Conserved quantities in general relativity and anomalies

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Outline

Noether's theorem

Phase space formulation

Killing symmetries

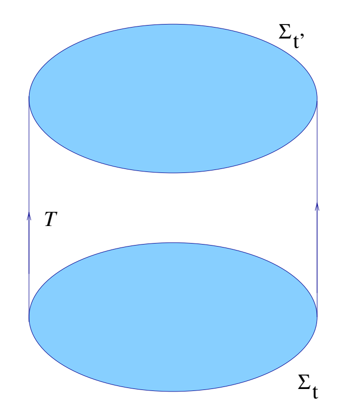
Double differential complex

Anomalies

Noether's Theorem

- Lagrangian $\boldsymbol{L}(F^I, \partial_\mu F)$, dynamical fields F^I (metric, matter),
$$\delta \boldsymbol{L}(F^I) = E_J(F^I)\delta F^J + d\boldsymbol{\theta}(F^I, \delta F^I)$$
where $E_I(F^I) = 0$ are the equations of motion.
- If $\delta_{\mathcal{Q}} = \epsilon \mathcal{T}_{\mathcal{Q}}$ generates a symmetry of L, $\delta_{\mathcal{Q}} L = 0$, then
$$d\boldsymbol{\theta} = 0$$
on shell, $\theta(S^I, \delta S^I)$ with $S^I \subset \mathcal{S}$: solutions of EoM.
$$\int_{\mathcal{M}} d\boldsymbol{\theta} = \int_{\Sigma} \boldsymbol{\theta} - \int_{\Sigma'} \boldsymbol{\theta} = 0.$$
$\mathcal{M}$ bounded by hypersurfaces Σ and Σ'.

Noether's Theorem



Noether charge, $\boldsymbol{\theta} = *\boldsymbol{j} \quad (\theta_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\sigma} j^\sigma)$

$$\mathcal{Q} = \int_\Sigma *\boldsymbol{j} = \int_{\Sigma'} *\boldsymbol{j} \quad \left(= \int_\Sigma j_\mu d\Sigma^\mu \right)$$

Phase space and symplectic form

- Under a second variation define, on-shell $S^I \subset \mathcal{S}$,

$$\omega(S^I, \delta_1 S^I, \delta_2 S^I) := \delta_1 \boldsymbol{\theta}_2 - \delta_2 \boldsymbol{\theta}_1$$

where $\boldsymbol{\theta}_1 = \boldsymbol{\theta}(S^I, \delta_1 S^I)$ and $\boldsymbol{\theta}_2 = \boldsymbol{\theta}(S^I, \delta_2 S^I)$.

Define a symplectic form $\boldsymbol{\Omega}$

$$\boldsymbol{\Theta} = \int_\Sigma \boldsymbol{\theta}, \qquad \boldsymbol{\Omega} = \boldsymbol{\delta}\,\boldsymbol{\Theta} = \int_\Sigma \boldsymbol{\omega}$$

- $\boldsymbol{\delta}$ is the exterior derivative on the space of solutions, $\boldsymbol{\delta}^2 = 0$.
- $d\boldsymbol{\delta} = \boldsymbol{\delta}\, d$.
- $\boldsymbol{L} \to \boldsymbol{L} + d\boldsymbol{\alpha} \quad \Rightarrow \quad \boldsymbol{\theta} \to \boldsymbol{\theta} + \boldsymbol{\delta}\,\boldsymbol{\alpha}$ and $\boldsymbol{\Omega} = \boldsymbol{\delta}\,\boldsymbol{\Theta}$ is invariant.

Symplectic form for scalar field theory

- Scalar field ϕ

$$\boldsymbol{L} = -\frac{1}{2} d\phi \wedge * d\phi - m^2 \phi^2 * 1$$

- $\boldsymbol{\theta} = -(\boldsymbol{\delta}\,\phi) \wedge * d\phi$
- Conjugate momentum: $\boldsymbol{\pi} = \dot{\phi}$

$$\boldsymbol{\omega} = (\boldsymbol{\delta}\,\phi) \wedge * d(\boldsymbol{\delta}\,\phi).$$

- Choose $t = const$ hypersurface Σ,

$$\boldsymbol{\Omega} = \int_\Sigma (\boldsymbol{\delta}\,\phi) \wedge * \boldsymbol{\delta}\,(d\phi) = \int_\Sigma (\boldsymbol{\delta}\,\phi) \wedge (\boldsymbol{\delta}\,\boldsymbol{\pi})\, \tilde{*}\, 1,$$

$\tilde{*}\, 1 =$ volume form on Σ.

Diffeomorphisms

- $\delta\, S^I$ on the space of solutions $\mathcal{S}$ includes gauge transformations and diffeomorphisms, $\mathcal{G}$.
- Restrict $\mathcal{S}$ to $\widehat{\mathcal{S}} = \mathcal{S}/\mathcal{G}$.
- $\widehat{\Omega}$ on $T^*\widehat{\mathcal{S}}$ must pull-back to Ω on $T^*\mathcal{S}$ under this projection.
- $\Omega = \int_\Sigma \omega$ must vanish if one of the field variations is a diffeomorphism, $\delta_{\vec{X}} S^I = \varepsilon\, \mathcal{L}_{\vec{X}} S^I$,

  (ε a constant 1-form on $\mathcal{S}$, $\varepsilon\,\delta = -\delta\,\varepsilon$).
- This will be the case if

$$\omega = d\big(\varepsilon \wedge \phi(\vec{X})\big)$$

for some $\phi(\vec{X})$ and field variations vanish on $\partial\Sigma$

Crnković + Witten (1987), Lee + Wald (1990).

Hamiltonian

- If a Hamilton $h(\vec{X})$ exists that generates the flow $\vec{X}$ then

$$\omega = -\delta\left(\varepsilon\, h(\vec{X})\right) = d\left(\varepsilon \wedge \phi(\vec{X})\right)$$

$$\Rightarrow \qquad \boldsymbol{\Omega} = 0$$

  when one of the variations is a diffeomorphism.

- Define

$$\mathcal{H}[\vec{X}] := \int_\Sigma h(\vec{X})$$

  then $\delta\,\mathcal{H}[\vec{X}] = \int_\Sigma \delta\, h(\vec{X}) = \int_{\partial\Sigma} \phi(\vec{X}) = 0$ if $\vec{X}$ vanishes on $\partial\Sigma$.

- For example in general relativity, for a time-like vector field $\vec{X}$, $\mathcal{H}[\vec{X}] = 0$ is a constraint.

Killing symmetries

- If $\vec{X} = \vec{K}$ is a Killing vector then the symplectic density is invariant, $\omega = 0$, and ϕ is d-closed
$$d\phi(\vec{K}) = 0.$$
- $\vec{K}$ need not vanish on $\partial\Sigma$.
- If Σ has an inner and an outer boundary, $\partial\Sigma = \sigma_1 \cup \sigma_2$ then
$$\int_\Sigma d\phi(\vec{K}) = \int_{\sigma_2} \phi(\vec{K}) - \int_{\sigma_1} \phi(\vec{K}) = 0.$$
$\int_\sigma \phi(\vec{K})$ is independent of the surface σ.
- If $\varepsilon \wedge \phi(\vec{K}) = \delta\,\psi(\vec{K})$ is δ-exact then $\int_\sigma \phi(\vec{K}) = \delta\,\mathcal{Q}[\vec{K}]$.
- Let $\psi(\vec{K}) = \varepsilon * \mathcal{J}(\vec{K})$: Noether 2-form

Invariant — Wald [gr-qc/9307038]

$$\mathcal{Q}(\vec{K}) = \int_\sigma *\mathcal{J}$$

Summary

- (Pre)-symplectic form $\boldsymbol{\Omega} = \int_\Sigma \boldsymbol{\omega}$. (2-form on $\mathcal{S}$)
- $\boldsymbol{\omega} = \boldsymbol{\varepsilon} \wedge d\boldsymbol{\phi}(\vec{X})$ for a diffeo. ($\vec{X} = 0$ on $\partial\Sigma$)
- For a Killing symmetry $\boldsymbol{\omega} = 0$, $\phi(\vec{K})$ is d-closed.
  ($\vec{K} \neq 0$ on $\partial\Sigma$)
- If $\boldsymbol{\phi}$ is $\boldsymbol{\delta}$-exact, $\boldsymbol{\phi} = -\boldsymbol{\delta} * \mathcal{J}$, then

$$\mathcal{Q}[\vec{K}] = \int_\sigma *\mathcal{J}$$

is an invariant associated with the Killing vector $\vec{K}$.

Mass in General Relativity

$$L = \frac{1}{16\pi} \boldsymbol{R}_{ab} \wedge * \boldsymbol{e}^{ab}, \qquad \boldsymbol{R}_{ab} = \boldsymbol{d}\boldsymbol{\omega}_{ab} + \boldsymbol{\omega}_{ac} \wedge \boldsymbol{\omega}^{c}{}_{b}$$

$$(\boldsymbol{e}^{ab} = \boldsymbol{e}^{a} \wedge \boldsymbol{e}^{b}).$$

- $\boldsymbol{\theta} = (\boldsymbol{\delta}\, \boldsymbol{\omega}_{ab}) \wedge * \boldsymbol{e}^{ab}, \qquad \phi(\vec{K}) = \frac{1}{16\pi}\left(i_{\vec{K}}\boldsymbol{\theta} + \boldsymbol{\delta} * K\right).$
- Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

- Time-like Killing vector $\vec{K} = \frac{\partial}{\partial t}$, metric variation $m \to m + \boldsymbol{\delta}\, m$, $\qquad e^2 = r\, d\vartheta,\ e^3 = r \sin\vartheta d\varphi.$

$$\phi(\vec{K}) = \frac{1}{4\pi} \frac{\boldsymbol{\delta}\, m}{r^2} \boldsymbol{e}^{23} \quad \Rightarrow \quad \mathcal{Q}[\vec{K}] = \frac{1}{4\pi} \int_{S^2} \frac{m}{r^2} \boldsymbol{e}^{23} = m.$$

- ADM mass in general, Iyer+Wald [gr-qc/9403028].
- Brown-York mass; Bondi-Sachs mass, BPD [1804.10451].

Double Complex (commutative diagram)

$$
\begin{tikzcd}
L + d\alpha \arrow[r, "\delta"'] & \delta L = d\theta \arrow[r, "\delta"'] & 0 & \\
\alpha \arrow[u, "d"'] \arrow[r, "\delta"'] & \theta + \delta\alpha + d\psi \arrow[u, "d"'] \arrow[r, "\delta"'] & \begin{array}{c} \delta\theta + \delta d\psi \\ = \omega + d\delta\psi \end{array} \arrow[u, "d"'] \arrow[r, "\delta"'] & 0 \\
 & \psi \arrow[u, "d"'] \arrow[r, "\delta"'] & \delta\psi \arrow[u, "d"'] \arrow[r, "\delta"'] & 0
\end{tikzcd}
$$

BPD [arXiv:1804.07689]

Obstruction to defining $\mathcal{Q}[\vec{K}]$ if

$$\varepsilon \wedge \phi(\vec{K}) \neq \delta\psi(\vec{K}).$$

Quantum anomalies

- $\delta \boldsymbol{L} = d\boldsymbol{\theta}$ reminiscent of the Stora-Zumino descent equations for quantum anomalies in gauge theories
  Stora-Zumino (1983), Alvarez-Gaumé+Witten, (1983)
  Alvarez-Gaumé+Ginsparg (1985).
- The differential complex structure is the same for both classically conserved quantities and quantum anomalies.

Summary

- Wald (1993) constructed a generalised Noether charge associated with any diffeomorphic invariant field theory.
- Based on differential forms on the phase space of solutions (infinite dimensional).
- Depending on the field theory there may be cohomological obstructions to defining conserved charges.
- The mathematical framework is identical to that of the Stora-Zumino consistency conditions and their role in the study of quantum anomalies.

Double complex

- Let $W^{p,q}$ be the space of p-forms on $T^*\mathcal{M}$ and q-forms on $T^*\mathcal{S}$ and let
$$W^r = \bigoplus_{p+q=r} W^{p,q}$$
be the space of forms of total degree r.
- Differential operator
$$\boldsymbol{D} = \boldsymbol{\delta} + (-1)^p d$$
maps W^r to W^{r+1} with $D^2 = 0$. BPD [1804.07689]

Diffeomorphisms

- Under a diffeomorphism $\delta_{\vec{X}} S^I = \varepsilon \mathcal{L}_{\vec{X}} S^I$ (ε is an infinitesimal constant).

$$\delta_{\vec{X}} \boldsymbol{L} \to \varepsilon \mathcal{L}_{\vec{X}} \boldsymbol{L} = \varepsilon \, d i_{\vec{X}} \boldsymbol{L}$$

$\mathcal{L}_{\vec{X}} = d\, i_{\vec{X}} + i_{\vec{X}}\, d$, $i_{\vec{X}}$ = interior derivative, $d\boldsymbol{L} = 0$.

- $\boldsymbol{\theta}(\vec{X}) = \varepsilon\, i_{\vec{X}} \boldsymbol{L} + \varepsilon\, \boldsymbol{J}(\vec{X})$ with $d\boldsymbol{J}(\vec{X}) = 0$.
- If $\boldsymbol{J}(\vec{X}) = d\boldsymbol{Q}(\vec{X})$ then

$$\boldsymbol{\theta}(\vec{X}) = \varepsilon \left(i_{\vec{X}} \boldsymbol{L} + d\boldsymbol{Q}(\vec{X}) \right).$$

Cohomology

- $\delta L = d\theta$, $L \to L + d\alpha$,
  $\delta L \to \delta L + \delta\, d\alpha$, $\theta \to \theta + \delta\, \alpha$ and $\Omega = \delta\, \theta$ is invariant because $\delta^{\,2}\alpha = 0$.

$$\begin{aligned}\omega &= \delta_{\vec{X}}\theta + \delta\, \theta(\vec{X}) \\ &= \varepsilon \wedge (i_{\vec{X}} d\theta + d i_{\vec{X}}\theta) + \delta\, \varepsilon \left(i_{\vec{X}} L + d\, Q(\vec{X})\right) \\ &= \varepsilon \wedge (i_{\vec{X}} d\theta + d i_{\vec{X}}\theta) - \varepsilon \wedge \left(i_{\vec{X}}\delta\, L + d\delta\, Q(\vec{X})\right) \\ &= \varepsilon \wedge d\left(i_{\vec{X}}\theta - \delta\, Q(\vec{X})\right) := \varepsilon \wedge d\phi(\vec{X})\end{aligned}$$

where we can take

$$\phi(\vec{X}) = i_{\vec{X}}\theta - \delta\, Q(\vec{X}).$$

Killing symmetries and Noether forms

- If $\vec{X} = \vec{K}$ is a Killing vector then

$$\omega = \varepsilon \wedge d\phi(\vec{K})$$

  is invariant and $d\phi(\vec{K}) = 0$.
- If Σ has an inner and an outer boundary, $\partial\Sigma = \sigma_1 \cup \sigma_2$
  e.g. foliate Σ with a radial co-ordinate r, then $\sigma_{r_1} = \sigma_{r_1}$ and $\sigma_2 = \sigma_{r_2}$ with $r_1 < r_2$
  then

$$\int_\Sigma d\phi(\vec{K}) = \int_{\sigma_1} \phi(\vec{K}) = \int_{\sigma_2} \phi(\vec{K}) = 0$$

  and

$$\Phi[\vec{K}] = \int_\sigma \phi(\vec{K}) = \int_\sigma \left(i_{\vec{X}}\boldsymbol{\theta} - \boldsymbol{\delta}\, \boldsymbol{Q}(\vec{X})\right)$$

  is independent of which surface it is evaluated on.
- If $i_{\vec{K}}\boldsymbol{\theta} = \boldsymbol{\delta}\, \boldsymbol{\mu}(\vec{K})$ for some $\boldsymbol{\mu}(\vec{K})$ then

$$\Phi[\vec{K}] = \boldsymbol{\delta}\, \mathcal{Q}[\vec{K}] \qquad \text{with} \qquad \mathcal{Q}[\vec{K}] = \int_\sigma \left(\boldsymbol{\mu}(\vec{K}) - \boldsymbol{Q}(\vec{K})\right).$$