Humboldt Kolleg
Corfu, 16/09/2019

THE STRING GEOMETRY BEHIND TOPOLOGICAL AMPLITUDES

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to appear … soon!

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THE STRING GEOMETRY BEHIND TOPOLOGICAL AMPLITUDES

OVERVIEW

TOPOLOGICAL AMPLITUDES

FLUX-TUBE (MELVIN) GEOMETRY

HETEROTIC STRING ON FLUX TUBES

INTRODUCTION

In the mid 90's, Antoniadis, Gava, Narain and Taylor studied a series of higher derivative couplings in string theory to put under non-trivial scrutiny string/string dualities.

$$\int d^4\theta\, F_g(X)\, W^{2g} \sim F_g\, R^2\, T^{2g-2}$$

$W = T + \theta R \theta$

These F-term couplings are computed by the genus-g topological string on a suitable Calabi-Yau or, using the heterotic/type II duality, by suitable topological amplitudes in the heterotic side.

INTRODUCTION

In the early 2000's a renewed interest on these couplings was triggered by the work on Nekrasov on the solution of the dynamics of N=2 gauge theories

$$ds^2 = A\,dz\,d\bar{z} + g_{IJ}\left(dx^I + \Omega^I{}_K x^K\,dz + \bar{\Omega}^I{}_K x^K\,d\bar{z}\right)\left(dx^J + \Omega^J{}_L x^L\,dz + \bar{\Omega}^J{}_L x^L\,d\bar{z}\right)$$

This background lifts the instanton moduli space, leaving only a finite number of isolated points as a full set of supersymmetric minima of the action. One is thus left to compute ratios of determinants near each critical point

INTRODUCTION

Nekrasov and Okounkov observed a puzzling coincidence in the case of a single parameter background

Note that up to the terms of instanton degree zero the function $\gamma_{\hbar}(x|\beta;\Lambda)$ *coincides with the all-genus free energy of the type A topological string on the resolved conifold, with* βx *being the Kähler class of the* $\boldsymbol{P}^1$*, and* $\beta\hbar$ *the string coupling.*

Alternatively, the perturbative contribution is captured by topological amplitudes computing (higher-derivative) F-terms of the form $F_g W^{2g}$.

Why topological amplitudes have anything to do with the (perturbative) Nekrasov free energy?

What is the string realisation of the Omega background?

TOPOLOGICAL AMPLITUDES

The topological amplitudes corresponding to higher derivative F-term of the form $F_g W^{2g}$ can be computed in heterotic string as

$$\mathscr{A}_g = \langle V_{\rm grav}^2 \, V_{\rm gph}^{2g-2} \rangle$$

graviton

graviphoton

in a background which preserves N=2 supersymmetries, *i.e.* K3xT^2

TOPOLOGICAL AMPLITUDES

A suitable choice of space-time momenta drastically simplifies the computation of the amplitude, although one is still left with

$$G_g \equiv \left\langle \prod_{i=1}^{g} \int d^2x_i \, Z^1 \bar{\partial} Z^2(x_i) \prod_{j=1}^{g} \int d^2y_j \, \bar{Z}^2 \bar{\partial} \bar{Z}^1(y_j) \right\rangle$$

To this end, one actually computes the generating function

$$G(\lambda) = \sum_{g=1}^{\infty} \frac{1}{(g!)^2} \left(\frac{\lambda}{\tau_2}\right)^{2g} G_g$$

TOPOLOGICAL AMPLITUDES

The advantage of having introduced G is that it can be expressed as the normalised Gaussian functional integral

$$G(\lambda) = \frac{\int[\mathscr{D}Z\mathscr{D}\bar{Z}]\exp\left(-S_0 + \frac{\lambda}{\tau_2}\int d^2x(Z^1\bar{\partial}Z^2 + \bar{Z}^2\bar{\partial}\bar{Z}^1)\right)}{\int[\mathscr{D}Z\mathscr{D}\bar{Z}]\exp(-S_0)}$$

Free action for Z^1 and Z^2

It can be straightforwardly computed (using zeta-function regularisation) to yield

$$G(\lambda) = \left(\frac{2\pi i\lambda\bar{\eta}^3}{\bar{\vartheta}_1(\lambda|\bar{\tau})}\right)^2 e^{-\pi\lambda^2/\tau_2}$$

TOPOLOGICAL AMPLITUDES

Back to the amplitude, the generating function of topological amplitudes is

$$
\begin{aligned}
F(\lambda) &= \sum_{g=1}^{\infty} \lambda^{2g} F_g \\
&= \int_{\mathscr{F}} \frac{d^2\tau}{\tau_2} F(\bar{\tau}) \sum_{m,n} \left(\frac{2\pi i \lambda \bar{\eta}^3}{\bar{\vartheta}_1(\tilde{\lambda}|\bar{\tau})} \right)^2 e^{-\pi \lambda^2 \tau_2}\, q^{\frac{1}{2}|p_L|^2}\, \bar{q}^{\frac{1}{4}|p_R|^2}
\end{aligned}
$$

$\tilde{\lambda} \propto \lambda p_L \tau_2$

encodes the contribution of the internal coordinates

TOPOLOGICAL AMPLITUDES

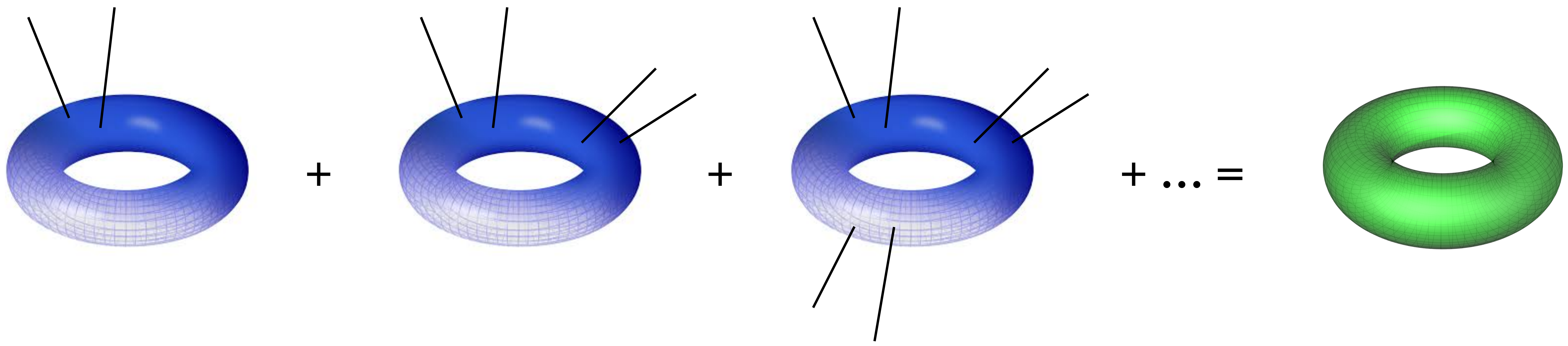
What can/do we “learn” from all this?

$$F(\lambda) = \int [\mathscr{D}Z\mathscr{D}\bar{Z}][\mathscr{D}X_{K3}][\mathscr{D}\lambda_{\text{gauge}}] \exp\left(-S_{K3} - S_{\text{gauge}} - S_0 + \frac{\lambda}{\tau_2}\int d^2x (Z^1\bar{\partial}Z^2 + \bar{Z}^2\bar{\partial}\bar{Z}^1)\right)$$

Are topological amplitudes equivalent to
“free energies” of strings on “non-trivial” backgrounds?

TOPOLOGICAL AMPLITUDES

After all …



the vertex operators effectively modify the geometry of space-time!

FLUX-TUBE (MELVIN) GEOMETRY

What is then the geometry behind the topological amplitudes?

The graviphoton vertex operators correspond to anti-self-dual gauge field configurations (in Euclidean space-time)

The generating function, on the other hand, involves (anti-chiral) rotations on the non-compact directions

How to combine these two aspects?

FLUX-TUBE (MELVIN) GEOMETRY

Consider the four-dimensional geometry

$$
\begin{aligned}
ds_4^2 &= d\rho_1^2 + \rho_1^2 \tilde{G}(1+q_2^2\rho_2^2)d\varphi_1^2 + d\rho_2^2 + \rho_2^2\tilde{G}(1+q_1^2\rho_1^2)d\varphi_2^2 - 2\tilde{G}q_1 q_2\rho_1^2\rho_2^2 d\varphi_1 d\varphi_2\,,\\
A &= \tilde{G}(q_1\rho_1^2 d\varphi_1 + q_2\rho_2^2 d\varphi_2)\,,\\
\phi &= \phi_0\,,\\
e^{2\sigma} &= \tilde{G}^{-1}
\end{aligned}
$$

$$
\tilde{G}^{-1} = 1 + q_1^2\rho_1^2 + q_2^2\rho_2^2
$$

FLUX-TUBE (MELVIN) GEOMETRY

It includes uniform magnetic fields

$$F^{ab} = 2\tilde{G}\begin{pmatrix} 0 & -q_1 & 0 & 0 \\ q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q_2 \\ 0 & 0 & q_2 & 0 \end{pmatrix} \qquad \tilde{F}_{ab} = 2\tilde{G}\begin{pmatrix} 0 & -q_2 & 0 & 0 \\ q_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q_1 \\ 0 & 0 & q_1 & 0 \end{pmatrix}$$

which are (anti)self-dual (at leading order) if $\boldsymbol{q_1=q_2}$

FLUX-TUBE (MELVIN) GEOMETRY

The sigma model associated to this background geometry is

$$\mathscr{L} = \partial\rho_1\bar{\partial}\rho_1 + \rho_1^2(\partial\varphi_1 + q_1\partial y)(\bar{\partial}\varphi_1 + q_1\bar{\partial} y) + (1 \to 2) + \partial y\bar{\partial} y$$

and corresponds to free fields if

$$\varphi_i \to \phi_i = \varphi_i + q_i y$$

$$\mathscr{L} = \sum_{i=1,2} \partial\rho_i\bar{\partial}\rho_i + \rho_i^2\partial\phi_i\bar{\partial}\phi_i + \partial y\bar{\partial} y = \sum_i \partial Z_i\bar{\partial}\bar{Z}_i + \partial y\bar{\partial} y$$

FLUX-TUBE (MELVIN) GEOMETRY

The new angular coordinate is not a *true angular coordinate,*
and therefore Z is not periodic

$$Z_i(\sigma+\pi,\tau) = e^{2\pi i n q_i R}\, Z_i(\sigma,\tau)$$

Z is rotated by the angle $\boldsymbol{2\pi n q_i R}$

The (anti)self-duality of the magnetic field
is equivalent to opposite (equal) rotations of the two planes.

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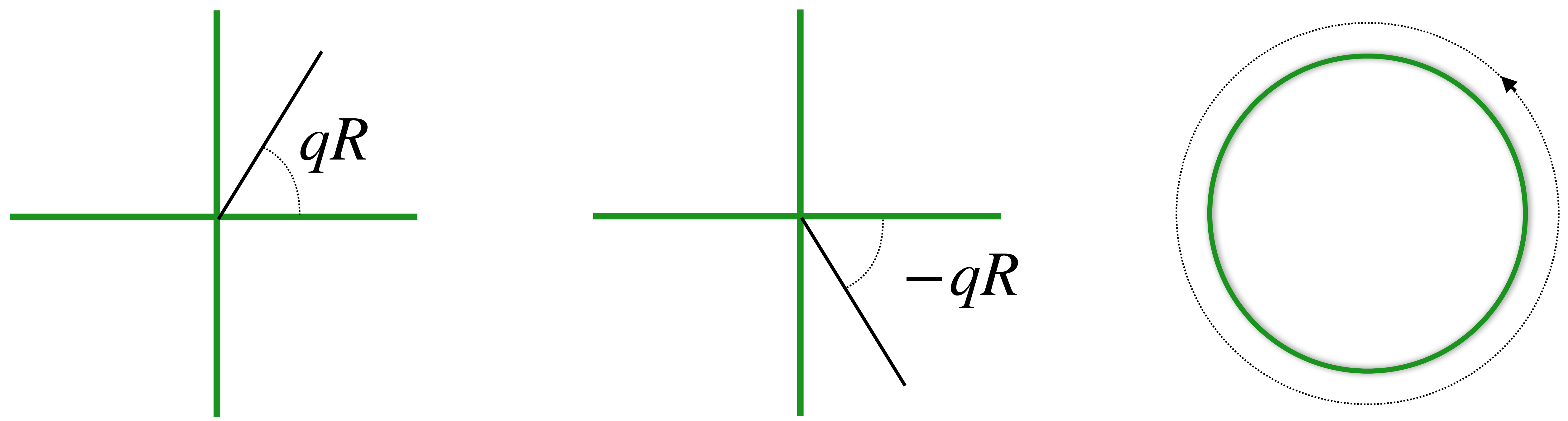
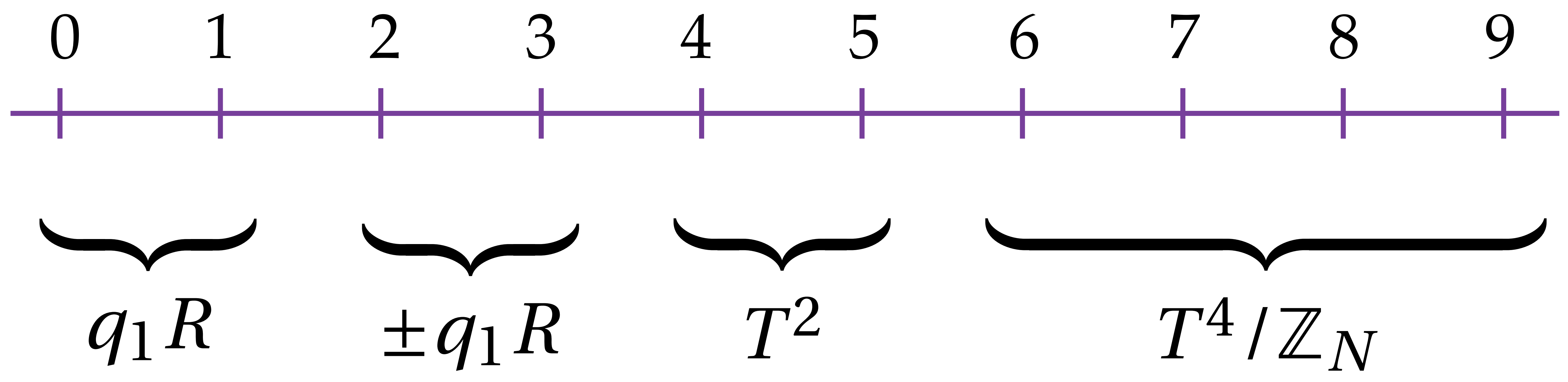


half of the supersymmetries are preserved

THE GEOMETRY BEHIND TOPOLOGICAL AMPLITUDES



THE HETEROTIC STRING FREE ENERGY

… can be calculated using standard techniques and Riemann identity

$$\mathscr{F}(\lambda) = -\int_{\mathscr{F}} \frac{d^2\tau}{\tau_2} \sum_{g,h} \frac{\left(\sum_{k,l} \bar{\theta}^6\left[{k \atop l}\right] \bar{\theta}\left[{k+g/2 \atop l+h/2}\right] \bar{\theta}\left[{k-g/2 \atop l-h/2}\right]\right) \left(\sum_{\rho,\sigma} \bar{\theta}^8\left[{\rho \atop \sigma}\right]\right)}{\bar{\eta}^{18} \bar{\theta}\left[{1/2+g/2 \atop 1/2+h/2}\right] \bar{\theta}\left[{1/2-g/2 \atop 1/2-h/2}\right]}$$

$$\times \sum_{m_i,n_i} \left(\frac{2\pi i \lambda \bar{\eta}^3}{\bar{\vartheta}_1(\tilde{\lambda}|\tau)}\right)^2 e^{-\pi \tilde{\lambda}^2/\tau_2}\; q^{\frac{1}{4}|p_{\mathrm{L}}|^2}\, \bar{q}^{\frac{1}{4}|p_{\mathrm{R}}|^2},$$

THE HETEROTIC STRING FREE ENERGY

$$\mathscr{F} = -\int_{\mathscr{F}} \frac{d^2\tau}{\tau_2} F(\bar{\tau}) \sum_{m_i,n_i} \left(\frac{2\pi i \lambda \bar{\eta}^3}{\bar{\vartheta}_1(\tilde{\lambda}|\bar{\tau})}\right)^2 e^{-\pi\tilde{\lambda}^2/\tau_2}\, q^{\frac{1}{4}|p_{\rm L}|^2}\, \bar{q}^{\frac{1}{4}|p_{\rm R}|^2}$$

This expression matches the generating function for the topological amplitude

In the field theory limit reproduces the (perturbative) result of Nekrasov

$$\sum_{i,j} \int_0^\infty \frac{dt}{t} \sum_n \frac{1}{\sin^2(\hbar n t)}\, e^{-\pi t(n+a_i-a_j)^2/R^2}$$

A similar calculation can be performed also for open strings.

In this case, BPS states are not stringy and the resulting expression is

$$\mathscr{A} = 2\pi R_1 \sum_{i,j=1}^{M} \int_0^\infty \frac{dt}{t^{3/2}} \sum_{\tilde{k}_1 \neq 0} \sum_{k_2 \in \mathbb{Z}} \frac{e^{-\pi (R_1 \tilde{k}_1)^2/\alpha' t - \pi t \left(\frac{k_2}{R_2} + a_i - a_j \right)^2}}{\left[2 \sin(\pi q R_1 \tilde{k}_1) \right]^2} \, ,$$

Making the K3 orbifold freely acting, describes the case of N=2* theories

THE (REFINED) STRINGY OMEGA BACKGROUND

Why is it important the correct identification of the Nekrasov background?

$$ds^2 = A\,dz d\bar{z} + g_{IJ}\left(dx^I + \Omega^I{}_K x^K\, dz + \bar{\Omega}^I{}_K x^K\, d\bar{z}\right)\left(dx^J + \Omega^J{}_L x^L\, dz + \bar{\Omega}^J{}_L x^L\, d\bar{z}\right)$$

Generally, the eigenvalues of the Omega deformation can be different.

How to describe this situation in string theory?

How to implement this refinement in topological string/amplitudes?

THE (REFINED) STRINGY OMEGA BACKGROUND

Despite many attempts, both in topological strings and topological amplitudes, the ***world-sheet description*** of the refinement is still an open issue.

A correct identification of the Omega Background in string theory can definitely shed some light on the correct way to refine the world-sheet!

THANK YOU