Melonic CFTs

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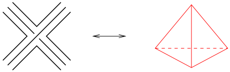
Tensor models – Origins

Tensor models provide a definition of discretized Euclidean quantum gravity
['91: Ambjorn,Durhuus,Jonsson; Sasakura]

Zero-dimensional field theory with fields in a tensor representation of $O(N)$:

e.g.: $T_{abc} \to \sum_{a',b',c'}^{1\ldots N} R_{aa'} R_{bb'} R_{cc'} T_{a'b'c'}$, $R \in O(N)$

Vertex: $\lambda T_{abc} T_{ab'c'} T_{a'bc'} T_{a'b'c}$



$\Rightarrow$ duality between Feynman diagrams and simplicial manifolds

Several limitations:

- No large-N limit
- No known methods of solution
- Very degenerate geometries
- Numerical simulations (fixed topology):
  no semiclassical limit, no 2nd order phase transition

Large-N limit of tensor models

Tensor models were revived by the discovery of "colored tensor models", admitting a large-N expansion [Gurau (2010); Gurau,Rivasseau (2011)]

- Models of a single tensor, but with no symmetry on the indices
  Complex ($U(N)^D$): [Bonzom, Gurau, Rivasseau (2012)]; Real ($O(N)^D$): [Carrozza, Tanasa (2015)]
- Models of a single tensor in an irreducible representation of $O(N)$ (e.g. symmetric traceless or antisymmetric tensors)
  [DB, Carrozza, Gurau, Kolanowski (2017); Carrozza (2018)]

⇒ A new type of large-N limit: **the melonic limit**

More complicated than the vector case, but simpler than the matrix case

A simple model of holography

- A new boost came from the SYK model, a model of N Majorana fermions in $d = 1$, whose coupling constant is a random tensor [Sachdev, Ye (1992); Kitaev (2015)]
- SYK: melonic large-N limit $\Rightarrow$ CFT_1 in the IR $\Rightarrow$ AdS_2 dual $\Rightarrow$ microscopics of near extremal black holes

The same melonic limit, and hence conformal symmetry in the IR, can be obtained without disorder, but with fermions in a tensor representation

[Witten (2016); Klebanov, Tarnopolsky (2016)]

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[Witten (2016); Klebanov, Tarnopolsky (2016)]

Advantages:

- No quenched disorder
- in tensor models $O(N)$ symmetry is there from the beginning, so it can be gauged
- Subleading corrections better understood (no tricky issues with replica limit)

Large-N limit: the general idea

For theories with N degrees of freedom, the large-N limit can provide a tractable limit

General idea: choose a rescaling of the couplings with N such that

1. the large-N limit of the free energy (or of the effective action) exists and it is non-trivial;
2. only a subset of the Feynman diagrams survives in the limit.

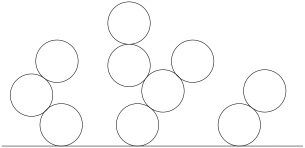
Large-N limit of vectors

e.g. fields in the fundamental representation of $O(N)$ ("$O(N)$ model")

[Stanley (1968); Wilson (1972); Coleman, Jackiw, Politzer (1974); Gross, Neveu (1974);Parisi (1975); ...]

$$\phi_a \to \sum_{a'}^{1\ldots N} R_{aa'}\phi_{a'} \ , \quad R \in O(N)$$

$\Rightarrow$ Large-N limit: **Cactus diagrams** (aka daisy or bubble diagrams)



$\to$ Closed Schwinger-Dyson equation for 2-point function
$=$ mass gap equation (no anomalous dimension)

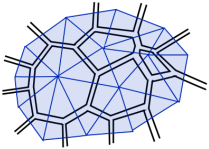
Large-N limit of matrices

e.g. fields in the adjoint representation of $U(N)$ (Hermitian matrix model)

[Wigner (1951); ... 't Hooft (1974); Brezin, Itzykson, Parisi, Zuber (1978); ...]

$$M_{ab} \to \sum_{a',b'}^{1\ldots N} R_{aa'} M_{a'b'} R^{\dagger}_{b'b} \,, \qquad R \in U(N)$$

$\Rightarrow$ Large-N limit: **planar diagrams**



$\to$ No closed Schwinger-Dyson equation; still very difficult

In zero dimension there are many techniques for solving matrix models, but they typically become very hard in higher dimensions

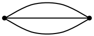
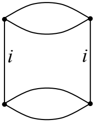
$O(N)^3$ tensor model [Carrozza, Tanasa (2015)]

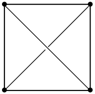
- Statistical model for tensors transforming in the fundamental representation of $O(N) \times O(N) \times O(N)$:

$$T_{abc} \to \sum_{a',b',c'}^{1\dots N} R^{(1)}_{aa'} R^{(2)}_{bb'} R^{(3)}_{cc'} T_{a'b'c'} \ , \qquad R^{(i)} \in O(N)$$

- Invariant action

$$S[T] = \frac{1}{2} T_{abc} T_{abc} + \frac{\lambda_p}{N^2} T_{abc} T_{ab'c'} T_{a'b'c'} T_{a'bc} + \frac{\lambda_t}{N^{3/2}} T_{abc} T_{ab'c'} T_{a'bc'} T_{a'b'c} + \dots$$

$= \frac{1}{2}$  $+ \frac{\lambda_p}{N^2}$ 

$+ \frac{\lambda_t}{N^{3/2}}$  $+ \dots$

(p =pillow; t =tetrahedron)

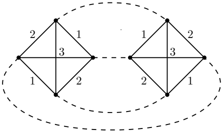
- There exists a scaling of the couplings with N, s.t. the large-N expansion is governed by a non-negative half-integer, the degree ω (not a topological): [Carrozza, Tanasa (2015)]

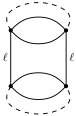
$$\ln \int [dT] e^{-S[T]} = \sum_{\omega \in \mathbb{N}/2} N^{3-\omega} F_\omega$$

Feynman graphs

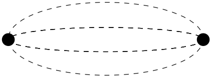
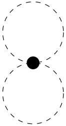
Perturbative expansion:

- Represent Wick contraction of two tensors by dashed line, e.g.:





- Ordinary Feynman diagrams, tracking only spacetime propagators, are obtained by shrinking interaction bubbles to a point:

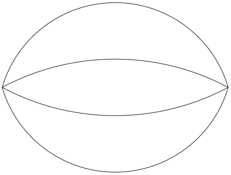



- The two graphs above are examples of melonic graphs (with $\omega = 0$):
  the tetrahedron graph is melonic in the the spacetime diagram representation, while the pillow graph is melonic in the solid+dashed graph representation.

The world of melons

$\omega = 0 \Rightarrow$ **Melonic diagrams**: a special subclass of planar graphs

[Bonzom, Gurau, Riello, Rivasseau (2011)]



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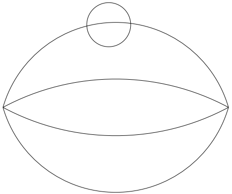

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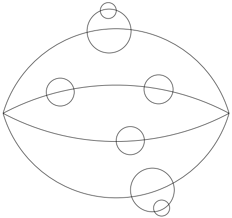

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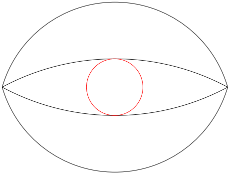

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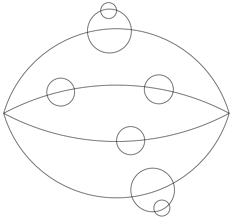


$\Rightarrow$ Combinatorially: tree structure (melons are generated by repeated insertions)

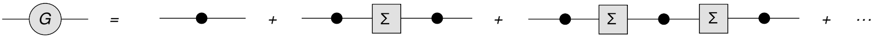
$\Rightarrow$ Melons can be counted: the large-N limit is solvable!

$\Rightarrow$ Bad for Euclidean QG (melons $\simeq$ "branched polymers", no semiclassical regime) but with possibly many other applications

Melonic Schwinger-Dyson equations in $d = 0$

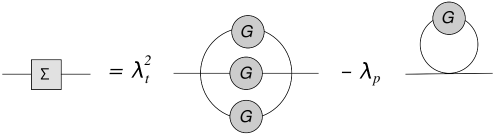
The standard Schwinger-Dyson equation for the 2-point function:

$$G^{-1} = 1 - \Sigma$$



In large-N limit, the self-energy is:

$$\Sigma = +\lambda_t^2 G^3 - \lambda_p G$$



$\rightarrow$ for $\lambda_p = 0$, $G(\lambda_t)$ is the generating function of the $A_p(4,1)$ Fuss-Catalan numbers:

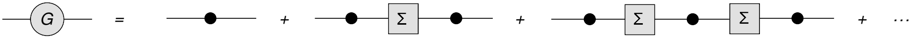
$$G(\lambda_t) = \sum_{p=0}^{\infty} \frac{1}{4p+1} \binom{4p+1}{p} \lambda_t^{2p} \sim (\lambda_{t,\mathrm{crit}} - \lambda_t)^{1/2}$$

where $\quad \lambda_{t,\mathrm{crit}}^2 = \frac{3^3}{4^4}$

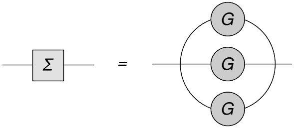
Conformal limit in SYK-like tensor models ($d = 1$)

The structure of the Schwinger-Dyson equations at large-N is the same as before, but with $d = 1$ integrals:

$$G(\omega) = (-i\omega - \Sigma(\omega))^{-1}$$



$$\Sigma(\omega) = J^2 \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} G(\omega_1) G(\omega_2) G(\omega - \omega_1 - \omega_2)$$



Conformal limit in SYK-like tensor models ($d = 1$)

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$$\frac{1}{G(\omega)} = -i\omega - J^2 \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} G(\omega_1) G(\omega_2) G(\omega - \omega_1 - \omega_2)$$

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- Same structure as Fuss-Catalan generating function, but with two loop integrals because we are in $d = 1$
- In the UV limit $\omega \to \infty$ the self energy $\Sigma(\omega)$ can be neglected: the theory is asymptotically free
- In the IR limit $\omega \to 0$ the free inverse propagator ("$-i\omega$") can be neglected and one obtains a conformal invariant solution:

$$G(\omega) \sim \omega^{2\Delta - 1}\,, \qquad \Delta = 1/q$$

($q =$ order of the interaction)

2PI formalism

Polynomial SD equations $\Rightarrow$ 2PI formalism is well suited for tensor models [DB, Gurau (2018)]

Starting from

$$\mathbf{W}[j,k] = \ln \int [d\varphi] \, \exp\left\{ -\mathbf{S}[\varphi] + j_{\mathbf{a}}\varphi_{\mathbf{a}} + \frac{1}{2}\varphi_{\mathbf{a}} k_{\mathbf{ab}} \varphi_{\mathbf{b}} \right\}$$

the 2PI effective action is obtained by a double Legendre transform

$$\mathbf{\Gamma}[\phi, G] = \underbrace{\mathbf{S}[\phi]}_{\text{tree level}} + \underbrace{\frac{1}{2}\mathrm{Tr}[\ln G^{-1}] + \frac{1}{2}\mathrm{Tr}[\mathbf{S}_{\phi\phi}[\phi] G]}_{\text{one loop}} + \underbrace{\mathbf{\Gamma}_2[\phi, G]}_{\text{two or more loops}}$$

with the following Feynman rules:

vertices: $\mathbf{S}_{\mathrm{int}}[\phi, \varphi] = \mathbf{S}[\phi + \varphi]_{\text{starting at cubic order in } \varphi}$

propagator: $G(x,y)$

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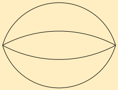
$\mathbf{\Gamma}_2[\phi, G]$ is given by the sum of all the $(n \geq 2)$-loops **two-particle irreducible** vacuum graphs. On shell: $\phi = \langle \varphi \rangle$ and $G = \langle \varphi\varphi \rangle_c$

[Cornwall, Jackiw, Tomboulis - '74]

2PI effective action for tensor models [DB, Gurau (2018)]

Counting traces = counting faces in stranded graph
⇒ **leading order is given by melons**

There is only one melon graph which is also 2PI: the fundamental melon



For SYK-like tensor model in symmetric phase ($\phi = 0$):

$$\frac{1}{N^3}\mathbf{\Gamma}[0,G] = -\frac{1}{2}\mathrm{Tr}[\ln G^{-1}] - \frac{1}{2}\mathrm{Tr}[\partial_t G(t,t')] - \frac{1}{8}\lambda^2 \int_{t,t'} G(t,t')^4$$

⇒ • SD equations recovered as equations of motion $\frac{\delta \mathbf{\Gamma}}{\delta G} = 0$

• 4-point function from inverse of $\frac{\delta^2 \mathbf{\Gamma}}{\delta G \delta G} = G^{-1}G^{-1}(1-K)$, i.e. geometric series in K (ladder diagrams)

The melonic limit in $d \geq 2$

Motivations:

- Melonic SD equations admit a conformal solution also in higher dimensions.
- AdS_{d+1}/CFT_d correspondence for tensor models?
- The melonic limit as an analytic tool for QFT?

Key differences and difficulties:

- In $d \geq 2$ one has to deal with renormalization
- The four-fermion interaction is marginal in $d = 2$
  $\Rightarrow$ only trivial fixed point
- Typical bosonic models have unstable potentials

Could a non-trivial CFT be found in a tensor-valued field theory?

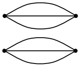
Tensorial Gross-Neveu model [DB,Carrozza,Gurau,Sfondrini (2017); DB, Delporte (2018)]

Standard free theory, which has $U(N^3)$ invariance:

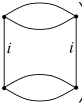
$$S_{\text{free}}[\psi] = \int d^d x \sum_{a,b,c}^{1\dots N} \bar{\psi}_{abc} \partial\!\!\!/ \psi_{abc}$$

Introduce interactions with tensor structure, breaking $U(N^3)$ to $U(N)^3$:

$$S_{\text{int}}[\psi] = \int d^d x \left(\frac{\lambda_d}{4N^3}\right.$$



$$+ \frac{\lambda_p}{4N^2} \sum_{i=1,2,3}$$



$$\left.\right)$$

Notice:

- Only bipartite interactions for $U(N)^3$ symmetry

$\Rightarrow$ Only tadpole diagrams

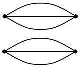
$\Rightarrow$ SD equation $=$ mass gap equation
e.g. leading to $m = \Lambda \exp\left(-\frac{\pi}{2(\lambda_0+3\lambda_1)}\right)$ in $d=2$ (critical dimension)

- In $d>2$, new tricritical UV fixed points (with symmetric, chiral-breaking, and new $U(N)$-breaking phases), but still very similar to ordinary GN model

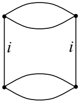
Fermions with tetrahedron interaction [DB,Carrozza,Gurau,Sfondrini (2017)]

For Majorana fermions with $O(N)^3$ symmetry we can add the tetrahedron interaction

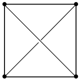
$$S_{\text{int}}[\psi] = \int d^d x \left(\frac{\lambda_d}{4N^3} \right.$$



$$+ \frac{\lambda_p}{4N^2} \sum_{i=1,2,3}$$



$$+ \frac{\lambda_t}{4N^{3/2}}$$



$$\left. \right)$$

The Schwinger-Dyson equation get modified by melonic contribution

$\Rightarrow$ situation in $d = 2 - \epsilon$ is reversed:

$$\beta_t = -\epsilon\lambda_t + \frac{3}{\pi^2}\lambda_t^3$$

$\Rightarrow$ real IR fixed point of order $\sqrt{\epsilon}$ for $\epsilon > 0$ $(d < 2)$,
and no real fixed point for $\epsilon < 0$ $(d > 2)$

For $\epsilon = 1$ we expect to recover an SYK-like model

Bosonic $O(N)^3$ tensor model

- In the bosonic case, the $O(N)^3$ model's action is unbounded from below, due to the tetrahedron interaction.
- In principle it still makes sense to study the model in the large-N limit, but in fact complex operator dimensions are found in $d = 4 - \epsilon$ [Klebanov et al. (2017-2019)]

A real CFT can be found with two modifications [DB, Gurau, Harribey (2019)]

- A non-local free propagator reproducing the expected conformal scaling

$$S_{\text{free}}[\phi] = \frac{1}{2} \int d^d x \, \left(\phi_{abc} (-\partial^2)^{d/4} \phi_{abc} + m^{d/2} \phi_{abc} \phi_{abc} \right)$$

  (similarly to Brydges-Mitter-Scoppola model (ϕ_3^4 theory), or conformal SYK model of Gross-Rosenhaus)

- A purely imaginary tetrahedron coupling: $\lambda_t = \mathrm{i}\, |\lambda_t|$ (similarly to Lee-Yang model with $\mathrm{i}\, \lambda \phi^3$ interaction)

Bosonic $O(N)^3$ tensor model: main results

[DB, Gurau, Harribey (2019); + Suzuki (2019)]

- 2-point function: only mass renormalization for $d < 4$
  Setting renormalized mass to zero, SD equation is solved by $G^{-1}(p) = Zp^{d/2}$ with Z a finite rescaling
- No vertex correction to the tetrahedron $\Rightarrow \beta_t = 0$
  Other beta functions: $\beta_i = \beta_0(-\lambda_t^2) - 2\beta_1(-\lambda_t^2)g_i + \beta_2(-\lambda_t^2)g_i^2$.
  $\Rightarrow$ • λ_t is an exactly marginal coupling
  • Other couplings have a λ_t-dependent fixed point, which is real and with real critical exponents for $\lambda_t^2 < 0$
- spectrum of bilinear operators:
  $h_{m,J} = d/2 + J + 2m + f_{m,J}(-\lambda_t^2)$ real for $\lambda_t^2 < 0$
  and consistent with unitarity bounds
- The OPE coefficients $C_{\phi\phi\mathcal{O}_{m,J}}$ are all real

$\Rightarrow$ Despite the imaginary coupling, the model is so far compatible with unitarity

Conclusions and outlook

- The melonic limit is a new entry in the QFT world
- Tensors are easier than matrices, but richer than vectors
- In $d = 1$ they provide a possible holographic model of black holes
- New fixed points and new phases for fermions in $d > 2$
- Tensorial Gross-Neveu model in $d = 2 - \epsilon \Rightarrow$ approach to $d = 1$ SYK-like model?
- For bosons we find interacting IR fixed points in $d < 4$
  [Giombi, Klebanov, Tarnopolsky (2017); DB, Gurau, Harribey (2019)]

Many open questions, e.g.:

- Unitarity of bosonic melonic CFTs?
- Classification of melonic CFTs?
- Holographic dual of melonic CFTs?
- Other limits hidden in tensor models?