Role of Weyl Connections in Holography

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Conference on Recent Developments in Strings and Gravity, Corfu

based on works with Rob Leigh

September 13, 2019

Overview

Introduction I

AdS holography is geometrically built using the Fefferman-Graham gauge [Fefferman, Graham '85-. . .]

- ✓ mathematically rigorous
- ✓ bulk solvable order by order from the boundary
- ✗ not Weyl covariant

The boundary couples to a conformal class of metrics: Weyl needed
Weyl pillar in other formulation of holography, like fluid/gravity

[Loganayagam et al '08-. . . , LC, Petkou, Petropoulos, Siampos '17-'18]

Introduction II

Our enhancement: Weyl-Fefferman-Graham gauge

- Weyl diffeomorphisms are linearly implemented
- New bulk Weyl gauge field
- Appearance of a Weyl connection in the boundary
- Holographic anomaly couples to Weyl covariant curvatures tensors
- Rich boundary structure to explore

Weyl-Fefferman-Graham gauge I

Fefferman-Graham in Poincaré coordinates with boundary $z \to 0$:[FG '85]

$$\mathrm{d}s^2 = L^2 \frac{\mathrm{d}z^2}{z^2} + h_{\mu\nu}(x,z)\mathrm{d}x^\mu \mathrm{d}x^\nu$$

near the boundary it can be expanded ($d =$ boundary dimension)

[Skenderis '02]

$$h_{\mu\nu} = \frac{L^2}{z^2}\left[\gamma^{(0)}_{\mu\nu}(x) + \frac{z^2}{L^2}\gamma^{(2)}_{\mu\nu}(x) + ...\right] + \frac{z^{d-2}}{L^{d-2}}\left[\pi^{(0)}_{\mu\nu}(x) + \frac{z^2}{L^2}\pi^{(2)}_{\mu\nu}(x) + ...\right]$$

$\gamma^{(0)}_{\mu\nu}$ boundary metric, $\pi^{(0)}_{\mu\nu}$ boundary energy-momentum tensor.
The boundary is located at

$$\frac{z^2}{L^2}\mathrm{d}s^2 \underset{z\to 0}{\longrightarrow} \gamma^{(0)}_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}s^2_{bdy}$$

defined up to a Weyl transformation $\Rightarrow$ conformal boundary.

Weyl-Fefferman-Graham gauge II

Def: bulk *Weyl diffeomorphism*

$$z \mapsto z' = z/\mathcal{B}(x) \qquad x^\mu \mapsto x'^\mu = x^\mu$$

It induces a Weyl rescaling of the boundary

$$\gamma^{(0)}_{\mu\nu} \mapsto \frac{\gamma^{(0)}_{\mu\nu}}{\mathcal{B}(x)^2}$$

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Fefferman-Graham is sensitive to this diffeomorphism

$$\mathrm{d}s^2 = L^2 \left(\frac{\mathrm{d}z'}{z'} + \partial_\mu \ln \mathcal{B}(x)\ \mathrm{d}x^\mu \right)^2 + h_{\mu\nu}(x, z'\mathcal{B}(x))\mathrm{d}x^\mu \mathrm{d}x^\nu$$

Weyl-Fefferman-Graham gauge III

Two ways out

1. Usual one: redefine x in subleading orders $x' = x + f(x)z^2 + \dots$.
   Problem: blurs Weyl covariance of subleading tensors (well-known)

   [Mazur, Mottola '01,...]

2. Our story: enhance the Fefferman-Graham gauge to the *Weyl-Fefferman-Graham* one, where Weyl diffs are allowed.

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2. Our story: enhance the Fefferman-Graham gauge to the *Weyl-Fefferman-Graham* one, where Weyl diffs are allowed.

$$\mathrm{d}s^2 = L^2 \left(\frac{\mathrm{d}z}{z} - a_\mu(x,z)\mathrm{d}x^\mu \right)^2 + h_{\mu\nu}(x,z)\mathrm{d}x^\mu \mathrm{d}x^\nu$$

Novelty of this gauge: Weyl gauge field one form a_μ, with expansion near the conformal boundary

$$a_\mu(z,x) = \left[a_\mu^{(0)}(x) + \frac{z^2}{L^2} a_\mu^{(2)}(x) + \dots \right] + \frac{z^{d-2}}{L^{d-2}} \left[p_\mu^{(0)}(x) + \frac{z^2}{L^2} p_\mu^{(2)}(x) + \dots \right]$$

$a_\mu^{(0)}$ boundary Weyl source, $p_\mu^{(0)}$ boundary Weyl current.

Weyl-Fefferman-Graham gauge IV

Weyl diffeomorphism on the Weyl-Fefferman-Graham gauge

$$\gamma^{(k)}_{\mu\nu}(x) \mapsto \gamma^{(k)}_{\mu\nu}(x)\mathcal{B}(x)^{k-2},\ \pi^{(k)}_{\mu\nu}(x) \mapsto \pi^{(k)}_{\mu\nu}(x)\mathcal{B}(x)^{d-2+k}$$

$$a^{(k)}_{\mu}(x) \mapsto a^{(k)}_{\mu}(x)\mathcal{B}(x)^{k} - \delta_{k,0}\partial_\mu \ln \mathcal{B}(x),\ p^{(k)}_{\mu}(x) \mapsto p^{(k)}_{\mu}(x)\mathcal{B}(x)^{d-2+k}$$

Weyl covariant subleading tensors!
Define the non-coordinatized basis

$$\{e = L\Big(\frac{\mathrm{d}z}{z} - a_\mu(x,z)\mathrm{d}x^\mu\Big), \mathrm{d}x^\mu\} \ \Rightarrow \ g = e \otimes e + h_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$$

Dual vectors $\{\underline{e} = \frac{z}{L}\partial_z, \underline{D}_\mu = \partial_\mu + a_\mu(x,z)z\partial_z\}$
The bulk LC connection in this frame is given by

$$\nabla_{\underline{D}_\mu}\underline{D}_\nu = \Gamma^\lambda_{\mu\nu}\underline{D}_\lambda + \Gamma_{\mu\nu}\underline{e}$$

Weyl-Fefferman-Graham gauge V

$\Gamma^{\lambda}_{\mu\nu}$ induces on the boundary a *Weyl connection*, instead of the usual LC one:

$$\Gamma^{\lambda}_{\mu\nu} = \gamma^{(0)}{}^{\lambda}_{\mu\nu} + O(z^2/L^2)$$

with

$$\gamma^{(0)}{}^{\lambda}_{\mu\nu} = \frac{1}{2}\gamma^{\lambda\rho}_{(0)}\Big((\partial_\mu - 2a^{(0)}_\mu)\gamma^{(0)}_{\nu\rho} + (\partial_\nu - 2a^{(0)}_\nu)\gamma^{(0)}_{\mu\rho} - (\partial_\rho - 2a^{(0)}_\rho)\gamma^{(0)}_{\mu\nu}\Big)$$

Weyl enhancement: induced boundary connection not the $\gamma^{(0)}_{\mu\nu}$'s LC.

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BULK: $g_{AB} = \{a_\mu, h_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu}{}^{LC}\}$ $\Rightarrow$ BOUNDARY: $\{a^{(0)}_\mu, \gamma^{(0)}_{\mu\nu}, \gamma^{(0)}{}^{\lambda}_{\mu\nu}{}^{Weyl}\}$

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Remark: e defines a distribution $D(e) \subset TM$ (M = bulk manifold)

$$D(e) = \ker(e) = \{\underline{X} | i_{\underline{X}} e = 0\} = \text{span}\{\underline{D}_\mu\}$$

$D(e)$ integrable $\Leftrightarrow$ Frobenius condition $f_{\mu\nu} = D_\mu a_\nu - D_\nu a_\mu = 0$
This is not required in our holographic setup.

Intermezzo: What is a Weyl Connection?

Levi-Civita connection ∇ is the unique connection satisfying

$$\text{i) metricity: } \nabla g = 0, \qquad \text{ii) torsionless: } T(\underline{X}, \underline{Y}) = 0$$

Tensorial expressions $\Rightarrow$ its curvatures are tensors

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Weyl transformation $g \to \Omega^{-2} g$. Not a diffeomorphism
i) is not Weyl invariant. Usually: require the Γs to shift under Weyl.
Problem: curvatures not Weyl tensors $\Rightarrow$ Promote to Weyl metricity

$$\text{i*)}\ \text{Weyl metricity: } \nabla g - 2a \otimes g = 0$$

With a a one form, shifting non-linearly under Weyl $a \to a - \mathrm{d} \ln \Omega$
Theorem: ∇ satisfying i*) and ii) is the unique *Weyl connection*

[Folland '70, Hall '92]

Curvature tensors Weyl covariant, impo for holography

Back to Holography: Weyl Anomaly I

Weyl anomaly obtained comparing bulk partition functions in different coordinates (Weyl diffeomorphism) to its boundary dual

$$\frac{Z_{bulk}\left[g', z', x'; \gamma'_{(0)}\right]}{Z_{bulk}\left[g, z, x; \gamma_{(0)}\right]} = 1 \qquad \frac{Z_{bdy}[x; \gamma'_{(0)}, a'_{(0)}, ...]}{Z_{bdy}[x; \gamma_{(0)}, a_{(0)}, ...]} \stackrel{?}{=} 1$$

In even boundary dimension $d = 2k$, an anomaly $\mathcal{A}_k$ arises

$$\begin{aligned} 0 &= S_{bulk}[g'; \gamma'_{(0)}, ...|z', x] - S_{bulk}[g; \gamma_{(0)}, ...|z, x] \\ &= S_{bdy}[x; \gamma'_{(0)}, a'_{(0)}, ...] - S_{bdy}[x; \gamma_{(0)}, a_{(0)}, ...] + \mathcal{A}_k \end{aligned}$$

Difference of bulk actions in Weyl-diffs-related coordinate systems = difference of boundary actions in Weyl-related backgrounds,

up to an anomalous term

[Henningson, Skenderis ’98]

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Back to Holography: Weyl Anomaly II

$$S_{bulk}[g;\gamma_{(0)},...|z,x] = \frac{1}{16\pi G}\int_M e\wedge \mathrm{d}^d x\sqrt{-\det h}(R-2\Lambda)$$

On shell, it evaluates to

$$S_{bulk} = -\frac{d}{8\pi GL}\int_M \frac{\mathrm{d}z}{z}\wedge \mathrm{d}^d x\sqrt{-\det h}$$

Expand $\sqrt{-\det h}$, defining $X^{(1)} = \gamma^{\mu\nu}_{(0)}\gamma^{(2)}_{\mu\nu}$

$$\sqrt{-\det h(z,x)} = \left(\frac{L}{z}\right)^d \sqrt{-\det \gamma^{(0)}}\left[1+\frac{1}{2}\frac{z^2}{L^2}X^{(1)}+...\right]$$

Go to $d=2$ and evaluate the difference of bulk Weyl-related actions:

$$\mathcal{A}_1 = \frac{1}{8\pi GL}\int_\Sigma \ln\mathcal{B}X^{(1)}\sqrt{-\det\gamma^{(0)}}\,\mathrm{d}^d x$$

Back to Holography: Weyl Anomaly III

The resolution of Einstein equations gives

$$X^{(1)} = -\frac{L^2}{2(d-1)} R^{(0)}$$

where $R^{(0)}$ is the *Weyl Ricci scalar* of $\gamma^{(0)}$, weight-2 Weyl covariant

Therefore

$$\boxed{\mathcal{A}_1 = -\frac{L}{16\pi G} \int_\Sigma \ln \mathcal{B}\; R^{(0)} \sqrt{-\det \gamma^{(0)}}\; \mathrm{d}^d x}$$

Back to Holography: Weyl Anomaly IV

Comments

- It leads to the correct central charge $c = \frac{3L}{2G}$ **[Brown, Henneaux '85]**
- However, not the usual LC Ricci but its Weyl covariant version
- Weyl covariance has to be a feature of $\mathcal{A}_k$, $\forall k$, by construction

Back to Holography: Weyl Anomaly IV

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The Weyl anomaly can be expressed cohomologically [Mazur, Mottola '01]

- The bulk on-shell action is proportional to the total (divergent) volume $e \wedge vol_S$, with $vol_S = \sqrt{-\det h}\mathrm{d}^d x$
- Is a top form so it is closed
- Weyl anomaly: two Weyl-related volumes differ by a top form

$$(e\wedge Tr(\gamma^{(2)})Vol_S)' - (e\wedge Tr(\gamma^{(2)})Vol_S) = -\frac{L}{16\pi G}\mathrm{d}(\ln\mathcal{B}\; R^{(0)}\; Vol_S)$$

Outlook

The Weyl-Fefferman-Graham gauge has led to a rich holographic structure. It would be worth to:

- Pursue the computation to higher dimensions
- Weyl boundary field theory, Ward identities and properties
- Non-Einsteinian modifications of gravity
- Holography with a non involutive boundary

Questions