

Commuting Pairs of Generalized Structures and 2D Sigma Models

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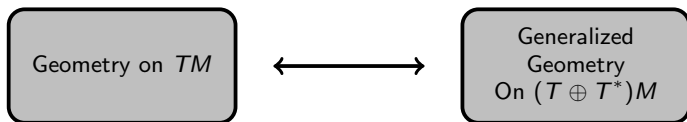
This talk is based on [\[arXiv:1909.04646\]](#)

Generalized geometry

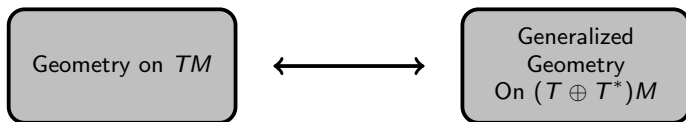
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 - J is equivalent to a choice of a spacetime metric (i.e. metric on L)

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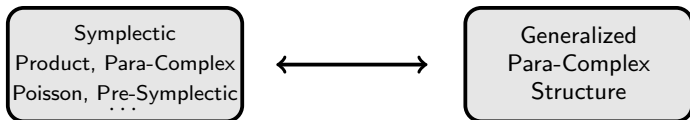
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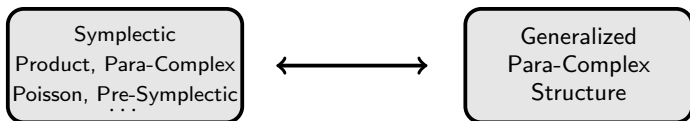
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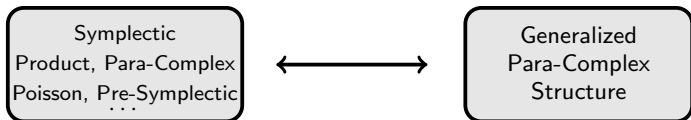
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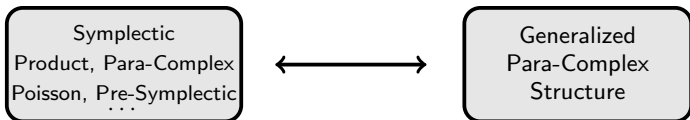
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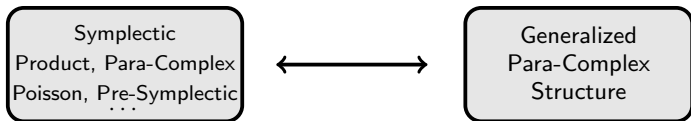
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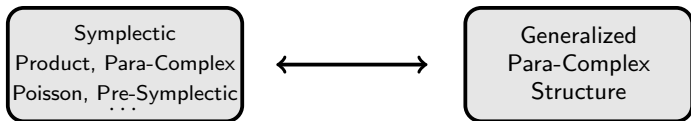
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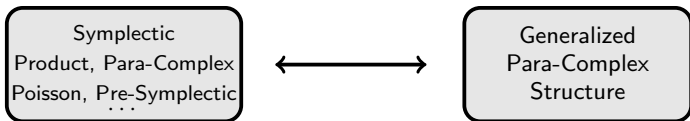
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- Extension to (2, 2) **twisted** SUSY amounts to adding supercharges Q_{\pm}^2 that satisfy

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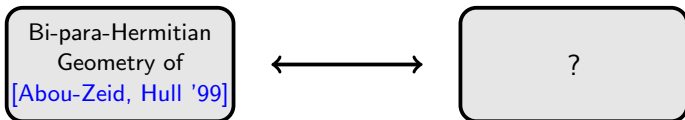
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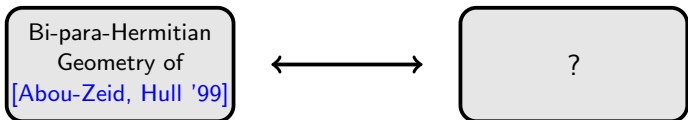
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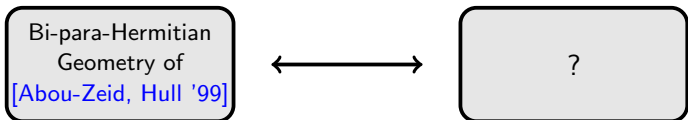
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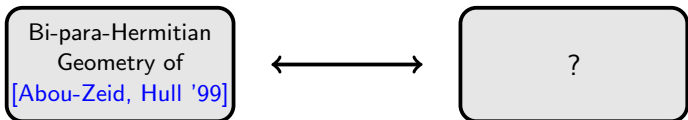
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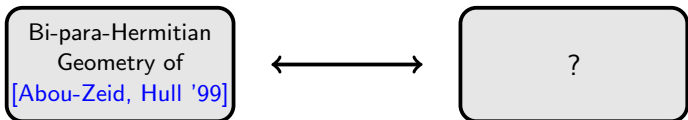
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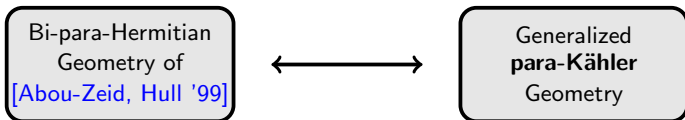
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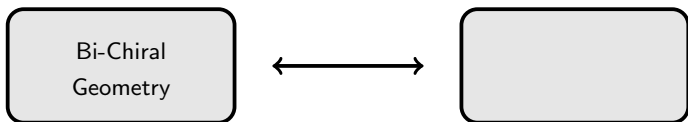
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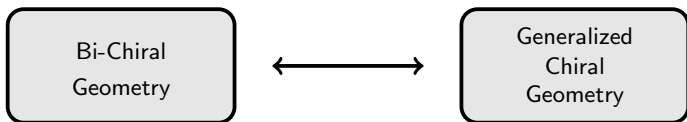
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