# An action for dual gravity and graded Poisson algebra

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on-going work

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# Motivations

$\Rrightarrow$ To introduce **interactions with gauge fields** but through
*deformations of the commutation relations of phase space coordinates*

- quantum charged particle in an ext. magnetic field $F_{ij} = \epsilon_{ijk} B^k$:

$$[p_i, x^j] = i\hbar\, \delta_i{}^j, \quad [p_i, p_j] = i\hbar e F_{ij}$$

Hamiltonian $H = \frac{p^2}{2m}$ and dynamics

$$\dot{\vec{x}} = -\frac{i}{\hbar}[\vec{x}, H] = \frac{\vec{p}}{2m}, \quad \dot{\vec{p}} = \frac{i}{\hbar}[\vec{p}, H] = \frac{e}{m}\vec{p} \times \vec{B};$$

- This is not a canonical transformation but an application of Moser lemma (or generalizations of it) $\varphi_t^* \omega_t = \omega_0, \quad \frac{d\varphi_t}{dt} = V_t \circ \varphi_t;$
- Interested in curved spacetime manifolds and symmetric 2-tensors (metric). How about them? $\rightarrow$ need of $\mathbb{Z}$-grading: odd and even coordinates
($\Rrightarrow \Lambda^2 T^*(M[1]) \cong \vee^2 T^* M$).
Looking for a *metric* theory with general covariance (gravity theory without matter)

# Review of dg-manifolds

$\Rrightarrow$ $T^*[2]T[1]M$ with canonical symplectic form $\omega = dx^i \wedge dp_i + d\xi_J \delta^{JK} \wedge d\xi_K \Leftrightarrow O(d,d)$ symmetry of degree-1 coordinates for $TM \oplus T^*M$:

- canonical transformation $\varphi \in End(T^*[1]M \oplus T[1]M)$,

$$\{\varphi\xi_J, \varphi\xi_K\} = \{\xi_J, \xi_K\}, \quad \delta^{JK} \equiv \begin{pmatrix} 0 & \mathbb{1}^j{}_k \\ \mathbb{1}_j{}^k & 0 \end{pmatrix} \mapsto \varphi^T \delta \varphi = \delta$$

$\varphi$ serving as $O(d,d)$ element;

$\Rrightarrow$ consider Hamiltonian function $\Theta$ and its vector field $Q = \{\cdot, \Theta\}$:
$\to$ $(U, V \in \mathcal{O}_1)$ $\{QU, V\} = [U, V]$ derived bracket;
$\to$ ($f$ function on $M$) $\{QU, f\} = \rho(U)f$ where $\rho(U) \in \Gamma(TM)$

$[U, V]$ is the non-skewsymmetric bracket of a Courant algebroid (CA), $\rho$ the *anchor*

## Review of dg-manifolds $\Leftrightarrow$ Courant algebroids

*Axioms*

- $[U, fV] = f[U, V] + \rho(U)f\, V$
- $\rho(W)\delta(U, V) = \delta([W, U], V) + \delta(U, [W, V])$
- $\rho(W)\delta(U, V) = \delta([U, V] + [V, U], W)$
- $[W, [U, V]] = [[W, U], V] + [U, [W, V]]$

*Proofs*

- $\{QU, fV\} = \{QU, f\}V + f\{QU, V\}$
- $\{\{QU, V\}, W\}$ & graded Jacobi id.
- $\{\{QU, V\}, W\}$ & Leibniz rule for $Q$ & graded skew-symmetry of $\{\cdot, \cdot\}$
- $Q^2 = 0 = \{\Theta, \Theta\}$ classical master equation in BRST, BV-BFV

"$\Leftarrow$" implication can also be proven;

*dg-symplectic manifolds of degree 2 with a Hamiltonian function are in 1:1 correspondence with Courant algebroids*

[Ševera '00, Roytenberg '02]

# Stringy fluxes in graded algebra picture

$$\Theta = \xi_I \rho^{Ij}(x) p_j - \frac{1}{3!} C^{IJK}(x) \xi_I \xi_J \xi_K \quad (1)$$

$$\xi_I \cong \left(\partial_i, dx^i\right) \qquad \{Q\xi_J, \xi_K\} = [\xi_J, \xi_K]_{\text{twisted}} = H_{jkm} dx^m + R^{jkm}\partial_m + 3 f_{jk}{}^m \partial_m + 3 Q_j{}^{km}\partial_m$$

Also: start with the simpler Hamiltonian $\Theta = \xi_I \rho^{Ij}(x) p_j$, it can be twisted via adjoint action!

[Deser, Heller, Ikeda, Watamura, Carow-Watamura ...]

Bianchi identities are automatical outcome of closure of algebroid structure $[\mathcal{L}_{\xi_J}, \mathcal{L}_{\xi_K}] = \mathcal{L}_{[\xi_J, \xi_K]}$, where $\mathcal{L}$ is Dorfman derivative.

- $H_{jkm} \xrightarrow{T_m} f_{jk}{}^m \xrightarrow{T_k} Q_j{}^{km} \xrightarrow{T_j} R^{jkm}$, last T-duality is just formal;

  [Shelton, Taylor, Wecht '05]

- $R$-flux can be a source of non-associativity (as seen by an open string ending on a brane in presence of such non-geometric flux);

  [Lüst, ...]

- compactifications with fluxes and dualities, symplectic gravity, $\beta$-gravity.

# Gauge theory from deformation

**So far**:

- canonical symplectic structure of $T^*[2]T[1]M$ with Hamiltonian (w/o fluxes)
- $B$, gauge field ($H = dB$) of the connective gerbe structure of $TM \oplus T^*M$; $H$ is Ševera class of exact CA;
- T-duality chain as structure constants (point-dependent!) of the CA

**Program**:

- non-canonical transformations (Moser lemma) on $(T^*[2]T[1]M, \omega)$: generalize $\delta$ to some $2d$-metric $\mathbb{G}(x)$
- give rise to interactions with gauge field $B(x) \in \Gamma(\Lambda^2 T^*M)$
- Seiberg-Witten closed-open strings relation $g + B = \left(G^{-1} + \Pi\right)^{-1} \Longrightarrow$ gauge potentials for $Q$-, $R$-flux

[Seiberg, Witten '99]

Locally!

# Deformed structure of dg-manifold

$$\textbf{symplectic form:}\quad \omega = dx^i \wedge dp_i + d\left(E_J{}^K \xi_K\right)\delta^{KL} d\left(E_L{}^M \xi_M\right)$$

$$\Downarrow$$

$$\mathbb{G}(x) = E^T(x)\delta E(x)$$

$$\{p_i, \xi_J\} = \Gamma_{iJ}{}^K \xi_K, \quad \{p_i, p_j\} = 0$$

$\Rightarrow$ physics content: $E$ depends on $g(x), B(x)$ and $G^{-1}(x), \Pi(x)$ (through open-closed strings relation) $\Longrightarrow$ $\Gamma$ depends on their derivatives, $\mathbb{G}$ depends on $g, G^{-1}$.

$$E = \begin{pmatrix} \mathbb{1} & -(g+B)^{-1}(x) \equiv -\left(G^{-1}+\Pi\right)(x) \\ g(x) - B(x) & \mathbb{1} \end{pmatrix}$$

$E$ is isomorphism $TM \oplus T^*M \cong C_+ \oplus C_-$, where $C_\pm$ are eigenbundles of generalized metric:

$$\mathcal{H}^{JK} \equiv \begin{pmatrix} g^{jk} & g^{jl}B_{lk} \\ -B_{jl}g^{jk} & g_{jk} - B_{jl}g^{lm}B_{mk} \end{pmatrix}.$$

$$\textbf{Hamiltonian:}\ \Theta = \xi_I \mathbb{G}^{IK} \rho'_K{}^j p_j, \quad \rho'_K{}^j := E_K{}^L \rho_L{}^j.$$

# Differential calculus on $TM \oplus T^*M$

**Definition (generalized Lie commutator $[\![U,V]\!]$)**

Any binary operation that $[\![U,V]\!] = -[\![V,U]\!]$ and $[\![U,fV]\!] = \rho(U)fV$

**Definition (affine connection of first type $\Gamma(W;U,V) \equiv \langle \nabla_W U, V\rangle$)**

Must have the properties $\Gamma(fW;U,V) = f\Gamma(W;U,V) = \Gamma(W;U,fV)$ and $\Gamma(W;fU,V) = \rho(W)f\langle U,V\rangle + f\Gamma(W;U,V)$

**Theorem**

*Given a Dorfman br. $[,]$, a generalized Lie commutator $[\![,]\!]$ and an affine connection of first type $\Gamma(;,)$, metric wrt the same $\langle,\rangle$ in CA definition,*

$$\langle [U,V] - [\![U,V]\!], W\rangle = \Gamma(W;U,V). \tag{2}$$

**Definition (torsion $T(U,V)$)**

$T(U,V) := \nabla_U V - \nabla_V U - [\![U,V]\!]$ is torsion tensor of connection $\nabla$.

to be compared with other definitions (e.g. Gualtieri's...)

# Deformed Courant algebroid

$\langle , \rangle = \delta$ replaced by $\mathbb{G}(x)$;

$$\mathbb{G}\left([U,V],W\right) = \mathbb{G}\left(\nabla_U V, W\right) - \mathbb{G}\left(\nabla_V U, W\right) + \mathbb{G}\left(\nabla_W U, V\right)$$

Natural connection of first type follows from theorem (2) and definition of torsion:

$$\begin{aligned}\tilde{\Gamma}(W;U,V) &= \mathbb{G}\left([U,V] - [\![U,V]\!], W\right) \\ &= \Gamma(W;U,V) + \mathbb{G}(T(U,V),W)\end{aligned}$$

# Projected connections

$$\left(TM \oplus T^*M, \rho', [\cdot,\cdot], \mathbb{G}\right),\ \tilde{\Gamma}$$

$$\swarrow pr_{TM} \qquad\qquad pr_{T^*M} \searrow$$

$$\left(TM, \tilde{\Gamma}|_{TM}\right) \qquad\qquad \left(T^*M, \tilde{\Gamma}|_{T^*M}\right)$$

$\mathbb{G}$ has $GL(d) \times GL(d)$ symmetry; in local coordinates,

$$\left(\tilde{\Gamma}|_{TM}\right)^i{}_{jk} = \tfrac{1}{2} g^{il}\left(\partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk}\right) - \tfrac{1}{2} g^{il} H_{ljk};$$

$$\begin{aligned}\left(\tilde{\Gamma}|_{T^*M}\right)_i{}^{jk} = &-\frac{1}{2} G_{im}\left[\left(G^{-1}+\Pi\right)^{jl}\partial_l\left(G^{-1}+\Pi\right)^{km} + \left(G^{-1}+\Pi\right)^{kl}\partial_l\left(G^{-1}+\Pi\right)^{mj}\right] \\ &+ \frac{1}{2} G_{im}\left(G^{-1}+\Pi\right)^{ml}\partial_l\left(G^{-1}+\Pi\right)^{jk};\end{aligned}$$

$\rightarrow$ **includes** $R^{ijk} := 3!\Pi^{[i|l}\partial_l\Pi^{|jk]}$ **and** $Q_i{}^{jk} := \partial_i\Pi^{jk}$ **fluxes!**

Curvature tensor:

$$\mathbb{R}(W,V)U = \left(\tilde{\nabla}_W\tilde{\nabla}_V - \tilde{\nabla}_V\tilde{\nabla}_W\right)U - \tilde{\nabla}_{[\![W,V]\!]}U;$$

project all arguments and the resulting generalized vector too:

- $(pr_{TM}) \Rrightarrow\ \mathcal{R} - \frac{1}{12}H^2 \equiv \mathbb{R}^i{}_{kij}g^{km}\left(g-B\right)_{ml}g^{lj}$
  *[low energy effective Lagrangian of the bosonic sector (apart from dilaton) common to all string theories]*

[Coimbra, Strickland-Constable, Waldram '11]
[Jurco, Vysoky '17]
[Ševera, Valach '18]
[Schupp, Boffo '19]

- $(pr_{T^*M})$: define $\mathcal{X}_i{}^{jk} := -\frac{1}{2}Q_i{}^{jk} + G_{im}G^{[j|l}Q_l{}^{|k]m}$, $\mathcal{X}_i{}^{jk} = -\mathcal{X}_i{}^{kj}$ and $D_G$ symmetric part of $\tilde{\Gamma}|_{T^*M}$
  $\Rrightarrow\ \mathcal{R}_{D_G} - \frac{1}{12}R^2 - R_{ijk}\mathcal{X}^{ijk} - \mathcal{X}^2 - \Pi_{jl}D_G^m\mathcal{X}_m{}^{jl} \equiv -\mathbb{R}_i{}^{kij}G_{kl}\left(G^{-1}+\Pi\right)^{lm}G_{mj}$
  *[Lagrangian for the dual metric G with non-geometric fluxes]*

# Geometric action with non-geometric fluxes

1. [Andriot, Larfors, Hohm, Lüst, Patalong '12] : *symplectic gravity*

$$\begin{array}{ccc} \mathcal{L}_{DFT}(\mathcal{E},d) & \xrightarrow{\mathcal{E}\mapsto\mathcal{E}^{-1}} & \mathcal{L}_{DFT}(\mathcal{E}^{-1},\tilde{d}) \\ \big\downarrow{\scriptstyle \tilde{\partial}=0} & & \big\downarrow{\scriptstyle \tilde{\partial}=0} \\ \mathcal{L}_{NS}(g,B,\phi) & \dashrightarrow & \mathcal{L}(G,\Pi,\tilde{\phi}) \end{array}$$

   field redefinition, but general covariance for doubled diffeomorphisms is preserved
   **comparison**: different assignation of $Q$ in the symm/antisymm part of connection

2. [Andriot, Bethe '13]: $\beta$-*gravity*
   another different field redefinition from covariantization of the DFT action wrt half of the generalized diffeomorphisms

3. [Blumenhagen, Deser, Plauschinn, Rennecke '13] : Lie algebroid $T^*M$, generalized metric and an $O(d,d)$ action to select another frame: $\gamma : TM \mapsto T^*M$ remains defined and is used to pull-back the tensors and connections of standard differential geometry on $TM$, eventually applied to $\mathcal{L}_{NS}$.
   **comparison**: we have $\left(\mathbb{1}_{TM}, \left(G^{-1}+\Pi\right)(\cdot)\right)^T$ as anchor

# Results

- specific example of correspondence between dg-symplectic 2-manifolds - Courant algebroids but *deformed*, so to involve relevant physical fields (graviton, gauge potentials of the fluxes); the vielbein $E$ treats the T-dual fluxes from their respective gauge potentials on the same footing;
- new definitions of generalized tensors of $TM \oplus T^*M$ (torsion and curvature);
- the generalized connection projected onto $T^*M$ yields a new geometric action with non-geometric fluxes.

## Outlook

Follow up in the graded geometry setting:

- other deformations in the realm of gravitational theories;
- non-abelian gauge theories (R-R fields);
- non-associative structures.

Some possible questions:

- How to include the dilaton?
- Relation to DFT action?
- Local vs global picture?
- Relation to $T$-dual of low-energy effective actions for compactifications with fluxes (without DFT formulation)?

# Thank you for the attention!