EXACT RESULTS FOR AN STU-MODEL

in collaboration with Swapna Mahapatra and Gabriel Cardoso

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This STU-model was first encountered in the study of equivalent compactifications of different string theories.

Sen, Vafa, 1995
Gregori, Kounnas, Petropoulos, 1999

In its type-II description the dilaton belongs to a hypermultiplet, and therefore the *N*=2 vector multiplet moduli space does not receive quantum corrections. The symmetry group of the model is $\Gamma_0(2)_{\rm S}\times\Gamma_0(2)_{\rm T}\times\Gamma_0(2)_{\rm U}$, where $\Gamma_0(2)\subset \mathrm{SL}(2;\mathbb{Z})$.

This model has a high degree of symmetry, and therefore one may expect that important features of this model can be exactly determined. As we shall see, this is indeed the case.

arXiv: 1907:04077

Furthermore the relation with the topological string partition function can be studied in more detail based on a framework that we have presented some time ago.

Cardoso, dW, Mahapatra, 2014

The classical moduli space is a product of a special-Kähler and a quaternion-Kähler space:

$$\mathcal{M}_{\mathrm{vector}} = \frac{\mathrm{SL}(2)}{\mathrm{SO}(2)} \times \frac{\mathrm{SL}(2)}{\mathrm{SO}(2)} \times \frac{\mathrm{SL}(2)}{\mathrm{SO}(2)} \qquad \mathcal{M}_{\mathrm{hyper}} = \frac{\mathrm{SO}(4,4)}{\mathrm{SO}(4) \times \mathrm{SO}(4)}$$

The corresponding effective (supergravity) action for the vector supermultiplets is based on the holomorphic homogeneous function of degree 2, in order to have *N=2* local supersymmetry,

$$F(X) = -\frac{X^1\,X^2\,X^3}{X^0}$$

where X^0, X^1, X^2, X^3 belong to four different *N=2* vector multiplets. The exact dualities involve 8 electric and magnetic charges and constitutes the $[\Gamma_0(2)]^3$ subgroup of $\mathrm{SL}(2)_{\mathrm{S}} \times \mathrm{SL}(2)_{\mathrm{T}} \times \mathrm{SL}(2)_{\mathrm{U}}$.

EM duality

The special-Kähler coordinates are defined by

$$S = -\mathrm{i}\,\frac{X^1}{X^0} \qquad T = -\mathrm{i}\,\frac{X^2}{X^0} \qquad U = -\mathrm{i}\,\frac{X^3}{X^0}$$

To describe the coupling to the square of the Weyl tensor, we deform the function to

square of the Weyl tensor

$$F(X,A) = -\frac{X^1 X^2 X^3}{X^0} + 2\mathrm{i}\,\Omega(X,A)$$

The STU dualities (and triality!) should be preserved. This requirement is very restrictive and will enable us to explicitly determine $\Omega(X,A)$.

The actual requirement is that the *8*-component ‘period vector’ (X^I, F_J) transforms according to the (*2*,*2*,*2*) representation of the STU duality group.

This leads to the transformations equations for S-duality:

$$\begin{aligned}
X^0 &\to X^{0\,\prime} = \Delta_{\mathrm{S}}\, X^0 \\
S &\to S' = \frac{a\,S - \mathrm{i}b}{\Delta_{\mathrm{S}}} \\
T &\to T' = T + \frac{2}{\Delta_{\mathrm{S}}\,(X^0)^2}\,\frac{\partial \Delta_{\mathrm{S}}}{\partial S}\,\frac{\partial \Omega}{\partial U} \\
U &\to U' = U + \frac{2}{\Delta_{\mathrm{S}}\,(X^0)^2}\,\frac{\partial \Delta_{\mathrm{S}}}{\partial S}\,\frac{\partial \Omega}{\partial T}
\end{aligned}$$

+ triality

$$\Gamma_0(2)\ :\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$ad - bc = 1$$

$$a, d \in 2\,\mathbb{Z} + 1$$

$$b, c \in 2\,\mathbb{Z}$$

$$\Delta_{\mathrm{S}}(S) = d + \mathrm{i}c\,S$$

Leads to:

$$\left(\frac{\partial\Omega}{\partial T}\right)'_{\rm S} = \frac{\partial\Omega}{\partial T}\,, \qquad \left(\frac{\partial\Omega}{\partial U}\right)'_{\rm S} = \frac{\partial\Omega}{\partial U}$$

$$\left(\frac{\partial\Omega}{\partial S}\right)'_{\rm S} - \Delta_{\rm S}^2\,\frac{\partial\Omega}{\partial S} = \frac{\partial\Delta_{\rm S}}{\partial S}\left[-\Delta_{\rm S}\,X^0\,\frac{\partial\Omega}{\partial X^0} - \frac{2}{(X^0)^2}\,\frac{\partial\Delta_{\rm S}}{\partial S}\,\frac{\partial\Omega}{\partial T}\,\frac{\partial\Omega}{\partial U}\right]$$

$$\left(X^0\,\frac{\partial\Omega}{\partial X^0}\right)'_{\rm S} = X^0\,\frac{\partial\Omega}{\partial X^0} + \frac{4}{\Delta_{\rm S}\,(X^0)^2}\,\frac{\partial\Delta_{\rm S}}{\partial S}\,\frac{\partial\Omega}{\partial T}\,\frac{\partial\Omega}{\partial U}$$

+ triality

Cardoso, dW, Mahapatra, 2008

Local supersymmetry requires $\Omega(X,A)$ to be homogeneous and holomorphic:

$$\Omega(X,A) = A\left[\gamma\,\ln\frac{(X^0)^2}{A} + \omega^{(1)}(S,T,U) + \sum_{n=1}^{\infty}\left(\frac{A}{(X^0)^2}\right)^n\,\omega^{(n+1)}(S,T,U)\right]$$

Note the presence of the (necessary) logarithmic term, which can be interpreted as a contribution from the GV term.

Gopakumar, Vafa, 1998
Dabholkar, Denef, Moore, Pioline, 2005
Dedushenko, Witten, 2015
Denef, Moore, 2011
Sen, 2012

Duality requires that $\omega^{(1)}(S,T,U) = \omega(S) + \omega(T) + \omega(U)$ with

$$\omega(S') = \omega(S) - 2\gamma \, \ln \Delta_{\mathrm{S}}(S)$$

modular form

$$\boxed{\begin{aligned} \omega(S) &= (64\pi)^{-1} \ln \vartheta_2(S) \\ \gamma &= -\,(256\pi)^{-1} \end{aligned}}$$

Agrees with: Gregori, Kounnas, Petropoulos, 1999 !

Some higher-order contributions...

$$\omega^{(2)}(S,T,U) = \frac{1}{\gamma} \frac{\partial\omega}{\partial S} \frac{\partial\omega}{\partial T} \frac{\partial\omega}{\partial U}$$

$$\begin{aligned} \omega^{(3)} = & \frac{1}{4\gamma^3} \left(\frac{\partial\omega}{\partial S} \frac{\partial\omega}{\partial T} \frac{\partial\omega}{\partial U} \right)^2 \\ & + \frac{1}{2\gamma^2} \left[\frac{\partial^2\omega}{\partial S^2} \left(\frac{\partial\omega}{\partial T} \frac{\partial\omega}{\partial U} \right)^2 + \frac{\partial^2\omega}{\partial T^2} \left(\frac{\partial\omega}{\partial U} \frac{\partial\omega}{\partial S} \right)^2 + \frac{\partial^2\omega}{\partial U^2} \left(\frac{\partial\omega}{\partial S} \frac{\partial\omega}{\partial T} \right)^2 \right] \end{aligned}$$

$$
\begin{aligned}
\omega^{(4)} = {} & \frac{1}{6\,\gamma^5}\Big(\frac{\partial\omega}{\partial S}\,\frac{\partial\omega}{\partial T}\,\frac{\partial\omega}{\partial U}\Big)^3 \\
& + \frac{1}{2\,\gamma^4}\Big[\frac{\partial^2\omega}{\partial S^2}\,\frac{\partial\omega}{\partial S}\,\Big(\frac{\partial\omega}{\partial T}\,\frac{\partial\omega}{\partial U}\Big)^3 + \frac{\partial^2\omega}{\partial T^2}\,\frac{\partial\omega}{\partial T}\,\Big(\frac{\partial\omega}{\partial U}\,\frac{\partial\omega}{\partial S}\Big)^3 + \frac{\partial^2\omega}{\partial U^2}\,\frac{\partial\omega}{\partial U}\,\Big(\frac{\partial\omega}{\partial S}\,\frac{\partial\omega}{\partial T}\Big)^3\Big] \\
& + \frac{1}{\gamma^3}\,\Big[\Big(\frac{\partial\omega}{\partial S}\Big)^3\,\frac{\partial^2\omega}{\partial T^2}\,\frac{\partial\omega}{\partial T}\,\frac{\partial^2\omega}{\partial U^2}\,\frac{\partial\omega}{\partial U} + \Big(\frac{\partial\omega}{\partial T}\Big)^3\,\frac{\partial^2\omega}{\partial U^2}\,\frac{\partial\omega}{\partial U}\,\frac{\partial^2\omega}{\partial S^2}\,\frac{\partial\omega}{\partial S} \\
& \qquad\qquad + \Big(\frac{\partial\omega}{\partial U}\Big)^3\,\frac{\partial^2\omega}{\partial S^2}\,\frac{\partial\omega}{\partial S}\,\frac{\partial^2\omega}{\partial T^2}\,\frac{\partial\omega}{\partial T}\Big] \\
& + \frac{1}{6\gamma^3}\Big[\frac{\partial^3\omega}{\partial S^3}\,\Big(\frac{\partial\omega}{\partial T}\,\frac{\partial\omega}{\partial U}\Big)^3 + \frac{\partial^3\omega}{\partial T^3}\,\Big(\frac{\partial\omega}{\partial U}\,\frac{\partial\omega}{\partial S}\Big)^3 + \frac{\partial^3\omega}{\partial U^3}\,\Big(\frac{\partial\omega}{\partial S}\,\frac{\partial\omega}{\partial T}\Big)^3\Big]
\end{aligned}
$$

and....

$$
\begin{aligned}
\omega^{(5)} = \frac{1}{2\gamma^4}\Bigg[& \frac{\partial^3\omega}{\partial S^3}\frac{\partial\omega}{\partial S}\left[\left(\frac{\partial\omega}{\partial T}\right)^2\frac{\partial^2\omega}{\partial T^2}\left(\frac{\partial\omega}{\partial U}\right)^4+\left(\frac{\partial\omega}{\partial U}\right)^2\frac{\partial^2\omega}{\partial U^2}\left(\frac{\partial\omega}{\partial T}\right)^4\right] \\
& +\frac{\partial^3\omega}{\partial T^3}\frac{\partial\omega}{\partial T}\left[\left(\frac{\partial\omega}{\partial U}\right)^2\frac{\partial^2\omega}{\partial U^2}\left(\frac{\partial\omega}{\partial S}\right)^4+\left(\frac{\partial\omega}{\partial S}\right)^2\frac{\partial^2\omega}{\partial S^2}\left(\frac{\partial\omega}{\partial U}\right)^4\right] \\
& +\frac{\partial^3\omega}{\partial U^3}\frac{\partial\omega}{\partial U}\left[\left(\frac{\partial\omega}{\partial S}\right)^2\frac{\partial^2\omega}{\partial S^2}\left(\frac{\partial\omega}{\partial T}\right)^4+\left(\frac{\partial\omega}{\partial T}\right)^2\frac{\partial^2\omega}{\partial T^2}\left(\frac{\partial\omega}{\partial S}\right)^4\right]\Bigg] \\
+\frac{4}{\gamma^4} & \frac{\partial^2\omega}{\partial S^2}\left(\frac{\partial\omega}{\partial S}\right)^2\frac{\partial^2\omega}{\partial T^2}\left(\frac{\partial\omega}{\partial T}\right)^2\frac{\partial^2\omega}{\partial U^2}\left(\frac{\partial\omega}{\partial U}\right)^2 \\
+\frac{1}{24\,\gamma^4} & \left[\frac{\partial^4\omega}{\partial S^4}\left(\frac{\partial\omega}{\partial T}\right)^4\left(\frac{\partial\omega}{\partial U}\right)^4+\frac{\partial^4\omega}{\partial T^4}\left(\frac{\partial\omega}{\partial U}\right)^4\left(\frac{\partial\omega}{\partial S}\right)^4+\frac{\partial^4\omega}{\partial U^4}\left(\frac{\partial\omega}{\partial S}\right)^4\left(\frac{\partial\omega}{\partial T}\right)^4\right] \\
+\frac{1}{2\gamma^4} & \Bigg[\left(\frac{\partial\omega}{\partial S}\right)^4\left[\left(\frac{\partial^2\omega}{\partial T^2}\right)^2\frac{\partial^2\omega}{\partial U^2}\left(\frac{\partial\omega}{\partial U}\right)^2+\left(\frac{\partial^2\omega}{\partial U^2}\right)^2\frac{\partial^2\omega}{\partial T^2}\left(\frac{\partial\omega}{\partial T}\right)^2\right] \\
& +\left(\frac{\partial\omega}{\partial T}\right)^4\left[\left(\frac{\partial^2\omega}{\partial U^2}\right)^2\frac{\partial^2\omega}{\partial S^2}\left(\frac{\partial\omega}{\partial S}\right)^2+\left(\frac{\partial^2\omega}{\partial S^2}\right)^2\frac{\partial^2\omega}{\partial U^2}\left(\frac{\partial\omega}{\partial U}\right)^2\right] \\
& +\left(\frac{\partial\omega}{\partial U}\right)^4\left[\left(\frac{\partial^2\omega}{\partial S^2}\right)^2\frac{\partial^2\omega}{\partial T^2}\left(\frac{\partial\omega}{\partial T}\right)^2+\left(\frac{\partial^2\omega}{\partial T^2}\right)^2\frac{\partial^2\omega}{\partial S^2}\left(\frac{\partial\omega}{\partial S}\right)^2\right]\Bigg] \\
+\frac{1}{4\gamma^5} & \Bigg[\left(\frac{\partial\omega}{\partial S}\right)^4\left(\frac{\partial\omega}{\partial T}\right)^4\left[\left(\frac{\partial^2\omega}{\partial U^2}\right)^2+\frac{\partial^3\omega}{\partial U^3}\frac{\partial\omega}{\partial U}\right]+\left(\frac{\partial\omega}{\partial T}\right)^4\left(\frac{\partial\omega}{\partial U}\right)^4\left[\left(\frac{\partial^2\omega}{\partial S^2}\right)^2+\frac{\partial^3\omega}{\partial S^3}\frac{\partial\omega}{\partial S}\right] \\
& +\left(\frac{\partial\omega}{\partial U}\right)^4\left(\frac{\partial\omega}{\partial S}\right)^4\left[\left(\frac{\partial^2\omega}{\partial T^2}\right)^2+\frac{\partial^3\omega}{\partial T^3}\frac{\partial\omega}{\partial T}\right]\Bigg] \\
+\frac{2}{\gamma^5} & \Bigg[\frac{\partial^2\omega}{\partial S^2}\left(\frac{\partial\omega}{\partial S}\right)^2\left(\frac{\partial\omega}{\partial T}\right)^4\frac{\partial^2\omega}{\partial U^2}\left(\frac{\partial\omega}{\partial U}\right)^2+\frac{\partial^2\omega}{\partial T^2}\left(\frac{\partial\omega}{\partial T}\right)^2\left(\frac{\partial\omega}{\partial U}\right)^4\frac{\partial^2\omega}{\partial S^2}\left(\frac{\partial\omega}{\partial S}\right)^2 \\
& +\frac{\partial^2\omega}{\partial U^2}\left(\frac{\partial\omega}{\partial U}\right)^2\left(\frac{\partial\omega}{\partial S}\right)^4\frac{\partial^2\omega}{\partial T^2}\left(\frac{\partial\omega}{\partial T}\right)^2\Bigg] \\
+\frac{5}{8\gamma^6} & \Bigg[\frac{\partial^2\omega}{\partial S^2}\left(\frac{\partial\omega}{\partial S}\right)^2\left(\frac{\partial\omega}{\partial T}\right)^4\left(\frac{\partial\omega}{\partial U}\right)^4+\frac{\partial^2\omega}{\partial T^2}\left(\frac{\partial\omega}{\partial T}\right)^2\left(\frac{\partial\omega}{\partial U}\right)^4\left(\frac{\partial\omega}{\partial S}\right)^4 \\
& +\frac{\partial^2\omega}{\partial U^2}\left(\frac{\partial\omega}{\partial U}\right)^2\left(\frac{\partial\omega}{\partial S}\right)^4\left(\frac{\partial\omega}{\partial T}\right)^4\Bigg] \\
+\frac{5}{32\,\gamma^7} & \left(\frac{\partial\omega}{\partial S}\right)^4\left(\frac{\partial\omega}{\partial T}\right)^4\left(\frac{\partial\omega}{\partial U}\right)^4
\end{aligned}
$$

ETCETERA

Obviously, we are dealing with an infinite series. In principle this evaluation will thus become more and more involved, but we expect that, in principle, the effective Wilsonian action can be determined to any given order in perturbation theory.

However, we find it more interesting at this stage to exploit the connection with the corresponding topological string partition function, by following the prescription that was outlined a few years ago.

There is a subtle relation between the function $F(X, A)$ that encodes the effective Wilsonian action and the topological string partition function (which had been overlooked in the past).

Cardoso, dW, Mahapatra, 2014

Relation with the topological string partition function

As we have seen, the fields in the (Wilsonian) effective action transform under duality in a way that depends non-trivially on deformations. In the topological string partition function these fields are related to different moduli associated with the underlying string compactification, which are not affected by the deformations of the effective action.

This is reminiscent to the phenomenon that one observes when comparing invariances of a Lagrangian and its corresponding Hamiltonian. These two quantities are related by a Legendre transformation. For the Hamiltonian the invariance transformations constitute a subgroup of the canoncial transformations. Likewise, the effective action and the topological string partition function are related by a Legendre transformation.

The Legendre transformation of $F(X,A)$ is the so-called Hesse potential (known from real special geometry) which transforms as a real function invariant under the dualities. The Hesse potential can be decomposed into many different invariant functions,

$$\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(1)} + \mathcal{H}^{(2)} + \big(\mathcal{H}_1^{(3)} + \mathcal{H}_2^{(3)} + \text{h.c.}\big) + \mathcal{H}_3^{(3)} + \mathcal{H}_1^{(4)} + \mathcal{H}_2^{(4)} + \mathcal{H}_3^{(4)}$$
$$+ \big(\mathcal{H}_4^{(4)} + \mathcal{H}_5^{(4)} + \mathcal{H}_6^{(4)} + \mathcal{H}_7^{(4)} + \mathcal{H}_8^{(4)} + \mathcal{H}_9^{(4)} + \text{h.c.}\big) \ldots$$

They are functions of the 'canonical' variables $\mathcal{X}^I$ and for the STU-model their duality transformations are the ones associated with the classical function

$$\boxed{\mathcal{F}(\mathcal{X}) = -\,\frac{\mathcal{X}^1\mathcal{X}^2\mathcal{X}^3}{\mathcal{X}^0}}$$

The first function, $\mathcal{H}^{(0)}(\mathcal{X},\bar{\mathcal{X}}) = -\mathrm{i}\,\big[\bar{\mathcal{X}}^I\mathcal{F}_I(\mathcal{X}) - \mathcal{X}^I\bar{\mathcal{F}}_I(\bar{\mathcal{X}})\big]$, does not depend on the deformation $\Omega(\mathcal{X},A)$.

The second function $\mathcal{H}^{(1)}(\mathcal{X},\bar{\mathcal{X}})$ depends on $\Omega(\mathcal{X},A)$. From now on it is convenient to make the replacement $\Omega(\mathcal{X},A) \longrightarrow \Omega(\mathcal{X},A) + \bar{\Omega}(\bar{\mathcal{X}},\bar{A})$, which does not affect our previous analysis.

$\mathcal{H}^{(1)}(\mathcal{X},\bar{\mathcal{X}})$ can be written as an expansion in the deformation. It is the only function that is harmonic in terms of the deformation $\Omega(\mathcal{X},A)$!
This is characteristic for the topological string partition function.

Cardoso, dW, Mahapatra, 2014

These considerations have give rise to the following expression:

$$
\begin{aligned}
\mathcal{H}^{(1)} = {} & 4\,\Omega - 4\,N^{IJ}(\Omega_I\Omega_J + \Omega_{\bar I}\Omega_{\bar J}) \\
& + 16\,\mathrm{Re}\Big[\Omega_{IJ}(N\Omega)^I(N\Omega)^J\Big] + 16\,\Omega_{I\bar J}\,(N\Omega)^I(N\bar\Omega)^J \\
& - \tfrac{16}{3}\,\mathrm{Im}\,\big[\mathcal{F}_{IJK}(N\Omega)^I(N\Omega)^J(N\Omega)^K\big] \\
& - \tfrac{4}{3}\mathrm{i}\Big[\,\big(\mathcal{F}_{IJKL} + 3\mathrm{i}\mathcal{F}_{R(IJ}N^{RS}\mathcal{F}_{KL)S}\big)\,(N\Omega)^I(N\Omega)^J(N\Omega)^K(N\Omega)^L - \mathrm{h.c.}\Big] \\
& - \tfrac{16}{3}\,\big[\Omega_{IJK}(N\Omega)^I(N\Omega)^J(N\Omega)^K + \mathrm{h.c.}\big] \\
& - 16\,\big[\Omega_{IJ\bar K}(N\Omega)^I(N\Omega)^J(N\bar\Omega)^K + \mathrm{h.c.}\big] \\
& - 16\mathrm{i}\,\Big[\mathcal{F}_{IJK}N^{KP}\,\Omega_{PQ}(N\Omega)^I(N\Omega)^J(N\Omega)^Q - \mathrm{h.c.}\Big] \\
& - 16\Big[(N\Omega)^P\,\Omega_{PQ}\,N^{QR}\Omega_{RK}\,(N\Omega)^K + \mathrm{h.c.}\Big] \\
& - 16\Big[(N\Omega)^P\,\Omega_{PQ}\,N^{QR}\Omega_{R\bar K}\,(N\bar\Omega)^K \\
& \qquad\qquad + N^{IP}\,\Omega_{P\bar Q}\,N^{QR}\,\big(\Omega_{\bar R K}\,(N\Omega)^K + \Omega_{\bar R\bar K}\,(N\bar\Omega)^K\big)\,\Omega_I + \mathrm{h.c.}\Big] \\
& - 16\mathrm{i}\,\Big[\mathcal{F}_{IJK}N^{KP}\,\Omega_{P\bar Q}(N\Omega)^I(N\Omega)^J(N\bar\Omega)^Q - \mathrm{h.c.}\Big] + \mathcal{O}(\Omega^5)
\end{aligned}
$$

with $N_{IJ} \equiv 2\,\mathrm{Im}\,[\mathcal{F}_{IJ}]$.

$$\boxed{\mathcal{H}^{(1)} = 4A\,[-\gamma\ln\lambda + h(\omega,\lambda)\,] + \mathrm{h.c.}}$$

with $S = -\mathrm{i}\,\dfrac{\mathcal{X}^1}{\mathcal{X}^0}$ $\quad T = -\mathrm{i}\,\dfrac{\mathcal{X}^2}{\mathcal{X}^0}$ $\quad U = -\mathrm{i}\,\dfrac{\mathcal{X}^3}{\mathcal{X}^0}$ $\quad \lambda = \dfrac{A}{(\mathcal{X}^0)^2}$

topological string coupling constant

Generally true for deformations whose duality variation is harmonic

Comment: The logarithmic term can at a later stage be replaced by a term which is not holomorphic, but transforms in the same way under STU-duality,

$$\ln|\lambda|^2 \to \ln|\det N_{IJ}| = 2\,\ln|(S+\bar S)(T+\bar T)(U+\bar U)|$$

Hence we identify $\mathcal{H}^{(1)}(S,T,U,\lambda)$ with the topological string partition function.

The first two terms in this expansion:

$$\begin{aligned} h(S,T,U,\lambda) = {}& \omega(S)+\omega(T)+\omega(U)+\frac{\lambda}{\gamma}\Big[D_S\,\omega\;D_T\,\omega\;D_U\,\omega\Big] \\ &+\lambda^2\Bigg[\frac{1}{4\gamma^3}\,(D_S\,\omega)^2\,(D_T\,\omega)^2\,(D_U\,\omega)^2 \\ &\quad+\frac{1}{2\gamma^2}\Big[(D_S{}^2\omega)\,(D_T\,\omega)^2\,(D_U\,\omega)^2+(D_S\,\omega)^2\,(D_T{}^2\omega)\,(D_U\,\omega)^2 \\ &\qquad+(D_S\,\omega)^2\,(D_T\,\omega)^2\,(D_U{}^2\omega)\Big]\Bigg]+\mathcal{O}(\lambda^3) \end{aligned}$$

which is indeed STU-duality invariant!

$$D_S{}^n\omega\to\Delta_{\rm S}{}^{2n}\,D_S{}^n\omega \qquad S\to\frac{a\,S-{\rm i}b}{\Delta_{\rm S}} \qquad T,U\to T,U \qquad \lambda\to\frac{\lambda}{\Delta_{\rm S}{}^2}$$

Duality covariant derivatives: $D_S\,\omega = \dfrac{\partial\omega}{\partial S} - \dfrac{2\gamma}{S+\bar{S}}$ (non-holomorphic)

Note: $h(S,T,U,\lambda)$ **depends holomorphically on** ω **and** λ**, but not on** S**,** T **and** U**, because the covariant derivative is not holomorphic. This leads to a holomorphic anomaly equation,**

Bershadsky, Cecotti, Ooguri, Vafa,, 1993

$$\boxed{\frac{\partial h}{\partial \bar{S}} = \frac{2\lambda}{(S+\bar{S})^2}\, D_T\, h\, D_U\, h}$$

Equivalent to the the well-known holomorphic anomaly equation of the topological string!

(Partial) resummation of h

When suppressing the functions ω in the the topological string partition function $h(S,T,U,\lambda)$, we use

$$D_S{}^n \omega = -\frac{2\gamma}{(S+\bar{S})^n}$$

to write $h(S,T,U,\lambda)$ as a power series in terms of a single complex parameter $\tilde{\lambda}$,

$$h_0(\tilde{\lambda}) = \sum_{n=2} a_n\, \tilde{\lambda}^{n-1} = -8\,\gamma^2 \tilde{\lambda} - 32\,\gamma^3 \tilde{\lambda}^2 + \mathcal{O}(\tilde{\lambda}^3)$$

where $\tilde{\lambda} = \dfrac{\lambda}{(S+\bar{S})(T+\bar{T})(U+\bar{U})}$ *Alim, Yau, Zhou, 2015*

The holomorphic anomaly equation then yields

$$-(n-1)\,a_n = -8\gamma(n-2)\,a_{n-1} + 2\sum_{r=2}^{n-2} (r-1)(n-r-1)\,a_r\,a_{n-r}$$

which can be used to derive $\left(\tilde{\lambda}\,\dfrac{\partial h_0}{\partial \tilde{\lambda}}\right)^2 + \left(\dfrac{1}{2\,\tilde{\lambda}} - 4\gamma\right)\tilde{\lambda}\,\dfrac{\partial h_0}{\partial \tilde{\lambda}} + 4\,\gamma^2 = 0\,.$

Two solutions $\frac{\mathrm{d}h_0(\tilde{\lambda})}{\mathrm{d}\tilde{\lambda}} = \frac{2\gamma}{\tilde{\lambda}} - \frac{1}{4\,\tilde{\lambda}^2}\left[1 \mp \sqrt{1 - 16\,\gamma\,\tilde{\lambda}}\,\right]$

corresponding to the two sheets of a Riemann surface.
Instead of working in a single sheet we may be working with a single variable u on the Riemann surfaceso that

$$\tilde{\lambda} = -\frac{u^2 - 1}{16\gamma}$$

$$h_0(u) = 2\gamma\left(2\,\ln\frac{u+1}{2} - \frac{u-1}{u+1}\right)$$

with a logarithmic branchcut starting at $u = -1$ and a zero coupling point at $u = 1$.

However, this result is incomplete!

Covariantize $\tilde{\lambda} \longrightarrow \tilde{\lambda} = -\dfrac{\lambda}{8\,\gamma^3}\, D_S\omega(S)\, D_T\omega(T)\, D_U\omega(U)$

and compare to the terns we have already obtained. It then follows that there is yet an another invariant function that contributes:

$$h(S,T,U,\lambda) = \omega(S) + \omega(T) + \omega(U) + h_0(\tilde{\lambda}) + h_1(\lambda) \quad \textbf{with}$$

$$\begin{aligned}
h_1(\lambda) = \frac{\lambda^2}{2\,\gamma^2}\Big[& I^{(2)}(S)\,(D_T\,\omega)^2\,(D_U\,\omega)^2 + (D_S\,\omega)^2\, I^{(2)}(T)\,(D_U\,\omega)^2 + (D_S\,\omega)^2\,(D_T\,\omega)^2\, I^{(2)}(U)\Big] \\
+ \frac{\lambda^3}{\gamma^3}\Big[& (D_S\,\omega)^3\, I^{(2)}(T)\,(D_T\,\omega)\, I^{(2)}(U)\,(D_U\,\omega) + I^{(2)}(S)\,(D_S\,\omega)\, I^{(2)}(T)\,(D_T\,\omega)\,(D_U\,\omega)^3 \\
&+ I^{(2)}(S)\,(D_S\,\omega)\,(D_T\,\omega)^3\, I^{(2)}(U)\,(D_U\,\omega)\Big] \\
- \frac{\lambda^3}{\gamma^4}\Big[& I^{(2)}(S)\,(D_S\,\omega)\,(D_T\,\omega)^3\,(D_U\,\omega)^3 + (D_S\,\omega)^3\, I^{(2)}(T)\,(D_T\omega)\,(D_U\,\omega)^3 \\
&+ (D_S\,\omega)^3\,(D_T\,\omega)^3\, I^{(2)}(U)\,(D_U\,\omega)\Big] \\
+ \frac{\lambda^3}{6\,\gamma^3}\Big[& I^{(3)}(S)\,(D_T\,\omega)^3\,(D_U\,\omega)^3 + (D_S\,\omega)^3\, I^{(3)}(T)\,(D_U\,\omega)^3 \\
&+ (D_S\,\omega)^3\,(D_T\,\omega)^3\, I^{(3)}(U)\Big] + \mathcal{O}(\lambda^4)
\end{aligned}$$

where

$$I^{(2)}(S) = \frac{\partial^2 \omega}{\partial S^2} + \frac{1}{2\gamma} \left(\frac{\partial \omega}{\partial S}\right)^2 ,$$

$$I^{(3)}(S) = \frac{\partial^3 \omega}{\partial S^3} + \frac{3}{\gamma} \frac{\partial^2 \omega}{\partial S^2} \frac{\partial \omega}{\partial S} + \frac{1}{\gamma^2} \left(\frac{\partial \omega}{\partial S}\right)^3 ,$$

$$I^{(4)}(S) = \frac{\partial^4 \omega}{\partial S^4} + \frac{6}{\gamma} \frac{\partial^3 \omega}{\partial S^3} \frac{\partial \omega}{\partial S} + \frac{3}{\gamma} \left(\frac{\partial^2 \omega}{\partial S^2}\right)^2 + \frac{12}{\gamma^2} \frac{\partial^2 \omega}{\partial S^2} \left(\frac{\partial \omega}{\partial S}\right)^2 + \frac{3}{\gamma^3} \left(\frac{\partial \omega}{\partial S}\right)^4 .$$

are holomorphic invariants (*can be expressed in terms of linear combinations of products of Eisenstein series of* $\Gamma_0(2)$.

Cardoso, Nampuri, Polini, 1903.07586

It is convenient to choose the basis functions $\tilde{I}^{(n)}(S) = \dfrac{I^{(n)}(S)}{(D_S\,\omega)^n}$.

These terms have to be determined separately, again by imposing the holomorphic anomaly equation!

For instance,

$$
\begin{aligned}
h_1 = & \sum_{n=2} \frac{(2\gamma)^n}{n!} \left(\frac{u-1}{u+1}\right)^n \left[\tilde{I}^n(S) + \tilde{I}^n(T) + \tilde{I}^n(U)\right] \\
& + \sum_{m,n=2} \frac{(2\gamma)^{m+n-1}}{(m-1)!\,(n-1)!} \frac{1}{u} \left(\frac{u-1}{u+1}\right)^{m+n-1} \\
& \qquad \times \left[\tilde{I}^n(S)\,\tilde{I}^n(T) + \tilde{I}^n(T)\,\tilde{I}^n(U) + \tilde{I}^n(U)\,\tilde{I}^n(S)\right] \\
& + \cdots
\end{aligned}
$$

$1/u$ **pole**

Conclusions

The results that have been obtained apply only to a special STU-model, where duality symmetry is very restrictive and simplifies the calculations.

When suppressing all the non-holomorphic corrections the topological string partition function tends to the function that encodes the effective action.

The results confirm the consistency of the conjectured relation between the Wilsonian action and the topological string

The higher-order terms exhibit also poles at $u = 1$, whose implications are not yet quite clear.

It is possible to take limits in which the real part of some of the moduli are taken to infinity.

There have already been applications to BPS black holes

Cardoso, Nampuri, Polini, 1903.07586