Background structures in Noncommutative Geometry

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Introduction

- A *Spectral Triple*[1] is a family $(\mathcal{A}, \mathcal{H}, D, J, \chi)$, where $\mathcal{A} =$ algebra, $\mathcal{H} =$Hilbert space, $D, J, \chi =$ operators with special properties.
- Connes proved that commutative ST $\Leftrightarrow$ Spin manifolds[2],. . .
- and that if you tensorize the ST of spacetime with a well-chosen NC finite dimensional ST,. . .
- and write down a convenient action whose variables are "fluctuated Dirac operators" (bosonic) and $\mathcal{H}$-valued fields (fermionic) ,. . .
- . . . you obtain the SM.

The NCG approach in a nutshell

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Why is it interesting ?

- Elegant: Similar to KK: bosonic field $=$ metric (here metric $\leftrightarrow$ Dirac) on an extended manifold (here extension $=$ "quantum space" instead of compact manifold).
- Unifying: Higgs and gauge fields are all fluctuations.
- Predictive: less free parameters than in usual SM.

[1] Always even and real in this talk.

[2] A. Connes, *On the spectral characterization of manifolds*, J. Noncommut. Geom. **7** (2013) arXiv:0810.2088

What (currently) works and what doesn't (yet)

1. All the gauge and Higgs terms of the bosonic action can be unified by a single YM-type term Tr(F^2) (*Connes-Lott action*).
2. You can even include gravity with the *Spectral action* Tr$(f(D^2))$.
3. Works well in Euclidean signature[3,4]
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on the other hand...

1. There is a Fermion Doubling problem, but it can be solved[6], in an almost unique way[7], and the solution has an interesting link with neutrino mixing.
2. The SA predicts a Higgs mass 40% too large. (Can be corrected by the addition of a new scalar boson.)
3. The SA cannot be defined on a Lorentzian manifold.
4. In the current model with algebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, there is an anomalous extra $U(1)$ symmetry (unimodularity problem).

[3] A. Connes, J. Lott, *Particles models and noncommutative geometry*, Nucl. Phys. Proc. Suppl., **18B** (1991)

[4] A. H. Chamseddine, A. Connes, *The spectral action principle*, Commun. Math. Phys., **186** (1997)

[5] N. Bizi, *Semi-Riemannian Noncommutative Geometry, Gauge Theory, and the Standard Model of Particle Physics*, thesis, abs/1812.00038 (2018)

[6] J. W. Barrett, *A Lorentzian version of the non-commutative geometry of the standard model of particle*, J. Math. Phys., **48** (2007)

More details on Spectral Triples

Def: A (real, even) spectral triple is a multiplet $(\mathcal{A}, \mathcal{H}, \pi, D, J, \chi)$ with $\mathcal{A}$ a C^*-algebra, $\mathcal{H}$ a Hilbert space, π a rep. of $\mathcal{A}$, D, χ linear and J antilinear s.t.

1. $\chi^2 = 1$, $\chi^* = \chi$, $[\chi, \pi(\mathcal{A})] = 0$, $\{\chi, D\} = 0$,
2. $D^* = D$ + some analytical conditions
3. $J^2 = \pm 1$, $J^* J = 1$, $[J, D] = 0$, $J\chi = \pm \chi J$.

The signs depend on an integer $[8]$ (KO-dimension).

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In the semi-Riemannian case you have to:

1. replace $\mathcal{H}$ with a *pre-Krein space*[8] $\mathcal{K}$,
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The canonical triple of a spin manifold:

1. $\mathcal{A} =$ compactly supported smooth functions,
2. $\mathcal{K} =$ compactly supported smooth spinor fields,
3. $D =$ canonical Dirac operator associated with the metric and spin structure.

[8] FB, N. Bizi, *Doppler shift in semi-Riemannian signature and the non-uniqueness of the Krein space of spinors*, JMP, **60**, (2019) abs/1806.11283

[9] C. Brouder, N. Bizi, FB, *Space and time dimensions of algebras with application to Lorentzian noncommutative geometry and quantum electrodynamics*, JMP, **59**, 6 (2018) abs/1611.07062

Let $a_i, b_i \in \mathcal{A}$. Then

$$\omega = \sum_i a_i[D, b_i]$$

is a NC 1-form (where a_i, b_i are identified with their image under π).
The $\mathcal{A}$-bimodule of NC 1-forms is written Ω^1_D. Let $\omega \in \Omega^1_D$ be selfadjoint, then the *fluctuated Dirac* D_ω is

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- The Connes-Lott action is[10] $S_{CL}(D_\omega) = -\int_M \mathrm{Tr}(F(\omega)^2)$.
- The *Spectral Action* is $S_\Lambda(D_\omega) = \mathrm{Tr}(f(D_\omega^2))$, where f approximate $[0, \Lambda]$.

[10] For an AC manifold, and without taking care of the junk

GR or not GR ?

We should recover GR when we apply the SA to the canonical ST of a manifold but *we don't*, for a number of reasons.

1. Wrong configuration space: there is no fluctuation.
2. Wrong automorphism group: the unitary operators which commute with D, J, χ and stabilize $\pi(\mathcal{A})$ correspond to *isometries*.

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The latter point is the key to a solution. . .

Consider a *parallelizable* manifold, and[11]

1. A trivial bundle $M \times S$, $S = \mathbb{C}^4$,
2. gamma matrices $\gamma_a \in \mathrm{End}(S)$ (in Dirac or Weyl representation),
3. $\chi = \gamma_5$,
4. $J = \gamma_2 \circ c.c$,
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Then every tetrad $e = (e_a)$ defines at the same time a metric g_e such that e is pseudo-orthonormal, a g_e-spin structure with rep $\rho_e : \mathbb{C}\ell TM \to \mathrm{End}(S)$ s.t $\rho_e(e_a) = \gamma_a$, and so a Dirac operator $D(e) = i \sum \pm \gamma_a \nabla^e_{e_a}$.

Do like physicists !

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Let $\Gamma = \mathrm{Span}(\gamma_a | a = 0, \ldots, 3)$. Then:

1. $\Omega^1_{D(e)} := \Omega^1$ is independent of e and is the space of Γ-valued fields.
2. This space is invariant under diffeomorphisms and local Lorentz transformations.

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$\Rightarrow$ Ω^1 should be a background structure while D should not.

[11] Here $n = 4$.

Algebraic backgrounds

- An Algebraic Background is a ST - D + an odd bimodule $\Omega^1 \subset \mathrm{End}(\mathcal{K})$.
- A *compatible Dirac operator* on a background $\mathcal{B}$ is a Dirac operator like before $+$ satisfies $\Omega^1_D \subset \Omega^1$. (It is *regular* if $\Omega^1_D = \Omega^1$.)
- An automorphism of $\mathcal{B}$ is a Krein unitary U which commutes with χ and J and stabilizes $\mathcal{A}$ and Ω^1.

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The canonical background $\mathcal{B}(M)$ of a parallelizable manifold is constructed like before thanks to an origin metric g_0 of signature (p, q), only needed to define

$$(\Psi, \Psi') = \int_M H_S(\Psi_x, \Psi'_x)\mathrm{vol}_{g_0}$$

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Let $\theta : M \to M$ be a diffeo and $\Sigma : M \to \mathrm{Spin}(p,q)^0 \subset \mathrm{End}(S)$, then

$$V_\theta : \Psi \mapsto \sqrt{\frac{\mathrm{vol}_{\theta * g_0}}{\mathrm{vol}_{g_0}}}\Psi \circ \theta^{-1}, \text{ and } U_\Sigma : \Psi \mapsto \Sigma\Psi$$

are automorphisms of $\mathcal{B}(M)$. Moreover, they generate $\mathrm{Aut}(\mathcal{B}_M)$.
$\Rightarrow$ the automorphisms of $\mathcal{B}_M$ correspond exactly to the symmetries of tetradic GR (except LLT are lifted to the spin group).

The configuration space

- Let r be a field of invertible matrices: acts on tetrads $e \mapsto r \cdot e$.
- $S_r \in \mathrm{End}(\mathcal{K})$ is defined by $\Psi \mapsto |\det r|^{-1/2}\Psi$

Theorem The regular Dirac operators of the canonical background $\mathcal{B}(M)$ are

$$D = \delta_r + \zeta$$

where $\delta_r = S_r D(r \cdot e_0) S_r^{-1}$ and ζ is a multiplication operator $(\zeta\Psi)_x = \zeta_x \Psi_x$, s.t. $\zeta_x^\times = \zeta_x$, ζ commutes with J and anticommutes with χ.

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$\Rightarrow$ the config space is larger than in GR ! There are additional *centralizing fields*.

- In $1+3$ dim, there is a single centralizing pseudo-vector field.
- The spaces of δ_r's and ζ's are separately invariant under automorphisms.
- They are “orthogonal”.

$\Rightarrow$ centralizing fields can be removed safely.

$\mathcal{B}_{SM} = \mathcal{B}(M)\hat{\otimes}\mathcal{B}_F$ where $\mathcal{B}_F = (\mathcal{A}_F, \mathcal{K}_F, \pi_F, J_F, \chi_F, \Omega^1_F)$:

- $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$,
- $\mathcal{K}_F = \mathcal{K}_R \oplus \mathcal{K}_L \oplus \mathcal{K}_{\bar{R}} \oplus \mathcal{K}_{\bar{L}}$, $K_\sigma = \mathbb{C}^2 \otimes (\mathbb{C} \oplus \mathbb{C}^3_{\text{color}}) \otimes \mathbb{C}^3_{\text{gen}}$,
- Finite Krein product $(\psi, \psi') = \psi^\dagger \chi_F \psi$, with $\chi_F = [1_R, -1_L, -1_{\bar{R}}, 1_{\bar{L}}]$,
- $J_F = \begin{pmatrix} 0 & -1_{antipart} \\ 1_{part} & 0 \end{pmatrix} \circ c.c.$,
- $\pi_F(\lambda, q, m) = [\tilde{q}_\lambda, \tilde{q}, \lambda 1_2 \oplus 1_2 \otimes m, \lambda 1_2 \oplus 1_2 \otimes m] \otimes 1_3$, where $q_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix}$ and $\tilde{q} = q \oplus q \otimes 1_3 \simeq q \otimes 1_4$.
- $\Omega^1_F = \{ \begin{pmatrix} 0 & Y_0^\dagger \tilde{q}_1 & 0 & 0 \\ \tilde{q}_2 Y_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, q_1, q_2 \in \mathbb{H}\}$, where
$$Y_0 = \begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix} \oplus \begin{pmatrix} 1_3 \otimes Y_u & 0 \\ 0 & 1_3 \otimes Y_d \end{pmatrix}.$$

Choice of Ω^1_F constrained by: 1) odd $\mathcal{A}_F$-$\mathcal{A}_F$-bimodule, 2) non-vanishing config space, and 3) first-order condition: $[\Omega^1_F, J_F \pi_F(\mathcal{A}_F) J_F^{-1}] = 0$

Automorphisms of the SM background

For $T \in \mathrm{End}(\mathcal{K})$ define $T^o = JT^\times J^{-1}$. Then for u unitary in $\mathcal{A}$, define $\Upsilon(u) = uJuJ^{-1} = u(u^{-1})^o$.

- Υ is a group homomorphism from $U(\mathcal{A})$ into $\mathrm{Aut}(\mathcal{B}_M)$.
- This is true because $[\pi(\mathcal{A}_F), \pi(\mathcal{A}_F)^o] = 0$ and the first-order condition.
- Note that the weaker condition $\pi(u)^o\Omega^1\pi(u^{-1})^o = \Omega^1$ would suffice (weak order 1 cond.).
- $\Upsilon(U(\mathcal{A}))$ is the group of local gauge transformations $M \to U(1) \times SU(2) \times U(3)$.

Th: $\mathrm{Aut}(\mathcal{B}_{SM})$ is generated by

1. diffeo-spino-morphisms $U_\theta \otimes 1$, $U_\Sigma \otimes 1$ coming from the base manifold,
2. $\Upsilon(U(\mathcal{A}))$,
3. local $B-L$-transformations $1 \otimes g_{B-L}(t)$ where $g_{B-L}(t) = [A(t), A(t), A(t)^*, A(t)^*] \otimes 1_3$, $A(t) = e^{-it}1_2 \oplus e^{\frac{it}{3}}1_2 \otimes 1_3$

Note that the automorphism group of the SM Spectral Triple is *larger* if D is not fixed, and (much) *smaller* if it is.

For $q \in \mathbb{H}$ let $\Phi(q) = \begin{pmatrix} 0 & -Y_0^\dagger \tilde{q}^\dagger & 0 & 0 \\ \tilde{q} Y_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and for any symmetric M acting on generations let $\sigma(M) = \begin{pmatrix} 0 & 0 & -p_\nu \otimes M^\dagger & 0 \\ 0 & 0 & 0 & 0 \\ p_\nu \otimes M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, where $p_\nu = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is the projection on the space spanned by ν.

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The compatible Dirac operators are

$$\Phi(q) + \Phi(q)^o + \sigma(M)$$

- The $\Phi(q) + \Phi(q)^o$ part can be obtained by the “fluctuation formalism”.
- The $\sigma(M)$ part cannot.
- The latter is the one that had been put by hand (with only 1 dof) to correct the Higgs mass prediction.

A general compatible Dirac is of the form

$$D = \delta_r \hat{\otimes} 1 + \zeta_g + \zeta_X + \zeta_{B-L} + \zeta_H + \zeta_\sigma + \zeta_{\text{other}}$$

1. The ζ_{other} part contains centralizing fields which act on generations only.

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1. The ζ_{other} part contains centralizing fields which act on generations only.
2. The automorphisms act separately on $\delta_r \hat{\otimes} 1 + \zeta_g + \zeta_X + \zeta_{B-L} + \zeta_H$, ζ_σ and ζ_{other}.
3. Only $B - L$ acts non-trivially on ζ_σ, and by multiplication. The ζ_{other} are aut-invariant.
4. ζ_X is centralizing, and so is the e.m. field.

$\Rightarrow$ we can freely include from 0 to 6 ζ_σ fields, but we need at least one to have neutrino oscillations.
$\Rightarrow$ we can throw away ζ_{other} without harm.
$\Rightarrow$ We have to keep some centralizing fields, and throw some other away: not nice...
$\Rightarrow$ There is no known action in Lorentzian signature for these fields. But the Euclidean SA could be applied.

Just replace $\mathcal{A}_F$ by $\mathcal{A}_F^{\text{ext}} = \mathbb{C} \oplus \mathcal{A}_F = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, with

$$\pi_F^{\text{ext}}(\lambda, \mu, q, m) = [\tilde{q}_\lambda, \tilde{q}, \mu 1_2 \oplus 1_2 \otimes m, \mu 1_2 \oplus 1_2 \otimes m] \otimes 1_3$$

and Ω_F^1 by

$$(\Omega_F^1)^{\text{ext}} \ni \begin{pmatrix} 0 & Y_0^\dagger \tilde{q}_1 & z_1 p_\nu \otimes M_0^\dagger & 0 \\ \tilde{q}_2 Y_0 & 0 & 0 & 0 \\ z_2 p_\nu \otimes M_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, z_1, z_2 \in \mathbb{C}, q_1, q_2 \in \mathbb{H}$$

- Only satisfies *weak* order 1 condition.
- The compatible finite Dirac are $\Phi(q) + \Phi(q)^o + \sigma(zM_0)$.
- $\mathcal{B}_{\text{SM}}^{\text{ext}} = \mathcal{B}(M) \hat{\otimes} \mathcal{B}_F^{\text{ext}}$ has the same automorphism group as $\mathcal{B}_{\text{SM}}$.
- Its configuration space contains: SM fields $+$ anomalous X $+$ Z'_{B-L} $+$ 1 complex scalar $\sigma(zM_0)$, $+$ flavour changing ζ_{other}.
- All fields apart from ζ_{other}, are now fluctuations, so CL action is defined.
- The Higgs part is

$$\begin{aligned} S(H, z) = \quad & 16a|D_\mu H|^2 + 8b|D_\mu z|^2 - 8V_0(|H|^2 - 1)^2 - 8W_0(|z|^2 - 1)^2 \\ & -16K(|H|^2 - 1)(|z|^2 - 1) \end{aligned}$$

Conclusion

With the algebraic background framework:

- symmetries exactly correspond to those of tetradic GR,
- variable=Dirac operator (no more fluctuation),
- in the SM, $B - L$ pops up by itself,
- the needed scalar too !

But. . .

- The only known action defined on all Dirac is the SA (Euclidean).
- In Lorentzian signature, one needs to take a step back and use CL.
- Unimodularity is not solved.

Work to do:

- What is the SA prediction for the Higgs with this model ?
- What is the exact role of the centralizing fields ? Do we need to get rid of them ?
- Hint towards a link between centralizing fields and unimodularity in Pati-Salam: X is the only centralizing gauge field.

Thank you for your attention !

- FB, *A $U(1)_{B-L}$-extension of the Standard Model from Noncommutative Geometry*, coming soon !
- FB, *Algebraic backgrounds a framework for noncommutative Kaluza-Klein theory*, arXiv:1902.09387, (2019)
- C. Brouder, N. Bizi, FB, *Space and time dimensions of algebras with application to Lorentzian noncommutative geometry and quantum electrodynamics*, JMP, **59**, 6 (2018)
- N. Bizi, *Semi-Riemannian Noncommutative Geometry, Gauge Theory, and the Standard Model of Particle Physics*, thesis, abs/1812.00038 (2018)

Complete bosonic action:

$$
\begin{aligned}
\mathcal{L}_b \quad = \quad & -160\frac{N}{3}\mathbb{F}^Y_{\mu\nu}\mathbb{F}^{Y\mu\nu} - 32N\mathbb{F}^W_{\mu\nu a}\mathbb{F}^{W\mu\nu a} - 32N\mathbb{F}^C_{\mu\nu a}\mathbb{F}^{C\mu\nu a} \\
& -\frac{64}{3}NF^{Z'}_{\mu\nu}F^{Z'\mu\nu} - \frac{128}{3}N\mathbb{F}^Y_{\mu\nu}F^{Z'\mu\nu} + 16a|D_\mu H|^2 - 8bs|D_\mu z|^2 \\
& -8V_0(|H|^2-1)^2 - 8W_0(|z|^2-1)^2 + 16sK(|H|^2-1)(|z|^2-1)
\end{aligned}
$$

Normalization of kinetic terms: $\mathbb{B}^Y_\mu = \frac{1}{2}g_Y Y_\mu$, $\mathbb{B}^{Wa}_\mu = \frac{1}{2}g_w W^a_\mu$, $\mathbb{B}^{Ca}_\mu = \frac{1}{2}g_s G^a_\mu$, $Z'_\mu = \frac{1}{2}g_{Z'}\hat{Z}'_\mu$, $H = k\tilde{H}$, $z = l\tilde{z}$, with

$$
g_w^2 = g_s^2 = \frac{5}{3}g_Y^2 = \frac{2}{3}g_{Z'}^2 = \frac{1}{32N}, \quad \kappa = 64\tfrac{N}{3}g_Y g_{Z'} = \sqrt{\tfrac{2}{5}}
$$

$$
k^2 = \frac{1}{16a}, \qquad l^2 = \frac{1}{8b}
$$

$$
\begin{aligned}
M_W^2 &= \frac{1}{k^2}g_w^2 \\
&= \frac{1}{4}\frac{1}{32N}32\mathrm{Tr}(Y_eY_e^\dagger + Y_\nu Y_\nu^\dagger + 3M_u + 3M_d) \\
&= \frac{1}{4N}\sum \text{squared masses of fermions}
\end{aligned}
$$

In particular for $N = 3$, one obtains $M_{\text{top}} \leq 2M_W$.

An automorphism of the canonical ST not coming from a diffeo-spinomorphism is possible as soon as dim ≥ 6.
Example: multiplication by $\sinh t\gamma_1\gamma_2 + \cosh t\gamma_3 \ldots \gamma_6$.
Many automorphisms of the finite SM triple are not AB automorphism (Krein-unitary commuting with J and χ but not stabilizing Ω^1_F)
Ex: $U = [A, B, A^*, B^*]$ with arbitrary unitary matrices A, B. (need not be block-diagonal, other examples exist)