# (Hopf) bialgebroids as a model for description of quantum covariant phase space

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based on a joint work with Anna Pachoł

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19TH HELLENIC SCHOOL AND WORKSHOPS ON ELEMENTARY PARTICLE PHYSICS AND GRAVITY, CORFU, GREECE 2019

*Workshop on Quantum Geometry, Field Theory and Gravity*

CORFU SUMMER INSTITUTE, SEPTEMBER 18 - SEPTEMBER 25, 2019

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Plan:
I. Motivating example: extended Weyl-Heisenberg algebra
II. Algebraic structure: smash product construction
III. Bialgebroid structure
IV. Twisted deformations
V. Twisted deformations of QM

# Examples: extended Wey-Heisenber (CCR) alegbras

Any finite-dimensional representations of Lie algebra $\mathfrak{g}$ provides an example of module algebra. Given representation $\rho$ induces the action on the vector space $V$:

$$\rho : \mathfrak{g} \to End_{\mathbb{K}} V \quad \Leftrightarrow \quad \triangleright : U_{\mathfrak{g}} \otimes V \to V \tag{1}$$

$L \triangleright v \equiv \rho(L)(v)$. This action can be further extended to the action on the (commutative algebra) of smooth functions $C^\infty(V)$, i.e. elements of $EndC^\infty(V)$. Let us denote by $\rho(L) = [L_\alpha^\beta]$ as the corresponding matrices. Then we are in position to construct the first order differential operators acting on the manifold $V$

$$\hat{L} = -L_\alpha^\beta x^\alpha \partial_\beta \tag{2}$$

which are in fact coordinate independent objects determining an extended action $\triangleright : U_{\mathfrak{g}} \otimes C^\infty(V) \to C^\infty(V)$.

In this way $C^{\infty}(V)$ becomes a (commutative) module algebra over the Hopf algebra $U_{\mathfrak{g}}$ since

$$\hat{L}(f \cdot g) = \hat{L}(f) \cdot g + f \cdot \hat{L}(g) \tag{3}$$

for all $f, g \in C^{\infty}(V)$ is automatically satisfied provided one uses the primitive Hopf algebra structure $(\Delta(L) = L \otimes 1 + 1 \otimes L)$ and rewrite (3) as $L \triangleright (f \cdot g) = \left(L_{(1)} \triangleright f\right) \cdot \left(L_{(2)} \triangleright g\right)$ which provides the module algebra condition over $\mathcal{U}_{\mathfrak{g}}$. Realization (2) allows to merge the initial Lie algebra $\mathfrak{g}$ with a canonical Heisenberg algebra. Resulting algebra can be characterized by the following commutation relations ($\hat{x}^{\alpha}$ denotes multiplication by the coordinate $x^{\alpha}$ )

$$[\hat{L}_a, \hat{L}_b] = \gamma_{ab}^{c} \hat{L}_c, \quad [\hat{L}_b, \partial_{\beta}] = (L_b)^{\alpha}_{\beta} \partial_{\alpha}, \quad [\partial_{\alpha}, \partial_{\beta}] = 0 \tag{4}$$

$$[\hat{L}_a, x^{\beta}] = -(L_a)^{\beta}_{\alpha} \hat{x}^{\alpha}, \quad [\partial_{\alpha}, \hat{x}^{\beta}] = 1\, \delta^{\beta}_{\alpha}, \quad [\hat{x}^{\alpha}, \hat{x}^{\beta}] = 0 \tag{5}$$

The first line (17) represents a Lie subalgebra which can be identified as a inhomogeneous extension $\mathfrak{ig}_{\rho}$ of the initial Lie algebra $\mathfrak{g}$ with respect to the representation $\rho$.

Our aim is to study unital associative **abstract algebra** generated by the same set of relations (here $[A, B] \equiv AB - BA$)

$$[L_a, L_b] = \gamma_{ab}^{c} L_c, \quad [L_b, p_\beta] = (L_b)^{\alpha}_{\beta} p_\alpha, \quad [p_\alpha, p_\beta] = 0$$

$$[p_\alpha, x^\beta] = 1\, \delta^{\beta}_{\alpha}, \quad [L_a, x^\beta] = -(L_a)^{\beta}_{\alpha} x^\alpha, \quad [x^\alpha, x^\beta] = 0$$

and its possible quantum deformation.
Let us point out that while extending the Lie algebra by adding the vector space of some representation we go beyond the category of Lie algebras. However, we remain in a category of associative unital algebras which include Lie algebras as a subcategory. To be more precise, obtained algebra is not an enveloping algebra of some Lie algebra. The unit as a group-like element cannot belong to Lie algebra. Consequently, instead of (Hopf) bialgebras one gets (Hopf) bialgebroids.

Replacing the unit by a central (primitive) element we are back in the Lie algebra case. But both structures are not equivalent (isomorphic) in the algebraic sense. We claim that the first construction is more natural for Physics since in the undeformed case it is related to Quantum Mechanics and representations (infinitesimal version) of the so-called Mackey's imprimitivity systems.

# Smash products algebra

- $A \rtimes H$ - Extension of module algebra $A$ by Hopf algebra $H$ to create a new algebra

- Determined on the vector space $A \otimes H$ by multiplication:
  $$\boxed{(\mathbf{a} \otimes \mathbf{L}) \rtimes (\mathbf{b} \otimes \mathbf{J}) = \mathbf{a}(\mathbf{L}_{(1)} \triangleright \mathbf{b}) \otimes \mathbf{L}_{(2)}\mathbf{J}}$$
  $a, b \in A$ and $L, J \in H$

- Initial algebras are canonically embedded, $A \ni f \to f \otimes 1$ and $H \ni L \to 1 \otimes L$ as subalgebras in $A \rtimes H$.
- Cross product can be rewritten as
  $$(a \rtimes L) \star (b \rtimes J) = a \star (L_{(1)} \triangleright b) \rtimes L_{(2)} J.$$
  where $(A, \star, 1_A)$ - a left $H$-module algebra

Trivial action $L \triangleright g = \epsilon(L) g$ makes $A \rtimes H$ isomorphic to the ordinary tensor product algebra $A \otimes H$: $(f \otimes L)(g \otimes M) = fg \otimes LM$.

# Twisted smash product by a Drinfeld twist

Proposition:(D. Bulacu, F. Panaite and F. M. J. Van Oystaeyen, Comm. Algebra 28 (2000), no. 2, 631)
For any Drinfeld twist $F$:

$$A \rtimes H \cong A_F \rtimes H^F$$

even though $A \ncong A_F$ (as algebras) and $H \ncong H^F$ (as bialgebras).

- Both algebras are determined on the same $\mathbb{C}$-module $A \otimes H$ but differ by the multiplications:

$$\begin{aligned} (a \rtimes L) \star (b \rtimes J) &= a \star (L_{(1)} \triangleright b) \rtimes L_{(2)} J \\ (a \rtimes L) \star_F (b \rtimes J) &= a \star_F (L_{(1^F)} \triangleright b) \rtimes L_{(2^F)} J \end{aligned}$$

- where $\Delta^F(L) = F\Delta(L)F^{-1} = L_{(1^F)} \otimes L_{(2^F)}$ - twisted coproduct of the bialgebra $H^F$ and twisted $*$-product
- $a \star_F b = m \circ F^{-1} \triangleright (a \otimes b) = (\bar{F}_1 \triangleright a) \cdot (\bar{F}_2 \triangleright b)$

# (Hopf) Bialgebroids

J.H. Lu, Intern. Journ. Math. 7, 47 (1996); [q-alg/9505024];
P. Xu, Comm. Math. Phys. 216, 539 (2001);
G. Bohm, K. Szlachanyi, J. Algebra 274 (2004), no. 2, 708;
[math.QA/0302325]

Bialgebroids are bialgebras over noncommutative rings (Hopf=coinverse).

A left bialgebroid $\mathcal{M} = (M, A, s, t, \Delta, \epsilon)$ is a left bialgebroid together with an antipode .
The bialgebroid $\mathcal{M}$ consists of

1. a total algebra $M$
2. a base algebra $A$
3. two mappings:
   - an algebra homomorphism $s : A \to M$ - a source map
   - an algebra anti-homomorphism $t : A \to M$ - a target map
   - such that: $s(a)t(b) = t(b)s(a)$, for all $a, b \in A$

- coproduct and counit maps, with analogous axioms of a coalgebra but all mappings are $A$-bimodule homomorphisms and all $\otimes \Longrightarrow \otimes_A$.
- Left bialgebroid $\mathcal{M}$ - an $A-$bimodule (with the bimodule structure which prefers the left side):

$$a.m.b = s(a)\,t(b)\,m$$

for all $a, b \in A$, $m \in M$.

- The bialgebroid coproduct map $\Delta : M \to M \otimes_A M$ should be a $A$-bimodule map s.t.: $\Delta(mn) = \Delta(m)\Delta(n)$
- But $M \otimes_A M$ **is not an algebra in general**

$$(t(a)\,m) \otimes_A n = m \otimes_A (s(a)\,n).$$

Takeuchi product $M \times_A M$ :

- It is defined as a subgroup of invariant elements
$M \times_A M =$
$\{m_1 \otimes_A m_2 \in M \otimes_A M :$
$(m_1 t(a)) \otimes_A m_2 = m_1 \otimes_A (m_2 s(a)); \forall a \in A\}$

- has natural (component-wise) algebra structure

$$(m_1 \otimes_A m_2)(n_{1'} \otimes_A n_{2'}) = m_1 n_{1'} \otimes_A m_2 n_{2'}$$

- Both $M \otimes_A M$ and $M \times_A M$ inherit $A$-bimodule structure determined by the action

$$m_1 \otimes_A m_2 \mapsto (s(a)m_1) \otimes_A (t(b)m_2)$$

.

- Request the image of the coproduct map in $M \times_A M$ (i.e. $\Delta$ the algebra map):

$$\Delta(mn) = \Delta(m)\Delta(n) \equiv m_{(1)}n_{(1)} \otimes_A m_{(2)}n_{(2)}$$

- The counit map $\epsilon : M \to A$ has to satisfy:
$\epsilon(1_M) = 1_A$,
$\epsilon(mn) = \epsilon(ms(\epsilon(n))) = \epsilon(mt(\epsilon(n)))$,
$s(\epsilon(m_{(1)}))m_{(2)} = t(\epsilon(m_{(2)}))m_{(1)} = m$
- The axioms are similar to those of a Hopf algebra but are complicated by the possibility that $A$ is a noncommutative algebra or its images under $s$ and $t$ are not in the centre of $M$.

# Twisted bialgebroids

P. Xu, Comm. Math. Phys. 216, 539 (2001).

- Bialgebroid definition provides a canonical action $\blacktriangleright: M \otimes A \to A$ (also known as an anchor $M \ni m \to m \blacktriangleright \in EndA$): $m \blacktriangleright a = \epsilon(ms(a)) = \epsilon(mt(a))$,

Theorem [Xu]:
Assume that $(M, A, s, t, \Delta, \epsilon)$ is bialgebroid over the algebra $A$ and $F = F_1 \otimes_A F_2 \in M \otimes_A M$ is a "twistor" (Hopf algebroid twist). Then $(M, A_F, s_F, t_F, \Delta_F, \epsilon)$ is a bialgebroid over the algebra $A_F$, where

$$s_F(a) = s\left(\bar{F}_1 \blacktriangleright a\right)\bar{F}_2 \quad ; \qquad t_F(a) = t\left(\bar{F}_2 \blacktriangleright a\right)\bar{F}_1 \quad \forall a \in A.$$

and new twisted coproduct $\Delta_F : M \to M \otimes_{A_F} M$ :

$$\Delta_F(m) = F^{\#}\left(\Delta(m) F^{-1}\right), \quad \forall \;\; m \in M$$

# Smash product algebras as bialgebroids

Theorem [T. Brzezinski and G. Militaru, J. Algebra 251 (2002), no. 1, 279]:
Let $H = (H, \Delta, \epsilon)$ be a bialgebra, $A = (A, \star)$ is a left $H$-module algebra and $(A, \rho)$ a right $H$-comodule. Then $(A, \star, \rho)$ is a **braided commutative algebra** in ${}_H\mathfrak{YD}^H$, i.e. $a \star b = b_{<0>} \star (b_{<1>}) \triangleright a$ if and only if $\left(A \rtimes H, s, t, \tilde{\Delta}, \tilde{\epsilon}\right)$ is an $A$-bialgebroid with

- Source, target, coproduct and the counit given by:

$$\begin{aligned} s(a) &= a \rtimes 1_H, \quad t(a) \equiv \rho(a) = a_{<0>} \rtimes a_{<1>} \\ \tilde{\Delta}(a \rtimes L) &= (a \rtimes L_{(1)}) \otimes_A (1_A \rtimes L_{(2)}) \\ \tilde{\epsilon}(a \rtimes L) &= \epsilon(L)a \end{aligned}$$

for all $a \in A$ and $L \in H$.

# Triangular braided commutativity

- Any (left) module $A$ over quasitriangular bialgebra $(H, R)$ becomes automatically a (left-right) Yetter-Drinfeld module with the right coaction
$$\rho_R(a) = (R_2 \triangleright a) \otimes R_1$$
[L. A. Lambe and D. E. Radford, J. Algebra 154 (1993), no. 1, 228]
- Particularly, a module algebra $A = (A, \star, 1_A)$ is an algebra in ${}_H\mathfrak{YD}^H$ if and only if it is a braided commutative:
$$a \star b = (R_2 \triangleright b) \star (R_1 \triangleright a)$$
- **above condition is automatic in a triangular case**, i.e. when
$$R^F \equiv F_{21} F^{-1} \qquad \text{and} \qquad \star = \star_F$$

.

# IV.Main result

**Goal**: To compare two constructions of bialgebroids:

The bialgebroid obtained by bialgebroid twisting of the smash product algebra $(A \rtimes H)^{\tilde{F}}$

and

bialgebroid obtained from the smash product algebra of twisted bialgebra with twisted module algebra ${}_F A^F \rtimes H^F$

**Main result**: Both bialgebroids are equivalent (isomorphic):

$$
{}_F A^F \rtimes H^F \cong (A \rtimes H)^{\tilde{F}}
$$

**Remainder**: As algebras all three are equivalent

$$
{}_F A^F \rtimes H^F \cong A \rtimes H \cong (A \rtimes H)^{\tilde{F}}
$$

**Corollary**: First two are not isomorphic as bialgebras !

$$
{}_F A^F \rtimes H^F \ncong A \rtimes H
$$

AB, A. Pachol, J. Phys. A 50 (2017), 055205, arXiv: 1603.09280

Proposition:
Let $(H, R)$ be a quasi-triangular bialgebra and $A$ stands for braided commutative module algebra w.r.t. $(H, R)$. Assume that $F = F_1 \otimes F_2 \in H \otimes H$ is a normalized cocycle twist in $H$. Then

$$A_F \rtimes H^F \cong (A \rtimes H)^{\tilde{F}}$$

are isomorphic as bialgebroids over the algebra $A_F$, where $\tilde{F}$ denotes bialgebroid cocycle twist

$$F \to \tilde{F} = (1_A \rtimes F_1) \otimes_A (1_A \rtimes F_2)$$

obtained from Drinfeld's bialgebra twist $F \in H \otimes H$.

Proof: The isomorphism

$$\varphi : A_F \rtimes H^F \to A \rtimes H$$

$(\varphi(a \rtimes L) = (\bar{F}_1 \triangleright a) \rtimes \bar{F}_2 L)$
of total algebras makes commuting the following diagram

$$\begin{array}{ccc} A_F \rtimes H^F & \xrightarrow{\varphi} & A \rtimes H \\ \widetilde{\Delta^F} \downarrow & & \downarrow \tilde{\Delta}_{\tilde{F}} \\ (A_F \rtimes H^F) \otimes_{A_F} (A_F \rtimes H^F) & \xrightarrow{\varphi \otimes_{A^F} \varphi} & (A \rtimes H) \otimes_{A_F} (A \rtimes H) \end{array}$$

i.e. $\tilde{\Delta}_{\tilde{F}} \circ \varphi = (\varphi \otimes_{A_F} \varphi) \circ \widetilde{\Delta^F}$.

# Back to the example

The Hopf algebra is $H = U_{\mathfrak{ig}_\rho}$ generated by the relations

$$[L_a, L_b] = \gamma_{ab}^c L_c, \quad [L_b, p_\nu] = (L_b)_\nu^\alpha p_\alpha, \quad [p_\mu, p_\nu] = 0$$

The base algebra $A = C^\infty(V)$ is commutative $fg = gf$. Then the remaining relations are the results of the smash product construction:

$$[L_a, f] = -(L_a)_\alpha^\beta x^\alpha \partial_\beta f, \quad [p_\alpha, f] = \partial_\alpha f.$$

Source and target maps are equal

$$s(f) = t(f) = f \rtimes 1 \equiv f$$

Counit

$$\epsilon(f) = f\,, \quad \epsilon(L) = 0\,, \quad \epsilon(p) = 0$$

Bialgebroid coproduct

$$\Delta(f) = f \otimes_A 1 = 1 \otimes_A f\,,$$

$$\Delta(L) = l \otimes_A L + 1 \otimes_A L\,, \qquad \Delta(p) = l \otimes_A p + 1 \otimes_A p\,.$$

After (twisted) quantum deformation $A \to A_F$ gets noncommutative (albeit braided commutative) star product. $H^F$ has twist deformed coproduct. Bialgebroid coproduct in $A_F \rtimes H^F$ representation becomes:

$$\widetilde{\Delta}^F(a \rtimes L) = (a \rtimes L_{(1^F)}) \otimes_{\mathcal{A}_F} (1_{A_F} \rtimes L_{(2^F)})\,,$$

$$s^F(a) = a \rtimes 1_H\,, \quad t^F(a) = \left(R_2^F \triangleright a\right) \rtimes R_1^F$$

Several explicit examples can be found e.g. in recent Lukierski, Meljanac, Woronowicz papers.

# Conclusions

- Notion of bialgebroid unifies qunatum space with their quantum symmetry group in one object which captures some quantum mechanical properties.
- The covariant quantum phase space has a (Hopf) bialgebroid structure and in such framework can be quantum-deformed by Drinfeld twist
- Deformation concerns co-ring structure; algebraic sector does not change up to the isomorphism $A_F \rtimes H^F \cong A \rtimes H$
- For QM-applications the framework should be reformulated in $C^*$-algebras and Hilbert modules language.
- In progress: real bialgebroids.

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Thank you for your attention!