Classical dynamics on fuzzy space[1]

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[1]FG Scholtz, Phys. Rev. D 98 (2018) 104058

Outline

Fuzzy Space

- The fuzzy sphere commutation relations are

$$[\hat{x}_i, \hat{x}_j] = 2i\lambda\varepsilon_{ijk}\hat{x}_k.$$

where λ is the non-commutative length parameter. These commutation relations respect the rotational symmetry. The Casimir operator $\hat{x}^2 = \hat{x}_i\hat{x}_i$ is associated with the square of the radial distance and its eigenvalues are determined by the $su(2)$ representation under consideration: $j(j+1)$, $j = 0, 1/2, \ldots$.

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- Fuzzy space is the collection of fuzzy spheres with each allowed radius appearing once.

Fock Space Realisation of Fuzzy Space

- A concrete realisation of fuzzy space is provided by the Schwinger construction, which utilises two sets of boson creation and annihilation operators to build a representation of $su(2)$:

$$[\hat{a}_\alpha, \hat{a}^\dagger_\beta] = \delta_{\alpha\beta} \quad \text{and} \quad [\hat{a}_\alpha, \hat{a}_\beta] = [\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta] = 0, \ \alpha, \beta = 1, 2.$$

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- We denote this realisation of fuzzy space by $\mathcal{H}_{\mathrm{FS}}$.

Quantum Mechanics on Fuzzy Space 1/4

- The quantum Hilbert space $\mathcal{H}_Q$ is defined as the algebra of bounded operators on $\mathcal{H}_{FS}$ generated by the coordinates (the operators that commute with $\hat{x}^2$) [2] and have a finite norm with respect to a weighted Hilbert-Schmidt inner product [3]:

$$\mathcal{H}_Q = \left\{ \psi : [\psi, \hat{n}] = 0,\ \mathrm{tr}_{\mathrm{FS}} \left(\psi^\dagger\, \hat{r}\, \psi \right) < \infty \right\}.$$

Here $\mathrm{tr}_{\mathrm{FS}}$ denotes the trace over $\mathcal{H}_{FS}$ and $\hat{r} = \lambda(\hat{n} + 1)$. States in $\mathcal{H}_Q$ are denoted $|\psi)$.

[2] N Chandra et al, J.Phys. A: Math.Theor 47 (2014) 445203

[3] The choice of weight is motivated by the requirement that the projector that projects on all states in $\mathcal{H}_{FS}$ with radius less then or equal to R (the analogue of the characteristic function of a ball with radius R) gives the volume of a sphere of radius R for large R (V. Gáliková and P. Prešnajder, 2013 *J. Math. Phys.* **54** 052102)

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  - The angular momentum operators which act adjointly according to

$$\hat{L}_i|\psi) = |\frac{\hbar}{2\lambda}[\hat{x}_i,\psi]) \quad \text{with} \quad [\hat{L}_i,\hat{L}_j] = i\hbar\varepsilon_{ijk}\hat{L}_k.$$

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  - The non-commutative analogue of the Laplacian, which gives the kinetic energy, is defined as

$$\hat{\Delta}_\lambda|\psi) = |-\frac{1}{\lambda\hat{r}}[\hat{a}_\alpha^\dagger,[\hat{a}_\alpha,\psi]]).$$

    It commutes with the angular momenta and is symmetric on $\mathcal{H}_Q$.

Quantum mechanics on fuzzy space 3/4

- The Hamiltonian is given by

$$\hat{H} = -\frac{\hbar^2}{2m}\hat{\Delta} + V(\hat{R})$$

with $\hat{R}$ the radius operator that acts as

$$\hat{R}|\psi) = |\lambda(\hat{n}+1)\psi), \ \hat{n} = a^\dagger_\alpha a_\alpha.$$

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$$\hat{\Gamma}|\psi) = |[a_\alpha^\dagger a_\alpha, \psi]).$$

- It can easily be checked that Γ commutes with the Hamiltonian. Note that $\hat{\Gamma}|\psi) = 0, \ \forall\psi \in \mathcal{H}_Q$.

Brief summary of results 1/2

- The formalism above has been used to solve the free particle and particle in a well [4], to develop scattering theory on non-commutative spaces[5] and the statistical physics[6]. The main results are:

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- For $|\vec{k}|\lambda << 2$

$$E_{\vec{k}} = \frac{\hbar^2|\vec{k}|^2}{2m}$$

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- For $|\vec{k}|\lambda << 2$

$$E_{\vec{k}} = \frac{\hbar^2|\vec{k}|^2}{2m}$$

- Each single particle state occupies a finite volume $V_0 = 4\pi\lambda^3$. For Fermions this is an excluded volume.

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Brief summary of results 2/2

- The resulting equation of state has striking consequences for the mass-radius relationship of a white dwarf:

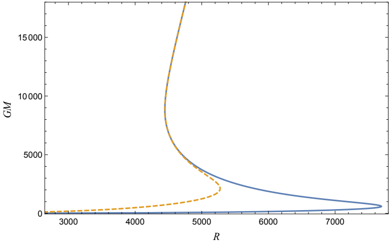

Figure: Mass-radius relationship for white dwarf at two temperatures in arbitrary units.

From Quantum to Classical

- Now that we have a quantum theory with a short length scale, manifest rotational symmetry and the appropriate low energy limit, we may ask what is the underpinning classical dynamics, i.e. how may Newton dynamics be altered? To answer this question, we must compute the path integral action and extract the equations of motion.

From Quantum to Classical

- Now that we have a quantum theory with a short length scale, manifest rotational symmetry and the appropriate low energy limit, we may ask what is the underpinning classical dynamics, i.e. how may Newton dynamics be altered? To answer this question, we must compute the path integral action and extract the equations of motion.
- To do this, we enlarge the quantum Hilbert space $\mathcal{H}_q$ to include all Hilbert-Schmidt operators acting on $\mathcal{H}_{\rm FS}$ and not just those commuting with the Casimir. We denote this enlarged space by $\mathcal{H}_q^0$. Clearly $\mathcal{H}_q \subset \mathcal{H}_q^0$. From the definition of $\mathcal{H}_q$, states that belong to the subspace $\mathcal{H}_q$ must satisfy the constraint

$$\hat{\Gamma}|\psi) = 0.$$

Note that since $\hat{\Gamma}$ is conserved, initial states that satisfy this condition, will do so at all times.

Path Integral Action: General Form

- Let $|\ell\rangle$ be a set of overcomplete coherent states, i.e.

$$\int d\mu(\ell)|\ell\rangle\langle\ell| = \mathbf{1},$$

then the transition amplitude can be represented as a path integral

$$\langle\ell_f, t_f|\ell_i, t_i\rangle = \int_{\ell(t_i)=\ell_i}^{\ell(t_f)=\ell_f}[d\mu(\ell)]e^{\frac{i}{\hbar}S},$$

with action

$$S = \int_{t_i}^{t_f} dt\langle\ell(t)|i\hbar\frac{\partial}{\partial t} - H|\ell(t)\rangle.$$

Path Integral Action: Fuzzy Space 1/4

- We can easily construct a set of coherent states on $\mathcal{H}_q^0$:

$$|z_\alpha, w_\alpha) = |z_\alpha\rangle\langle w_\alpha|.$$

where $|z_\alpha\rangle$ is a Glauber coherent state on $\mathcal{H}_c$ and

$$\int \frac{d\bar{z}_\alpha dz_\alpha d\bar{w}_\alpha dw_\alpha}{\pi^4}|z_\alpha, z_\alpha)(z_\alpha, w_\alpha| = \mathbf{1_q^0}.$$

Note though that in general $\Gamma|z_\alpha, w_\alpha) \neq 0$. However, if we want to compute transition amplitudes between states that satisfy $\Gamma|\psi) = 0$, we can safely use them to insert the identity at intermediate times in a time slicing procedure as Γ is conserved. Indeed, if this is done, the Γ must appear as a conserved quantity in the resulting action and we must simply require it to vanish to satisfy the condition of physicality of the initial state.

Path Integral Action: Fuzzy Space 2/4

- The general result for the path integral action can now be applied to obtain

$$S = \int_{T_i}^{T_f} dT \left[\frac{i}{2} \left(\bar{z}_\alpha \dot{z}_\alpha - \dot{\bar{z}}_\alpha z_\alpha + \dot{\bar{w}}_\alpha w_\alpha - \bar{w}_\alpha \dot{w}_\alpha \right) - H \right]$$

Here

$$H = \left(f_1(R)\bar{z}_\alpha z_\alpha - f_2(R)\left(\bar{z}_\alpha w_\alpha + z_\alpha \bar{w}_\alpha\right) + f_3(R)\bar{w}_\alpha w_\alpha\right) + W(R).$$

- with

$$
\begin{aligned}
R &= \bar{z}_\alpha z_\alpha \\
f_1(R) &= \frac{1}{2}\langle z_\alpha|\frac{1}{\hat{n}+2}|z_\alpha\rangle, \\
f_2(R) &= \frac{1}{2}\langle z_\alpha|\frac{1}{\sqrt{(\hat{n}+1)(\hat{n}+2)}}|z_\alpha\rangle, \\
f_3(R) &= \frac{1}{2}\langle z_\alpha|\frac{1}{\hat{n}+1}|z_\alpha\rangle, \\
W(R) &= \frac{1}{e_0}\langle z_\alpha|V(\hat{R})|z_\alpha\rangle + 2f_3(R) \equiv \tilde{V}(R) + 2f_3(R).
\end{aligned}
$$

Path Integral Action: Fuzzy Space 4/4

- Here we introduced the time and energy scales

$$t_0 = \frac{m\lambda^2}{\hbar}, \quad e_0 = \frac{\hbar}{t_0},$$

and the dimensionless quantities

$$T = \frac{t}{t_0}, \quad X_i = \frac{x_i}{\lambda}, \quad E = \frac{e}{e_0}.$$

Equations of Motion

- The equation of motion are easily found:

$$
\begin{aligned}
\dot{z}_\alpha &= -i\frac{\partial H}{\partial \bar{z}_\alpha}, \\
\dot{\bar{z}}_\alpha &= i\frac{\partial H}{\partial z_\alpha}, \\
\dot{w}_\alpha &= i\frac{\partial H}{\partial \bar{w}_\alpha}, \\
\dot{\bar{w}}_\alpha &= -i\frac{\partial H}{\partial \bar{w}_\alpha}.
\end{aligned}
$$

Constants of Motion

- There are five constants of motion

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\begin{aligned}
\Gamma &= \bar{z}_\alpha z_\alpha - \bar{w}_\alpha w_\alpha , \\
L_i &= \bar{z}_\alpha \sigma^{(i)}_{\alpha\beta} z_\beta - \bar{w}_\alpha \sigma^{(i)}_{\alpha\beta} w_\beta , \\
E &= H(z, \bar{z}, w, \bar{w}) .
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- It is important to note that we must require $\Gamma = 0$.

Equations of Motion for Coordinates 1/3

- We are interested in the equation of motion for the physical coordinates of a particle $X_i = \langle z_\alpha|\hat{X}_i|z_\alpha\rangle$. These can be extracted from the results above and read in the limit $R >> 1$ for the dimensionless coordinates

$$\ddot{\vec{X}}_\pm = \frac{W'(R)}{R}\left[\left(\vec{X}\times\dot{\vec{X}}\right) \pm \sqrt{1-\dot{\vec{X}}\cdot\dot{\vec{X}}}\,\vec{X}\right].$$

The dimensionless conserved quantities are

$$\begin{aligned}\vec{L}_\pm &= \sqrt{1-\dot{\vec{X}}\cdot\dot{\vec{X}}}\left(\vec{X}\times\dot{\vec{X}}\right) \pm \left(\left(\vec{X}\times\dot{\vec{X}}\right)\times\dot{\vec{X}}\right),\\ E_\pm &= 1 \pm \sqrt{1-\dot{\vec{X}}\cdot\dot{\vec{X}}} + W(R).\end{aligned}$$

Equations of motion for coordinates 2/3

- The dimensionful version of the equation of motion is

$$\ddot{\vec{x}}_{\pm} = \frac{w'(r)}{mr}\left[\frac{m\lambda}{\hbar}\left(\vec{x}\times\dot{\vec{x}}\right) \pm \sqrt{1-\left(\frac{m\lambda}{\hbar}\right)^2 \dot{\vec{x}}\cdot\dot{\vec{x}}}\;\vec{x}\right],$$

  and the conserved quantities

$$\vec{\ell}_{\pm} = \hbar\vec{L} = m\left[\sqrt{1-\left(\frac{m\lambda}{\hbar}\right)^2 \dot{\vec{x}}\cdot\dot{\vec{x}}}\;\left(\vec{x}\times\dot{\vec{x}}\right) \pm\frac{m\lambda}{\hbar}\left(\left(\vec{x}\times\dot{\vec{x}}\right)\times\dot{\vec{x}}\right)\right],$$

$$e_{\pm} = \frac{\hbar^2}{m\lambda^2}\left[1\pm\sqrt{1-\left(\frac{m\lambda}{\hbar}\right)^2 \dot{\vec{x}}\cdot\dot{\vec{x}}}\;\right] + w(r).$$

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  - They predict two branches, depending on the energy,
  - One branch reduces to standard Newton dynamics at speeds $v << v_0$.
  - From

$$\vec{L}_\pm \cdot \dot{\vec{X}} = 0, \quad \vec{L}_\pm \cdot \ddot{\vec{X}}_\pm = 0, \quad \vec{L}_\pm \cdot \vec{X} = \mp \vec{L} \cdot \vec{L} \equiv \mp L^2,$$

    we conclude that the motion is still planar, but displaced as $\vec{L} \cdot \vec{X} \neq 0$ as for central Newtonian dynamics.

Effective Radial Potential for Gravity

- From the conserved energy, we can construct an effective radial potential. In the case of gravity this reads in dimensionless units ($\beta = \frac{GMm^2\lambda}{\hbar^2} > 0$)

$$\dot{R}^2 + \frac{2(E-1)\beta}{R} + \frac{\beta^2 + L^2}{R^2} \equiv \dot{R}^2 + V_{\text{eff}} = E(2-E).$$

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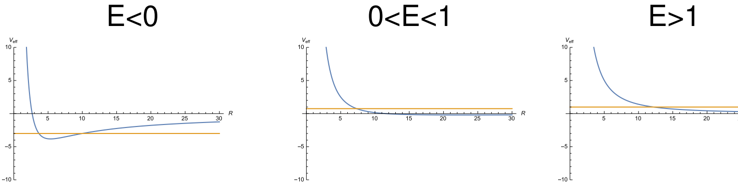
- We immediately observe that the energy must be limited by $E < 2$ for this to have a solution.



Precession in a Gravitational Potential

- The equations of motion in general lead to precession of the orbitals. In the case of gravity one can compute this to leading order in the non-commutative parameter, with the result

$$\Delta\phi = \pi + \frac{\pi GM}{8a\left(1-\epsilon^2\right)v_0^2}.$$

where a is the length of the semi-major axis and ϵ the eccentricity.

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$$\Delta\phi = \pi + \frac{3\pi GM}{c^2 a(1-\epsilon^2)},$$

- This comparison must be done with care as the noncommutative result is a noncommutative perturbation of flat space and not curved space as in the case of GR.

Stable Circular Orbitals in a Gravitational Potential 1/2

- Let us make the following ansatz for circular orbitals

$$x(t) = r\sin\theta\cos(\omega t),\ y(t) = r\sin\theta\sin(\omega t),\ z(t) = r\cos\theta.$$

Note that θ is time-independent.

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  Note that θ is time-independent.
- We can now compute the radial dependence of the velocity (velocity curve):

$$v(r) = v_0\sqrt{\frac{2}{1+\sqrt{1+4\left(\frac{r}{r_0}\right)^2}}},$$

$$\cot\theta = \sqrt{\frac{2}{\sqrt{1+4\left(\frac{r}{r_0}\right)^2}-1}}.$$

where v_0 is the limiting velocity and $r_0 = \frac{GM}{v_0^2}$.

Stable Circular Orbitals in a Gravitational Potential 2/2

- We note the following interesting behaviour

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v(r) &= v_0, \quad r << r_0, \\
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- If $v_0 < c$ and of the order of observed plateau velocities in galaxies (200-300 km.s^{-1}), we still need a much higher included mass to explain the flatness of the velocity curves on the observed length scales, but all the mass can now be concentrated in the centre of the galaxy.

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- In this scenario Newton dynamics applied to orbits of stars close to the centre will lead to a severe underestimation of included mass.

Lorentz Geometry 1/2

- The equations of motion can explicitly be written in covariant form

$$\frac{d^2x^\mu}{d\tau^2}+\tilde{\Gamma}^\mu_{\lambda\nu}\frac{dx^\lambda}{d\tau}\frac{dx^\nu}{d\tau}=S^\mu_{\lambda\nu}\frac{dx^\lambda}{d\tau}\frac{dx^\nu}{d\tau}$$

where $\tilde{\Gamma}^\mu_{\lambda\nu}$ are the Levi-Civita connections and $d\tau$ the proper time of the metric (to leading order in λ)

$$g_{\mu\nu}=\begin{pmatrix}1-\frac{\lambda}{r} & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}.$$

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- Note: We use as fiducial frame for the connections the one in which the metric has the above form.

Lorentz Geometry 2/2

- The $S^{\mu}_{\lambda\nu}$ are the differences of two sets of connections and explicitly read in the fiducial frame (to leading order in λ)

$$
\begin{aligned}
S^i_{00} &= \frac{\lambda}{2r^3}\left(1+\left(\frac{mv_0^2-e}{mv_0^2}\right)^2\right)x^i,\\
S^0_{0j} &= -\frac{\lambda}{2r^3}x_j,\\
S^i_{0j} &= \frac{\lambda\left(e-mv_0^2\right)}{2mv_0^2r^3}\epsilon_{ijk}x^k,\\
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- Note that all the $S^{\mu}_{\lambda\nu}$ vanish in the commutative limit and that $g_{\mu\nu}$ reduces to the Minkowski metric.

Concluding Remarks

- The emergence of a mass dependent limiting speed is problematic. This can be bypassed by assuming that λ is mass dependent such that v_0 is mass independent.

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- The more exiting scenario is one where $v_0 < c$, perhaps even locally, in which case there are observational consequences, but it is not clear that this can be done consistently.
- The emergence of the Lorentz geometry is a surprise.
- It is not that easy to introduce a short length sale and preserve the rotational symmetry and any such construction may share the features above. We therefore expect these to be rather generic.