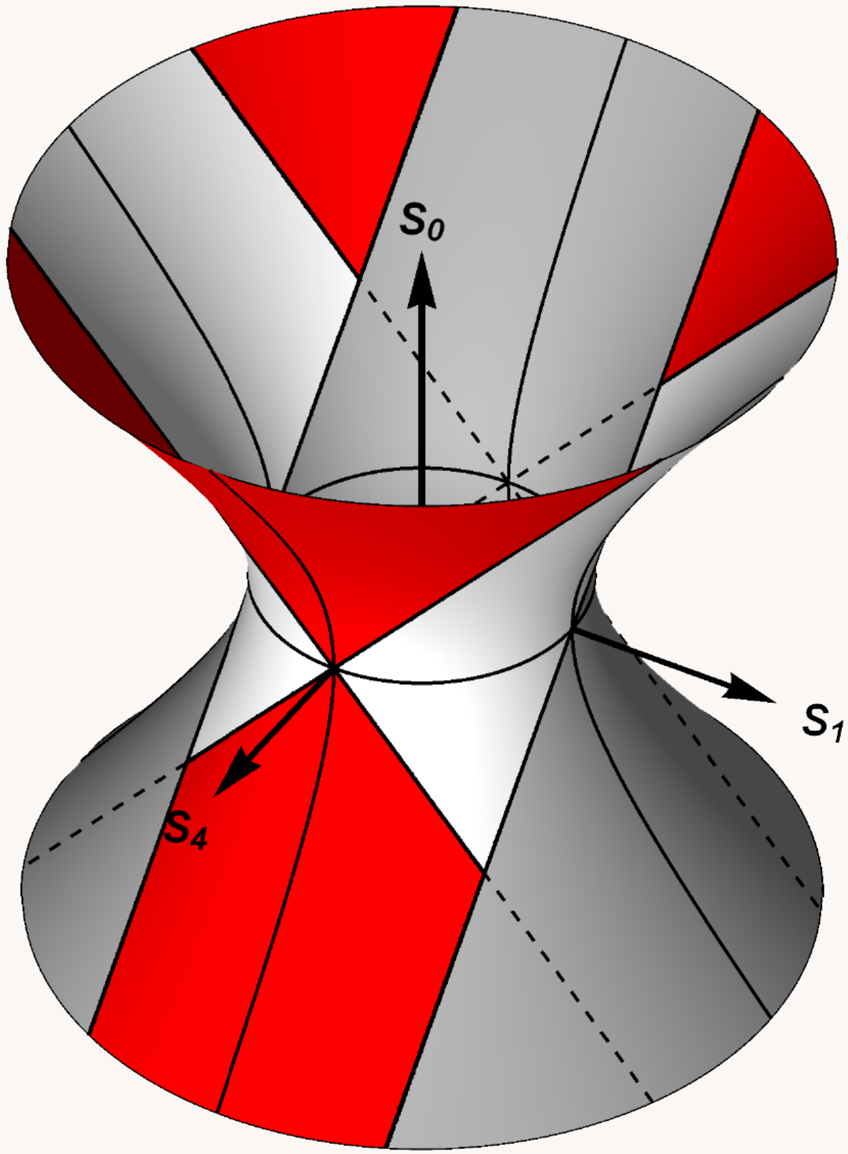
this is an embedding of 4D de Sitter spacetime in 5D Minkowski space:

$$-S_0^2 + S_1^2 + S_2^2 + S_3^2 + S_4^2 = 1 \, ,$$

or, actually, *half* of it:

$$S_0 + S_4 = e^{\frac{p_0}{\kappa}} > 0$$

[Kowalski-Glikman–Nowak , CQG 2003]



s_0
s_1
s_4

Lorentz transformations of momenta

Poincaré transformation of κ-Minkowski coordinates:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} \otimes x^{\nu} + a^{\mu} \otimes 1$$

transformation of ordered plane wave:

$$: e^{ip_\mu x'^\mu} :=: e^{ip'_\mu[\Lambda^\mu{}_\nu]\otimes x^\mu} :: e^{ip_\mu a^\mu \otimes 1} :$$

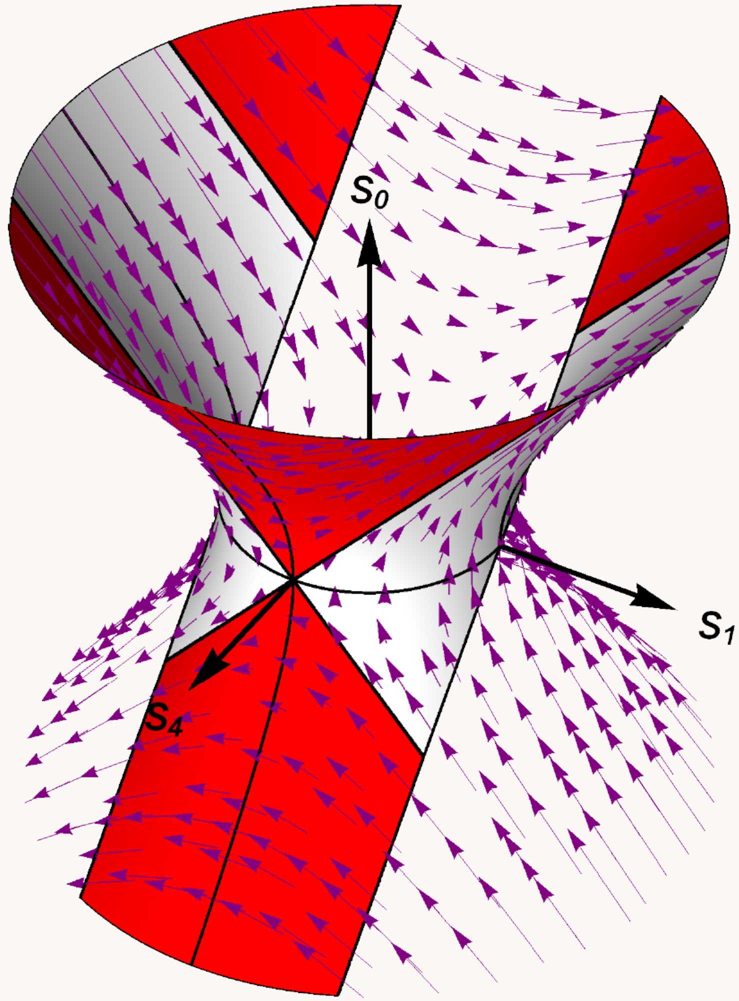
where $p'_\mu[\Lambda^\mu{}_\nu]$ = complicated nonlinear function of $\Lambda^\mu{}_\nu$ and p_μ
[FM–Sergola PRD 2018]

On the embedding coordinates they act linearly:

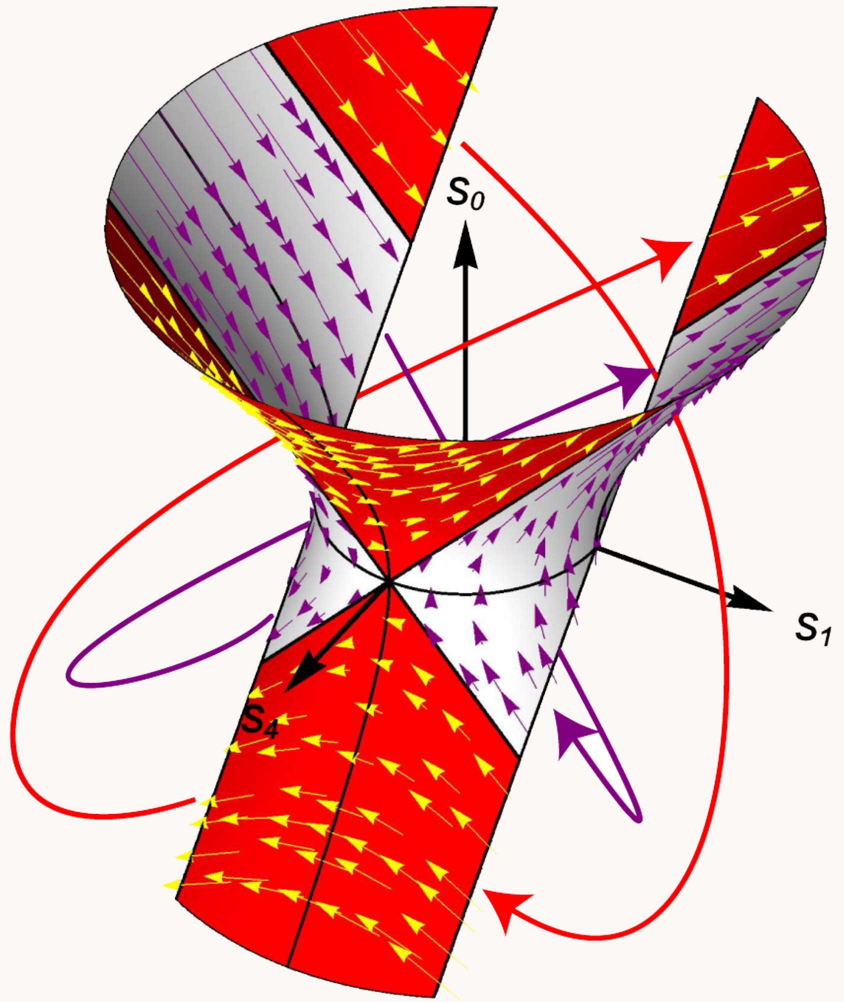
$$S'_\mu = \Lambda^\nu{}_\mu S_\nu \,, \qquad S'_4 = S_4 \;\Rightarrow\; \text{invariant! Mass Casimir?}$$

but the condition $S_0 + S_4 > 0$ is not invariant under Lorentz transformations!

[Freidel–Kowalski-Glikman–Nowak, IJMPA 2008]


S_0
S_1
S_4

A solution: Elliptic de Sitter topology: $dS/_{\sim}$, $\quad S^A \sim -S^A$,



[FM–Sergola, PRD 2018]

$$S_4 = \cosh\frac{p_0}{\kappa} - \frac{e^{\frac{p_0}{\kappa}}\|\vec{p}\|^2}{2\kappa^2} = 1 + \frac{1}{2\kappa}\left(p_0^2 - \|\vec{p}\|^2 + \mathcal{O}(\kappa^{-1})\right)$$

Lorentz-invariant function of momentum with the right $\kappa \to \infty$ limit.

It is tempting to interpret it as mass Casimir and do things like solve $S_4 =$ *const.* for p_0, and claim $\frac{\partial p_0}{\partial p_i}$ is the group velocity of something (modified dispersion relations, superluminal propagation etc...).

But if I use different coordinates on momentum space (a freedom that Hopf algebras warrant me), I can get any kind of dispersion relation.

I want to do something more refined: study **noncommutative field theory** on κ-Minkowski, and the **microcausality relations** it involves, to see if there is superluminal propagation.

Free quantum κ-Klein–Gordon field

From now on $v^\mu = \frac{1}{\kappa}\delta^\mu{}_0$.

Scalar field in time-to-the-right-ordered Fourier transform:

$$\phi(x) = \int d^4k \sqrt{-g(k)}\, \phi_r(k)\, e^{ik_i x^i} e^{ik_0 x^0}\,,$$

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$$\phi(x) = \int d^4k \underbrace{\sqrt{-g(k)}}_{\text{dS metric in comoving coordinates}} \phi_r(k) \overbrace{e^{ik_i x^i} e^{ik_0 x^0}}^{\text{time-to-the-right ordering}},$$

Changing ordering corresponds to a diffeomorphism in momentum space:

$$\underbrace{e^{ik_0x^0}e^{ik_ix^i}}_{\text{time-to-the-left}} = \underbrace{e^{ie^{\frac{k_0}{\kappa}}k_ix^i}e^{ik_0x^0}}_{\text{time-to-the-right}}, \qquad \underbrace{e^{ik_0x^0+ik_ix^i}}_{\text{Weyl ordering}} = \underbrace{e^{i\left(\frac{e^{k_0/\kappa}-1}{k_0/\kappa}\right)k_ix^i}e^{ik_0x^0}}_{\text{time-to-the-right}},$$

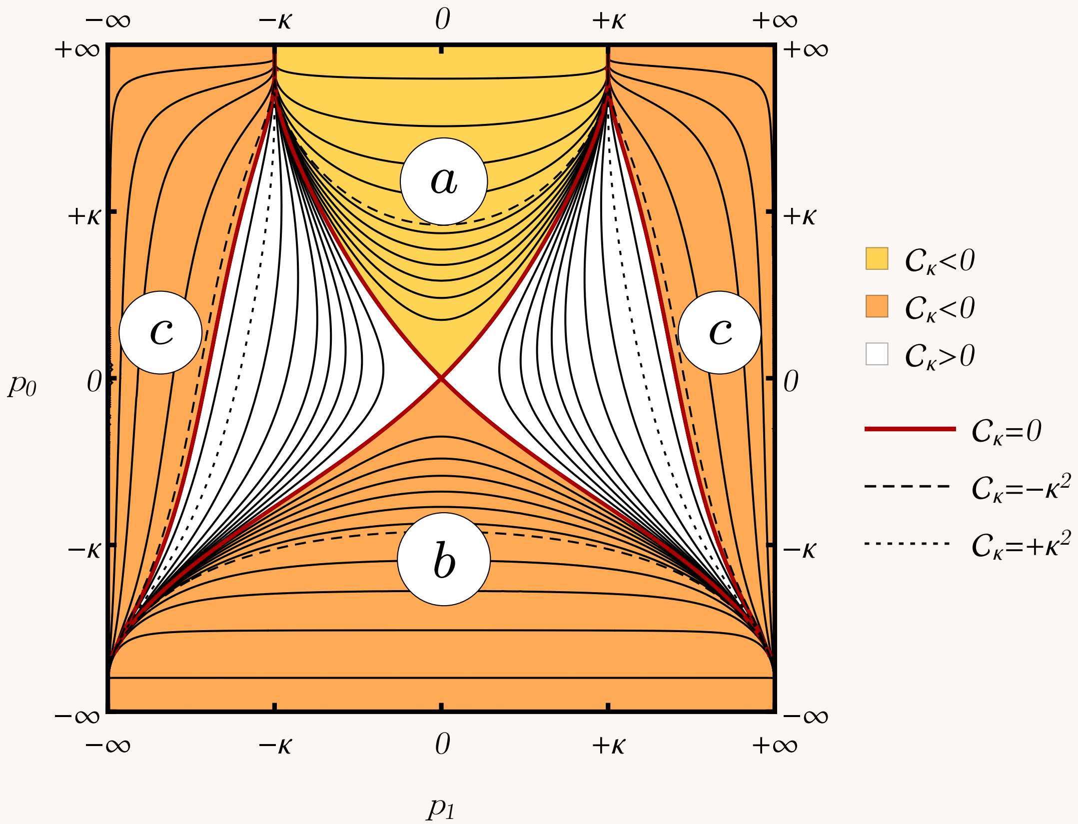
$$\begin{aligned}
\phi(x) &= \int_{\mathbb{R}^4} d^4k\,\sqrt{g(k)}\,\phi_r(k)\,e^{ik_ix^i}e^{ik_0x^0} \\
&= \int_{\mathbb{R}^4} d^4q\,\sqrt{g'(q)}\,\phi_l(q)\,e^{iq_0x^0}e^{iq_ix^i} \\
&= \int_{\mathbb{R}^4} d^4p\,\sqrt{g''(p)}\,\phi_w(p)\,e^{ip_0x^0+ip_ix^i}\,.
\end{aligned}$$

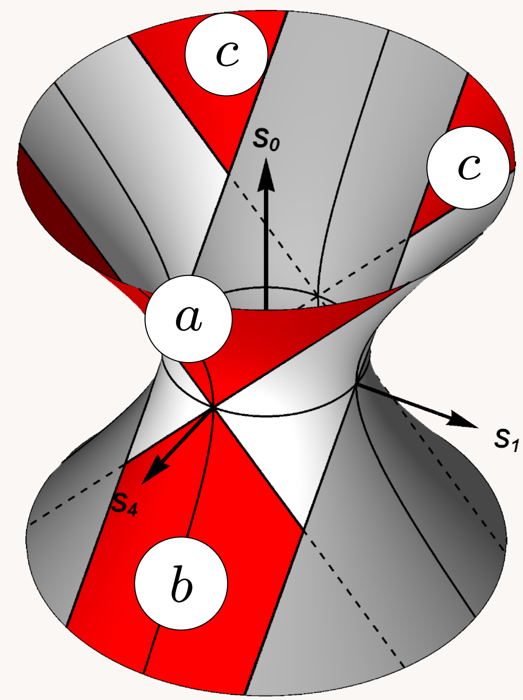
The Fourier transform $\tilde{\phi}_r(k)$ transforms, under diffeomorphisms of momentum space, like a *scalar field* on dS momentum space.

The κ-Klein–Gordon equation

S_4 is locally, not globally Lorentz-invariant on Elliptic dS.

Good mass Casimir: $\mathcal{C}_\kappa = (\kappa^2 - S_4^2) \sim -p_0^2 + |\vec{p}|^2 + \frac{p_0|\vec{p}|^2}{\kappa} + \mathcal{O}(\kappa^{-2})$





$|S_4| > \kappa$ in the mass shell (red region)

and $|S_4| < \kappa$ in the tachionic mass shell (white region).

Complex κ-Klein–Gordon equation:

$$\mathcal{C}_\kappa \triangleright \phi = -m^2\,\phi\,, \qquad \mathcal{C}_\kappa \triangleright \phi^\dagger = -m^2\,\phi^\dagger\,,$$

general solution (invariant under momentum-space diffeomorphisms):

$$\boxed{\phi(x) = \frac{1}{2m}\int d^4p\sqrt{-g(p)}\sqrt{-g^{\mu\nu}(p)\frac{\partial \mathcal{C}_\kappa}{\partial p^\mu}\frac{\partial \mathcal{C}_\kappa}{\partial p^\nu}}\,\delta\left(\mathcal{C}_\kappa(p)+m^2\right)\,\phi_r(p)\,\,:e^{ip_\mu x^\mu}:}$$

explicitly:

$$\phi(x) = \int_{|\vec{p}|<\kappa}\frac{d^3p\,e^{\frac{3\omega^+(|\vec{p}|)}{\kappa}}}{2\sqrt{m^2+|\vec{p}|^2}}\,\phi_a(\vec{p})\,e^{i\vec{p}\cdot\vec{x}}e^{i\omega^+(|\vec{p}|)x^0}+$$

$$\int_{\mathbb{R}^3}\frac{d^3p\,e^{\frac{3\omega^-(|\vec{p}|)}{\kappa}}}{2\sqrt{m^2+|\vec{p}|^2}}\,\phi_b(\vec{p})\,e^{i\vec{p}\cdot\vec{x}}e^{i\omega^-(|\vec{p}|)x^0}+$$

$$\int_{|\vec{p}|>\kappa}\frac{d^3p\,e^{\frac{3\omega^+(|\vec{p}|)}{\kappa}}}{2\sqrt{m^2+|\vec{p}|^2}}\,\phi_c(\vec{p})\,e^{i\vec{p}\cdot\vec{x}}e^{i\omega^+(|\vec{p}|)x^0}\,.$$

Quantization

Simply promote $\phi \in \mathcal{A}$ to an element of $\mathcal{A} \otimes \mathcal{H}$:

$$\hat{\phi}(x) = \int_{|\vec{p}|<\kappa} \frac{d^3p\, e^{\frac{3\omega^+(|\vec{p}|)}{\kappa}}}{2\sqrt{m^2+|\vec{p}|^2}}\, e^{i\vec{p}\cdot\vec{x}} e^{i\omega^+(|\vec{p}|)x^0} \otimes \hat{a}(\vec{p}) +$$

$$\int_{\mathbb{R}^3} \frac{d^3p\, e^{\frac{3\omega^-(|\vec{p}|)}{\kappa}}}{2\sqrt{m^2+|\vec{p}|^2}}\, e^{i\vec{p}\cdot\vec{x}} e^{i\omega^-(|\vec{p}|)x^0} \otimes \hat{b}^\dagger(\vec{p}) +$$

$$\int_{|\vec{p}|>\kappa} \frac{d^3p\, e^{\frac{3\omega^+(|\vec{p}|)}{\kappa}}}{2\sqrt{m^2+|\vec{p}|^2}}\, e^{i\vec{p}\cdot\vec{x}} e^{i\omega^+(|\vec{p}|)x^0} \otimes \hat{c}^\dagger(\vec{p})\,.$$

But what are the quantization rules?

Cannot use canonical quantization/equal-time commmutation relations:

$$[\dot{\phi}(0,\vec{x}),\phi(0,\vec{y})] = -i\delta^3(\vec{x}-\vec{y}).$$

Hamiltonian formalism ill-defined:

$$[x^0, x^i] = \frac{i}{\kappa}x^i \qquad \Rightarrow \qquad \delta x^0 \delta x^i \geq \frac{\langle x^i \rangle}{2\kappa}$$

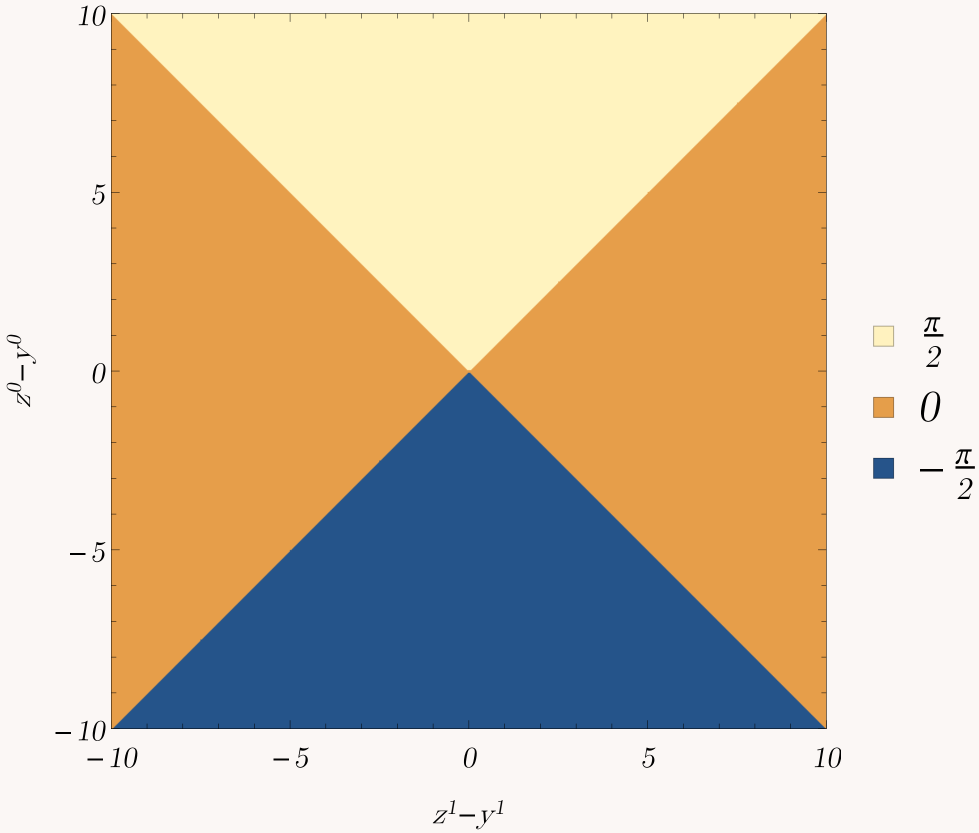
a fixed-x^0 slice has $\delta x^0 = 0 \Rightarrow$ either $\delta x^i = \infty$ or $\langle x^i \rangle = 0$.

Need a spacetime covariant notion of canonical commutators.

Pauli–Jordan function

In commutative Minkowski QFT:

$$[\hat{\phi}^\dagger(x), \hat{\phi}(y)] = i\,\Delta_{PJ}(x,y) = i \int d^4p \,\mathsf{sign}(p_0)\; \delta(p^2 - m^2) e^{ip_\mu(x^\mu - y^\mu)}$$



$\Delta_{PJ}(x,y)$ is a function of two variables. Natural way to generalize this notion to noncommutative spaces: element of $\mathcal{A}\otimes\mathcal{A}$.

Used it in [Amelino-Camelia–Gubitosi–Mercati, PLB 2009] and in [Amelino-Camelia–Bedic–Mercati, arXiv:19xx.xxxx] to define distance, area and volume operators.

The κ-Pauli–Jordan function will be $\Delta_{PJ}(z,y)$, where:

$$z^\mu = x^\mu \otimes 1\,, \qquad y^\mu = 1 \otimes x^\mu\,.$$

The momentum-space-diffeomorphism invariant expression of Δ_{PJ} is:

$$\Delta_{\mathrm{PJ}} = \frac{1}{2m}\int d^4p\sqrt{-g(p)}\varepsilon(p)\sqrt{-g^{\mu\nu}(p)\frac{\partial\mathcal{C}_\kappa}{\partial p^\mu}\frac{\partial\mathcal{C}_\kappa}{\partial p^\nu}}\delta(\mathcal{C}_\kappa+m^2) : e^{ip_\mu z^\mu}e^{iS(p)_\mu y^\mu} :$$

where $$\varepsilon(p) = \begin{cases} +1 & \text{in region } a \\ -1 & \text{in region } b \\ -1 & \text{in region } c \end{cases}.$$

Explicitly:

$$i\Delta_{\mathsf{PJ}}(z,y) = \int_{|\vec{p}|<\kappa} d^3p \frac{e^{3\frac{\omega^+}{\kappa}} e^{i\vec{p}\cdot\vec{z} - ie^{\frac{\omega^+}{\kappa}}\vec{p}\cdot\vec{y}} e^{i\omega^+(z^0-y^0)}}{2\sqrt{\left(m^2+|\vec{p}|^2\right)}}$$

$$- \int_{\mathbb{R}^3} d^3p \frac{e^{3\frac{\omega^-}{\kappa}} e^{i\vec{p}\cdot\vec{z}} e^{-ie^{\frac{\omega^-}{\kappa}}\vec{p}\cdot\vec{y}} e^{i\omega^-(z^0-y^0)}}{2\sqrt{\left(m^2+|\vec{p}|^2\right)}}$$

$$- \int_{|\vec{p}|>\kappa} d^3p \frac{e^{3\frac{\omega^+}{\kappa}} e^{i\vec{p}\cdot\vec{z}} e^{-ie^{\frac{\omega^+}{\kappa}}\vec{p}\cdot\vec{y}} e^{i\omega^+(z^0-y^0)}}{2\sqrt{\left(m^2+|\vec{p}|^2\right)}},$$

this expression is invariant under κ-Poincaré transformations of z and y, and satisfies the conjugacy relation:

$$\Delta^{\dagger}_{\mathsf{PJ}}(z,y) = \Delta_{\mathsf{PJ}}(y,z),$$

which is what you expect from the commutator $[\phi^\dagger(z), \phi(y)]$.

This allows us to write the creation and annihilation operator algebra:

$$[\hat{a}(\vec{p}), \hat{a}^\dagger(\vec{k})] = 2e^{-\frac{3\omega^+(\vec{k})}{\kappa}}\sqrt{m^2 + |\vec{S}_+(\vec{k})|^2}\,\delta^{(3)}[\vec{p} - \vec{k}]\,,$$

$$[\hat{b}(\vec{p}), \hat{b}^\dagger(\vec{k})] = 2e^{-\frac{3\omega^-(\vec{k})}{\kappa}}\sqrt{m^2 + |\vec{S}_-(\vec{k})|^2}\,\delta^{(3)}[\vec{p} - \vec{k}]\,,$$

$$[\hat{c}(\vec{p}), \hat{c}^\dagger(\vec{k})] = 2e^{-\frac{3\omega^+(\vec{k})}{\kappa}}\sqrt{m^2 + |\vec{S}_+(\vec{k})|^2}\,\delta^{(3)}[\vec{p} - \vec{k}]\,,$$

which is invariant under the Lorentz flow on momentum space.

This suggests a natural number operator:

$$\hat{N} = \int_{|\vec{p}|<\kappa} \frac{d^3p\, e^{\frac{3\omega^+(\vec{p})}{\kappa}}}{2\sqrt{m^2 + |\vec{S}_+(\vec{p})|^2}}\,\hat{a}^\dagger(\vec{p})\,\hat{a}(\vec{p}) + \int_{\mathbb{R}^3} \frac{d^3p\, e^{\frac{3\omega^-(\vec{p})}{\kappa}}}{2\sqrt{m^2 + |\vec{\vec{S}}_-(\vec{p})|^2}}\,\hat{b}^\dagger(\vec{p})\,\hat{b}(\vec{p})$$

$$+ \int_{|\vec{p}|>\kappa} \frac{d^3p\, e^{\frac{3\omega^+(|\vec{q}|)}{\kappa}}}{2\sqrt{m^2 + |\vec{S}_+(\vec{p})|^2}}\,\hat{c}^\dagger(\vec{p})\,\hat{c}(\vec{p})\,,$$

$\widehat{N}$ satisfies the standard commutation relations:

$$[\widehat{N}, \widehat{a}^\dagger(\vec{p})] = \widehat{a}^\dagger(\vec{p})\,, \;\; [\widehat{N}, \widehat{a}(\vec{p})] = -\widehat{a}(\vec{p})\,,$$
$$[\widehat{N}, \widehat{b}^\dagger(\vec{p})] = \widehat{b}^\dagger(\vec{p})\,, \;\; [\widehat{N}, \widehat{b}(\vec{p})] = -\widehat{b}(\vec{p})\,,$$
$$[\widehat{N}, \widehat{c}^\dagger(\vec{p})] = \widehat{c}^\dagger(\vec{p})\,, \;\; [\widehat{N}, \widehat{c}(\vec{p})] = -\widehat{c}(\vec{p})\,,$$

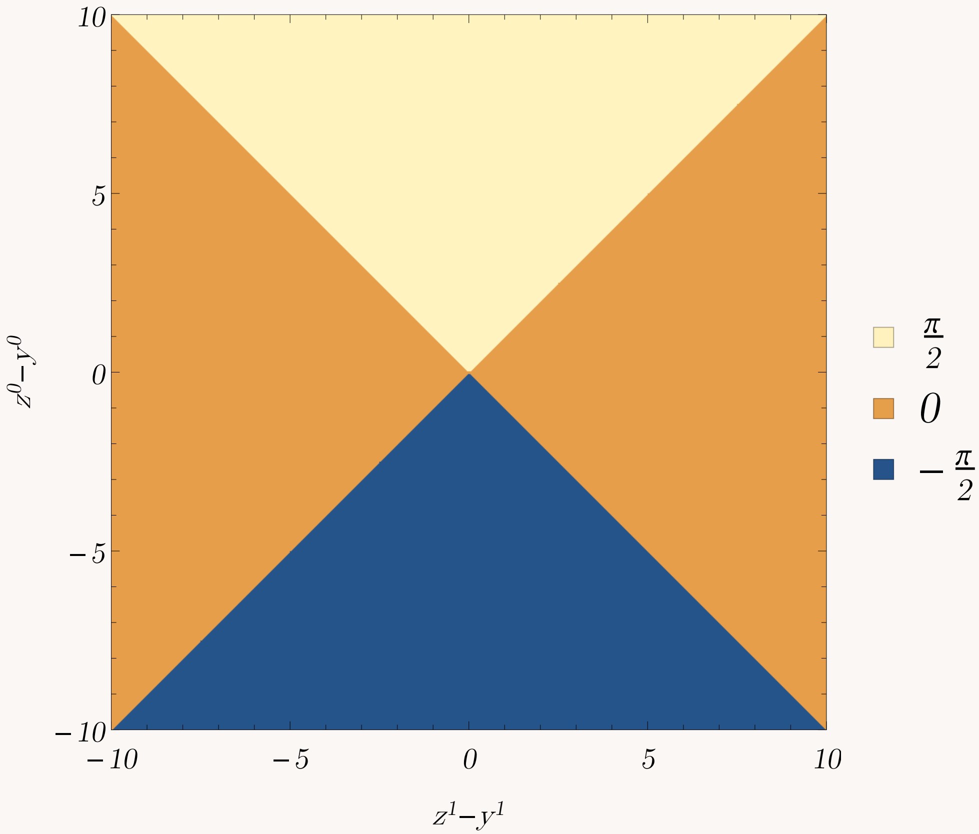
and it is explicitly Hermitian: $\widehat{N}^\dagger = \widehat{N}$.

Therefore the Fock space can be defined like in the commutative case.

However, finding a κ-Fock-space construction implementing the right statistics and a notion of total momentum is a complicated problem, see [Arzano–Marcianò PRD 2007, Arzano–Benedetti, IJMPA 2009, Arzano–Kowalski–Glikman CQG 2014].

‘Calculating’ the κ-Pauli–Jordan function

The commutative $\Delta_{PJ}(z, y)$ defines a light cone:



$\Delta_{PJ}(z, y)$

How to extract a light cone from $\Delta_{PJ}(z, y)$, a noncommuting operator?

Suggestion: introduce a differential representation for $\mathcal{A} \otimes \mathcal{A}$:

$$\widehat{x}^\mu \in \mathcal{A} \qquad [\widehat{x}^0, \widehat{x}^i] = \frac{i}{\kappa}\widehat{x}^i \, ,$$

and a infinite-dimensional Hilbert space of the 'states of points':

$$|\psi\rangle \in \mathcal{H} \, , \qquad \widehat{x}^\mu : \mathcal{H} \to \mathcal{H} \, ,$$

Then the expectation value of $\Delta_{PJ}(\widehat{z}, \widehat{y})$ on a localized state $|\psi\rangle$

$$\langle\psi| \, \Delta_{PJ}(\widehat{z}, \widehat{y}) \, |\psi\rangle =$$

expected value of Pauli–Jordan function in the region where $|\psi\rangle$ is localized.

The representation is simply:

$$\hat{x}^i = \frac{q^i}{\kappa} \quad (q \in \mathbb{R}) , \qquad \hat{x}^0 = \frac{i}{\kappa} \left(\sum_{i=1}^{3} q^i \frac{\partial}{\partial q^i} + \frac{3}{2} \right) ,$$

$$|\psi\rangle = \psi(\vec{q}) \in L^2(\mathbb{R}^3) , \qquad \langle \phi | \psi \rangle = \int d^3 q \, \bar{\phi}(\vec{q}) \psi(\vec{q}) ,$$

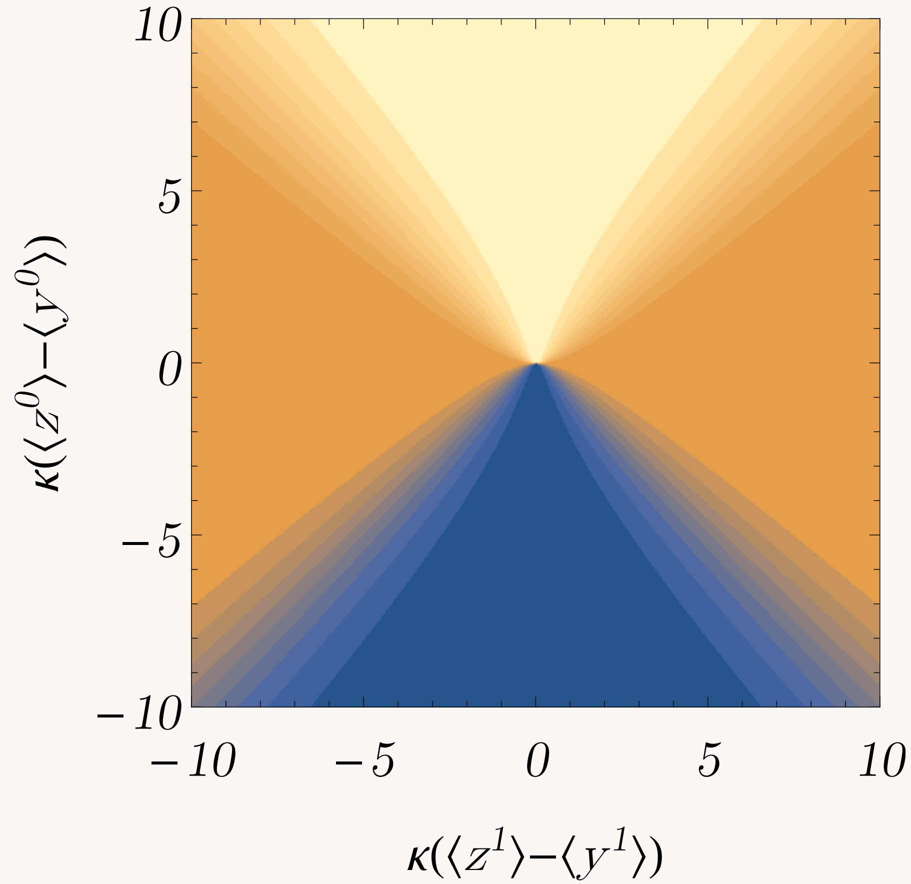
With F. Lizzi and M. Manfredonia, in [Lizzi–Manfredonia–Mercati, PRD 2018] we studied localized states of this representation, as well as an analogue representation of the κ-Poincaré group, and how to interpret everything in terms of fuzzy points/events and inertial observers related to each other by fuzzy transformations.

Check out Mattia Manfredonia's talk after mine!

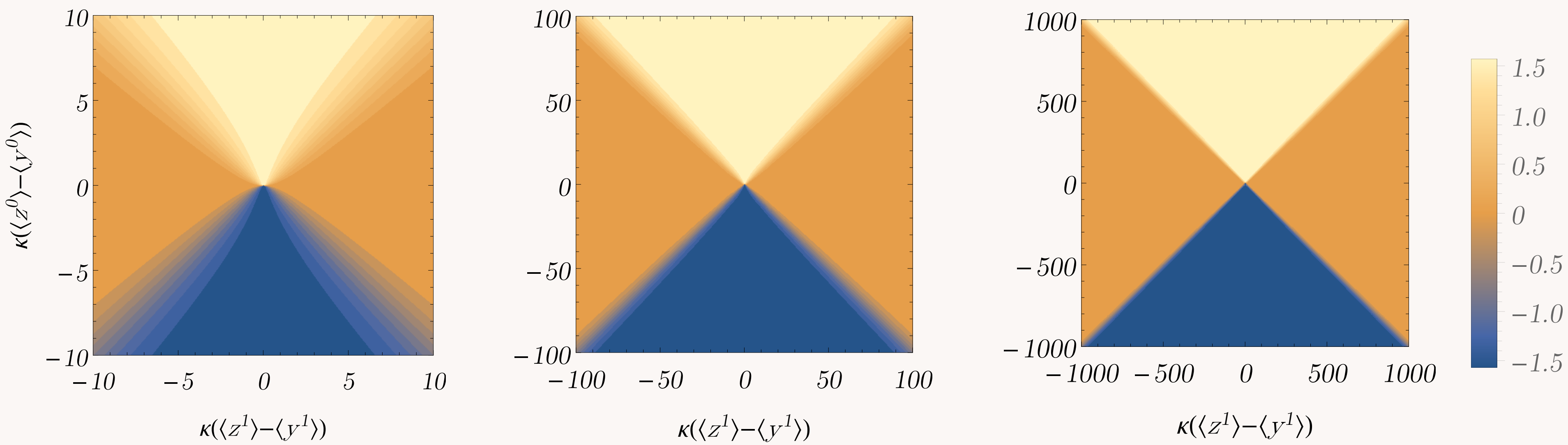
On optimally-localized states, *i.e.* $(\delta z^1)^2 + (\delta z^0)^2 \sim \frac{\langle z^1 \rangle}{\kappa}$

$$\langle \psi | \, \Delta_{PJ}(\hat{z}, \hat{y}) \, | \psi \rangle$$

as a function of the expectation values $\langle z^\mu \rangle$ and $\langle y^\mu \rangle$ is:

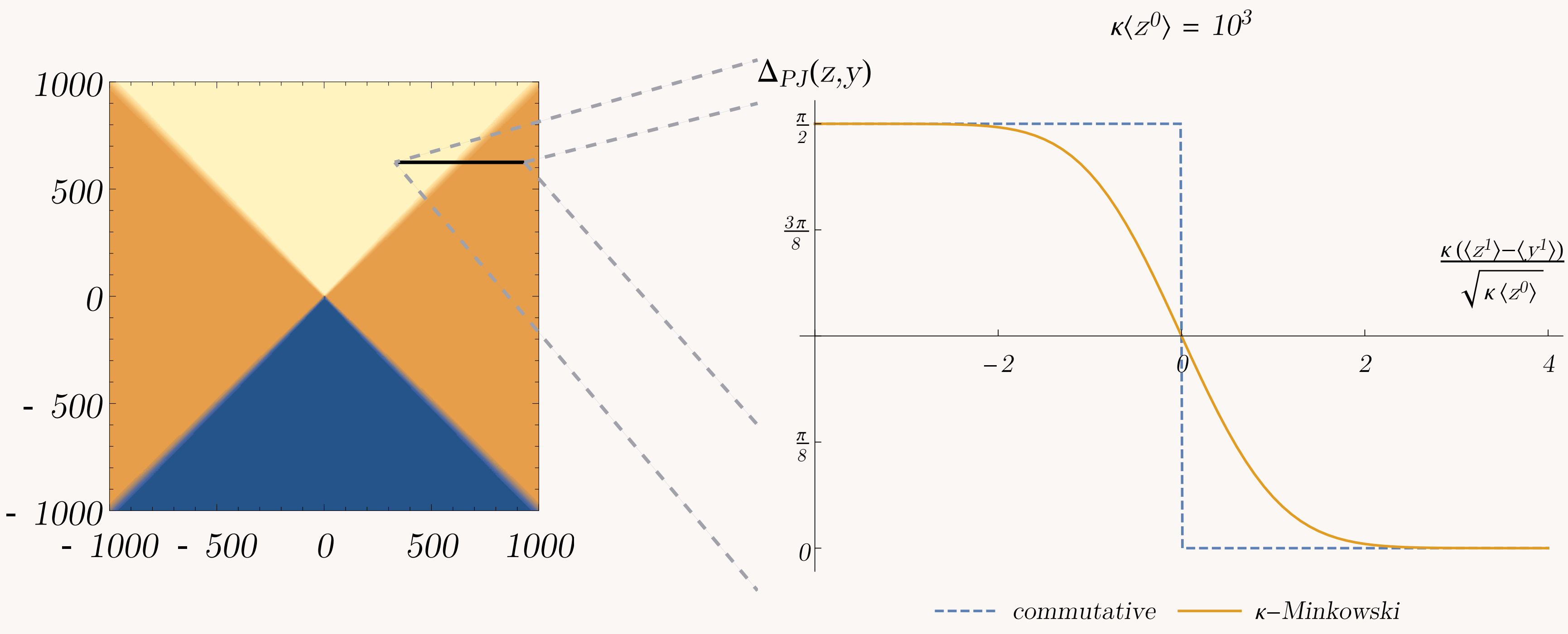


Zooming out:



$\kappa(\langle z^0\rangle - \langle y^0\rangle)$
$\kappa(\langle z^1\rangle - \langle y^1\rangle)$
$\kappa(\langle z^1\rangle - \langle y^1\rangle)$
$\kappa(\langle z^1\rangle - \langle y^1\rangle)$
1.5
1.0
0.5
0
−0.5
−1.0
−1.5

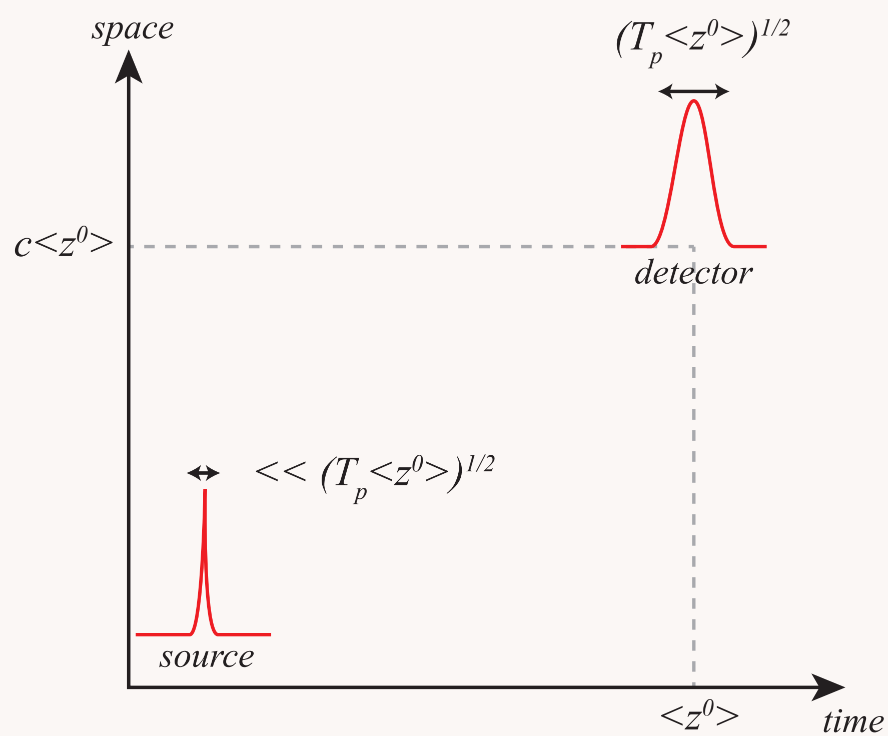
Taking a constant-time cross-cut:



The Pauli–Jordan functions falls off exponentially away from the classical light-cone, with a falloff radius of order:

$\sqrt{\frac{\langle z^0 \rangle}{\kappa}}$ = *geometric mean between* $\kappa^{-1} \sim L_p$ *and distance from the origin.*

A *pointlike* source one billion light-years away, $\langle z^0 \rangle \sim 10^9 y$ will be detected with a time uncertainty of $\sqrt{L_p \langle z^0 \rangle} = 10^{-14} s$ = 10 femtoseconds.



[FM–Sergola, PLB 2018, PRD 2018]

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'Coreductivity', or how to define a quantum homogeneous space:
[Ballesteros–FM–Gutierrez, arXiv:1909.01000]

Curved momentum spaces of κ-Minkowski:
[Lizzi–Manfredonia–FM, arXiv:1910.xxxx]

Momentum-space-diffeo-invariant QFT on κ-Minkowski:
[FM–Sergola Phys Rev. D 2018, arXiv:1801.01765]

Representations of κ-Minkowski/Poincaré, localization and observers:
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Light cone on κ-Minkowski:
[FM–Sergola, PLB 2018, Phys Rev. D 2018, arXiv:1810.08134]