Connections, Torsion & Curvature in Generalized Geometry

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Overview

Effective action:

$$S[g, B, \phi] =$$

$$\int_M e^{-2\phi} \left\{ R(g) - \frac{1}{2} \langle H', H' \rangle_g + 4 \langle d\phi, d\phi \rangle_g \right\}$$

g, B, ϕ - metic, B-field, dilaton

$H' = dB + H$, H - closed 3-form

Bosonic part of type II supergravity (RR-fields omitted)

Motivation: Understand the generalized Riemannian geomety of EAs and their (quasi-) PLTD

$M = P/G$, P - principal D-bundle

D - connected Lie, $\underline{d}$ - quadratic $\langle \cdot, \cdot \rangle_{\underline{d}}$

$G \subset D$ - connected, closed Lagrangian w.r. to $\langle \cdot, \cdot \rangle_{\underline{d}}$

In this talk: $P = D$, i.e., no spectators for simplicity

Courant algebroids

Definition CA data:

- Vector bundle $q: E \to M$
- Morphism $\rho: E \to TM$ - anchor
- Fibre-wise metric $\langle \cdot, \cdot \rangle_E$ on E
- $\mathbb{R}$-bilinear bracket $[\cdot, \cdot]_E$ on $\Gamma(E)$

Axioms:

- $[\psi, \cdot]_E$ is a differential operator

Leibniz $[\psi, f\psi']_E = f[\psi, \psi']_E + \rho(\psi)f \cdot \psi'$

- $[\psi, \cdot]_E$ is a derivation of the bracket

Jacobi

- The bracket $[\cdot, \cdot]_E$ and pairing $\langle \cdot, \cdot \rangle_E$ are compatible
- $\langle [\psi, \psi]_E, \psi' \rangle_E = \frac{1}{2} \rho(\psi') \langle \psi, \psi \rangle_E$

Example $E = TM \oplus T^*M$

ρ - projection to TM, $\langle \cdot, \cdot \rangle_E$ - canonical

$$[(X, \alpha), (Y, \beta)] = ([X, Y], L_X \beta - i_Y d\alpha - H(X, Y, \cdot))$$

$H \in \Omega^3_{cl}(M)$. H-twisted Dorfman bracket

Exact CA

Definition (generalized metric)

- a maximal positive subbundle

$$V_+ \subseteq E \quad \text{w.r. to} \quad \langle \cdot , \cdot \rangle_E \equiv g_E .$$

- $E = V_+ \oplus V_-$, $V_- := V_+^{\perp}$
- $\tau \in \mathrm{End}(E)$ $\quad \tau(V_\pm) = \pm 1 \quad \tau^2 = 1$

$$G(\psi, \psi') := \langle \psi, \tau(\psi') \rangle_E$$ – fibre-wise metric on E >0

- $(E, \langle \cdot , \cdot \rangle_E)$ orthogonal v.b. has a generalized metric

  $O(E, g_E)$ acts transitively on space of g.ms

- $TM \oplus T^*M$ $\quad V_+$ – graph of a bundle map $TM \to T^*M$.

$$\Gamma(V_+) = \{ (X, (g+B)(X)) , X \in \mathfrak{X}(M) \}$$

g – Riemannian metric , $B \in \Omega^2(M)$

CA connections

Definition CA conn $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$

s.t. $\nabla_\psi \equiv \nabla(\psi, \cdot) : \Gamma(E) \to \Gamma(E)$ satisfies

- $\nabla_\psi(f\psi') = f\nabla_\psi(\psi') + \rho(\psi)f \cdot \psi'$
- $\nabla_{f\psi}(\psi') = f\nabla_\psi(\psi')$

and is compatible with $\langle \cdot, \cdot \rangle_E$, i.e.,

$$\nabla g_E = 0.$$

Example ∇' - ordinary v.b. connection on E

$$\nabla_\psi := \nabla'_{\rho(\psi)} \text{ is a CA connection}$$

Definition (Gualtieri) Torsion 3-form

$$T_\nabla(\psi, \psi', \psi'') = \langle (\nabla_\psi \psi' - \nabla_{\psi'} \psi) - [\psi, \psi'], \psi'' \rangle + \langle \nabla_{\psi''}\psi, \psi' \rangle_E$$

- $C^\infty(M)$-linear, skew-symmetric

∇ is torsion-free if $T_\nabla = 0$.

– works for CA-connections

<u>Definition</u>

$V_+ \subseteq E$ — generalized metric.

∇ is a Levi-Civita w.r.t. to V_+, $\nabla \in LC(E, V^+)$

- $\nabla_\psi (V_+) \subseteq V_+$
- is torsion-free.

— $LC(E, V_+) \neq \phi$, there are many LC connections

<u>Definition</u> (Hohm, Zwiebach)

Curvature tensor

$$R_\nabla(\phi; \phi, \psi, \psi') =$$

$$\frac{1}{2} \langle ([\nabla_\psi, \nabla_{\psi'}] - \nabla_{[\psi, \psi']_E}) \phi, \phi' \rangle_E$$

$$+ \frac{1}{2} \quad (\psi \Leftrightarrow \phi \quad \psi' \Leftrightarrow \phi')$$

$$+ \frac{1}{2} \langle \nabla_{\psi_M} \psi, \psi' \rangle_E \langle \nabla_{\psi_E^M} \phi, \phi' \rangle_E$$

- geometrical meaning ?
- R_∇ has all the usual symmetries including alg. Bianchi

Definition

generalized Ricci tensor

$$Ric_\nabla(\psi, \psi') = R_\nabla(g_E^{-1}(\psi^n)\psi, \psi_n, \psi')$$

- symmetric, $C^\infty(M)$-bilinear.

∇ is Ricci-compatible with $V_+ \equiv$

$$Ric_\nabla(V_+, V_-) = 0$$

Definition Scalar curvatures R_∇, R_∇^+

$$R_\nabla = Ric_\nabla(\psi_n, g_E^{-1}(\psi^n))$$

$$R_\nabla^+ = Ric(\psi_n, G^{-1}(\psi^n)).$$

Lemma Define $div_\nabla(\psi) = \langle \nabla_{\psi_n}(\psi), \psi^n \rangle$.

If for $\nabla, \nabla' \in LC(E, V_+)$ $div_{\nabla'} = div_\nabla$

then

$$R_{\nabla'} = R_\nabla, \; R^+_{\nabla'} = R^+_\nabla$$

and $\overset{+-}{Ric}_{\nabla'} = \overset{+-}{Ric}_\nabla$

Theorem $E = TM \oplus T^*M$ with $[\cdot, \cdot]_H$.

Let V_+ corresponds to generalized metric (g, B) and $\nabla \in LC(E, V_+)$ satisfies $div_\nabla \psi = div_{\nabla_g^{LC}} \rho(\psi) - \rho(\psi)\phi$ for some $\phi \in C^\infty(M)$, $\nabla \in LC(E, V_+, \phi)$.

Then (g, B, ϕ) satisfies EOM given by $S[g, B, \phi]$ iff $R^+_\nabla = 0$ & ∇ is Ricci compatible with V_+. (7)

- Recall, R^+_∇ and Ric^{+-}_∇ do not depend on the choice of $\nabla \in LC(E, V_+, \phi)$
- Everything behaves nicely under CA isomorphism ("covariant" description).

KK reduction

- $$\begin{array}{c} P \leftarrow G \\ \pi \downarrow \\ M \end{array}$$ G - compact
  $c = (\cdot, \cdot)$ - Killing form

g_P - adjoint bundle

- A - connection on P, $A \in \Omega^1(P, g)$
  with curvature $F \in \Omega^2(M, g_P)$

$H_0 \in \Omega^3(M)$

- $E' = TM \oplus g_P \oplus T^*M$ (pre-) Courant algebroid
  - pairing is the canonical one on $TM \oplus T^*M$ and induced by $(\cdot, \cdot)$ on g_P.
  - anchor - projection to TM
  - bracket - combination of the H_0-twisted bracket and the Atiyah Lie-algebroid bracket on $TM \oplus g_P$

- It is a Courant algebroid if

$$dH_0 + \frac{1}{2}(F \wedge F)_g = 0$$

- generalised metric $V_+' \subset E'$

$\sim (g_0, B_0, \vartheta)$ $\quad \vartheta \in \Omega^1(M, g_P)$

- Effective action

$$S_0[g_0, B_0, \phi_0, A, \vartheta] =$$

$$= \int_M e^{-2\phi_0}\Big(R(g_0) + \tfrac{1}{2}\langle F', F'\rangle_{g_0,c} - \tfrac{1}{2}\langle H_0', H_0'\rangle_{g_0} + 4\langle d\phi_0, d\phi_0\rangle_{g_0} - 2\Lambda_0 \Big)$$

$$F' = F'(A, \vartheta) \,, \quad H'(B_0, F, \vartheta, H_0)$$

Einstein–Yang–Mills gravity

- $P = P_{YM} \times_M P_{spin}$
  $SO(32)$ or $E_8 \times E_8$ $\quad$ $Spin(9,1)$
  Heterotic supergravity
- Anomaly cancellation
  inflow Green–Schwarz mech.

- Every heterotic CA E' is obtained by reduction from the exact on $E = TP \oplus TP^*$ with $H = \pi^*(H_0) + \frac{1}{2} CS_3(A)$

- G-invariant generalized metric on E reduces to a generalized metric on E' $\quad (g, B) \to (g_0, B_0, \vartheta)$

Proposition

Let $\phi = \pi^* \phi_0$. Then there exist connections $\nabla \in LC(E, V_+, \phi)$ and $\nabla' \in LC(E', V_+', \phi')$ s.t. ∇ reduces to ∇'. ∇ is Ricci compatible with V_+, iff ∇' is Ricci compatible with V_+'

and $$R^+_\nabla = R^+_{\nabla'} \circ \pi + \frac{1}{6} \dim(\mathfrak{g})$$

Theorem (KK reduction)

For (g, B, ϕ) and (g_0, B_0, ϕ_0) related as above and for $\Lambda = \Lambda_0 + \frac{\dim \mathfrak{g}}{6}$ EOM for S are equivalent to those of S_0.

Poisson-Lie T-duality

- KLIMČÍK, ŠEVERA 1994
  ŠEVERA – in terms of CA and their reductions 2015, ...

- Simplest setting
  $(\underline{d}, g)$ – Manin pair
  $(\underline{d}, \langle\cdot,\cdot\rangle_{\underline{d}}, [\cdot,\cdot]_{\underline{d}})$ – quadratic Lie Alg
  $g \subset \underline{d}$ Lagrangian
  Assume (D, G) integrate $(\underline{d}, g)$
  $G \subset D$ – closed subgroup

- $$\begin{array}{ccc} P = D \leftarrow D & \qquad & P = D \leftarrow G \\ \pi \downarrow & & \pi_0 \downarrow \\ * & & N = D/G \end{array}$$

- CA $E = TP \oplus T^*P$, $H = \frac{1}{2} CS_3(\theta_L)$
  – reduction of of E by D
  $E'_{\underline{d}} = (\underline{d}, 0, \langle\cdot,\cdot\rangle_{\underline{d}}, -[\cdot,\cdot]_{\underline{d}})$
  – reduction by G Exact
  $E'_{g} = N \times \underline{d} \simeq TN \oplus T^*N$
  anchor given by the left action of $\underline{d}$ on N (dressing action)
  bracket and pairing – fibre-wise extension from $\underline{d}$

PLTD – sigma models

- Choose a g.m. $\mathcal{E}_+ \subset E'_{\underline{d}} = \underline{d}$, i.e., a maximal positive subspace w.r. to $\langle \cdot , \cdot \rangle_{\underline{d}}$
- $V'_+ := N \times \mathcal{E}_+$ is a g.m. in $E'_{\mathfrak{g}} = N \times \underline{d}$
- $E_{\mathfrak{g}} \cong TN \oplus T^*N$ $\quad H \in H^3_{cl}(N)$ – its Ševera class

  $V'_+ \sim (g, B)$
- Consider sigma model with WZW term targeted in $N = D/G$ with bacgrounds (g, B, H)

Proposition (Ševera 2017)

Fix $\mathcal{E}_+ \subset \underline{d}$. Then all sigma models (for any G) are (modulo some technical assumptions) equivalent.

– in particular for a ~~M~~anin triple $(\underline{d}, \mathfrak{g}, \mathfrak{g}^*)$ integrating to (D, G, G^*) this leads to PLT duality $(G \leftrightarrow G^*)$

PLTD - Effective actions

- in addition to the above fix a connection $\nabla^0 \in LC(\underline{d}, \mathcal{E}_+)$ similarly as above one finds $\nabla \in LC(TN \oplus T^*N, V_+)$. By construction $R^+_{\nabla^0} = R^+_{\nabla}$ and ∇^0 is Ricci compatible with $\mathcal{E}_+$ iff ∇ is Ricci compatible with V_+

- If $\underline{g}$ is unimodular, i.e.,

  $Tr(ad_x) = 0 \quad \forall x \in \underline{g}$ then

  we can choose a divergence-free ∇^0

  and the corresponding

  $\nabla \in LC(TN \oplus T^*N, \phi)$

  ϕ - an explicit expression terms of g, B and Π - the quasi-Poisson str. on N, and of Ad of D on $\underline{g}$.

  (unique up to an additive constant)

- For Manin triple $(\underline{d}, \underline{g}, \underline{g}^*)$ formulas agree with von Unge (2002, path $\int$)

Theorem

(g, B, ϕ) satisfy EOM on N iff $\nabla^0 \in LC(\underline{d}, E_+)$ is Ricci compatible with E_+ and $R^+_{\nabla_0} = 0$

- These are algebraic equations for E_+. From solutions of these we directly obtain solutions of EOM on $N = D/G$ for any choice of a Lagrangian G.

OUTLOOK

- Explicit solutions
- Including spectators
- topological T-duality
- Understand a role for the recent approach of ŠEVERA & VALACH

– Hopefully coming soon.