Commuting Pairs of Generalized Structures and 2D Sigma Models

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Geometry on TM $\longleftrightarrow$ Generalized Geometry On $(T \oplus T^*)M$

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  - J is equivalent to a choice of a spacetime metric (i.e. metric on L)

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Bi-Chiral Geometry $\longleftrightarrow$

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Thank you for your attention!