On Torsion and Curvature in Courant Algebroids

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Context and motivation

- Formulation of type II supergravities as gravity on the generalized tangent bundle (Coimbra, Strickland-Constable, Waldram 2011-2012) and general duality invariant gravity theories (Hull, Hohm, Zwiebach 2009-2013).
- Use of Courant algebroid connections, Courant algebroid-, or Gualtieri-torsion (Gualtieri 2007) and variants of Riemann curvature (Xu 2010,Hohm, Zwiebach 2013, Jurčo, Vysoký 2015).
- Compare to the definition of torsion and curvature:

  For a vector bundle $E \to M$ and a connection ∇, i.e. a degree-1 derivation of the dg module $\Omega(M, E) = \Gamma(\Lambda T^*M \otimes E)$ over $\Omega(M)$,

  $$\textit{The curvature of } \nabla \textit{ is} \quad R_\nabla = \nabla^2 \ .$$

  In case $E = TM$ and $\tau \in \Gamma(T^*M \otimes TM)$ the identity/solder form,

  $$\textit{The torsion of } \nabla \textit{ is} \quad T_\nabla = \nabla\tau \ .$$

Context and motivation

Alternative view on Courant algebroids:

Isomorphism classes of Courant algebroids correspond to degree-2 symplectic differential graded manifolds.

(Roytenberg, Ševera, 2002)

In this talk:

- Define the notion of connection $\mathcal{Q}_{\mathcal{E}}$ in a graded vector bundle $\mathcal{E}$ over a dg manifold $\mathcal{M}$.
- Find curvatue as $R_{\mathcal{Q}_{\mathcal{E}}} = 1/2\,[\mathcal{Q}_{\mathcal{E}}, \mathcal{Q}_{\mathcal{E}}]$ and torsion as $T_{\mathcal{Q}_{\mathcal{E}}} = \mathcal{Q}_{\mathcal{E}}(\tau)$ for an appropriate section τ.
- Compare these notions to standard torsion and curvature (e.g. on Lie algebroids by restricting to Dirac structures).
- The special case $\mathcal{M} = T^*[2]T[1]M$ is relevant for Courant algebroids.
- Compute Ricci and scalar curvature.

Setup: Graded vector bundles over Q-manifolds

Notations and conventions from graded geometry:
As e.g. in the lectures by Cattaneo, Schätz (2010)

- $\mathcal{M}$ denotes a graded manifold, i.e. a $\mathbb{Z}$-graded sheaf of commutative algebras over a smooth manifold M, locally isomorphic (in the category of $\mathbb{Z}$-graded algebras) to the local model $\mathcal{C}^\infty(U) \otimes SV$, where $U \subset M$ open in M, V a graded vector space and SV is the symmetric algebra.
- $C(\mathcal{M}) = \oplus_k C^k(\mathcal{M})$ denotes the graded commutative algebra of functions on $\mathcal{M}$, i.e. global sections of the underlying sheaf.
  $\mathrm{Vect}(\mathcal{M}) = \oplus_k \mathrm{Vect}^k(\mathcal{M})$ is the graded Lie algebra of derivations of $C(\mathcal{M})$, i.e. the graded vector fields.
- $\mathcal{E} \to \mathcal{M}$ denotes a graded vector bundle over $\mathcal{M}$, i.e. a graded manifold $\mathcal{E}$ together with an atlas of coordinates (y^A, s^α) such that y^A are coordinates of $\mathcal{M}$ (i.e. generators of the algebra of functions on the local model) and s^α transform linearly, i.e. they are fibre variables. The dual bundle is denoted by $\mathcal{E}^*$.
- $\Gamma(\mathcal{E}) \equiv C(\mathcal{E}^*)_{lin} \subset C(\mathcal{E}^*)$ denotes the space of functions on $\mathcal{E}^*$ linear in the fibre variables, i.e. the space of sections in $\mathcal{E}$.

Setup: Connection and curvature

For a $\mathcal{Q}$-manifold $\mathcal{M}$, i.e. a graded manifold equipped with a degree 1 cohomologial vector field $\mathcal{Q}$ (i.e. $\mathcal{Q}^2 = 1/2[\mathcal{Q}, \mathcal{Q}] = 0$) define:

Definition ($\mathcal{Q}$-connection)

A $\mathcal{Q}$-connection on $\mathcal{E} \to \mathcal{M}$ is a degree 1 vector field $\mathcal{Q}_{\mathcal{E}} \in \mathrm{Vect}^1(\mathcal{E}^*)$ satisfying the conditions

- $\mathcal{Q}_{\mathcal{E}}$ preserves $\Gamma(\mathcal{E})$,
- $\mathcal{Q}_{\mathcal{E}}$ projects to $\mathcal{Q}$.

We can take the commutator of $\mathcal{Q}_{\mathcal{E}}$ with itself to define the curvature:

Definition (Curvature of a $\mathcal{Q}$-connection)

The curvature of a $\mathcal{Q}$-connection $\mathcal{Q}_{\mathcal{E}}$ on $\mathcal{E} \to \mathcal{M}$ is the degree 2 vector field

$$R_{\mathcal{Q}_{\mathcal{E}}} = \mathcal{Q}_{\mathcal{E}}^2 = 1/2[\mathcal{Q}_{\mathcal{E}}, \mathcal{Q}_{\mathcal{E}}] \in \mathrm{Vect}^2(\mathcal{E}^*)\,. \tag{1}$$

In case the curvature vanishes, $(\mathcal{E}, \mathcal{Q}_{\mathcal{E}})$ is called $\mathcal{Q}$-bundle (Grützmann, Kotov, Strobl, 2014).

Setup: Torsion of a $\mathcal{Q}$-connection

Let now $\mathcal{M}$ be an $N\mathcal{Q}$-manifold, i.e. there is a fibration

$$\mathcal{M} = \mathcal{M}_n \xrightarrow{p_n} \mathcal{M}_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} \mathcal{M}_1 \xrightarrow{p_1} \mathcal{M}_0 \equiv M \, .$$

- $p := p_1 \circ p_2 \circ \cdots \circ p_n$.
- $(x^\mu, \psi^\alpha, \dots)$ coordinates on $\mathcal{M}$.
- ψ_α dual to the fibre coordinates ψ^α.
- $s_\alpha := p^* \psi_\alpha$.

$$\begin{tikzcd} p^*\mathcal{M}_1 \arrow[r, dashed] \arrow[d] & \mathcal{M}_1 \arrow[d] \\ \mathcal{M} \arrow[r, "p"] & M \end{tikzcd}$$

Then, "tautological section" $\tau_{\mathcal{M}} = \psi^\alpha s_\alpha \in \Gamma(p^*\mathcal{M}_1) = C(p^*\mathcal{M}_1^*)$, and

E.g. standard geometry, $\tau \in \Gamma(T^*M \otimes TM)$, $\tau(X) = X$, $\tau = dx^\mu \otimes \partial_\mu$

Definition (Torsion of a $\mathcal{Q}$-connection)

Let $\mathcal{Q}_\mathcal{E}$ be a connection on $\mathcal{E} = p^*\mathcal{M}_1 \to \mathcal{M}$. The torsion of $\mathcal{Q}_\mathcal{E}$ is the degree 1 section

$$T_{\mathcal{Q}_\mathcal{E}} = \mathcal{Q}_\mathcal{E}(\tau_\mathcal{M}) \in \Gamma(p^*\mathcal{M}_1) \, . \tag{2}$$

Immediate observations

- For a Lie algebroid $(\mathcal{M} = A[1], d_A)$ and a bundle $\mathcal{E} \to A[1]$, $\mathcal{Q}_\mathcal{E}$ gives the standard definition of a Lie algebroid connection, $\mathcal{Q}_\mathcal{E}^2$ gives the curvature of the Lie algebroid connection.
- In case $\mathcal{E} = p^* A[1]$, where $p : A[1] \to M$ is the bundle projection itself, we get the standard definition of Lie algebroid torsion.

- Now: Take the degree 2 dg manifold $\mathcal{M} = T^*[2]T[1]M$ which encodes Courant algebroids. The Ševera class $[H] \in H^3(M)$ is contained as twist of the cohomological vector field. We will investigate curvature and torsion in this case.

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Graded geometry of Courant algebroids

$\mathcal{M} = T^*[2]T[1]M$

- Degree 2: $(x^\mu, \psi^\mu, b_\mu, p_\mu)$ local coordinates.
- Canonical symplectic form: $\omega = dx^\mu dp_\mu + d\psi^\mu db_\mu$.
- Homological vector field is hamiltonian, $d_\mathcal{M} = \{\Theta, \cdot\}$,
  $\Theta = \psi^\mu p_\mu + \frac{1}{3!} H_{\mu\nu\rho}\psi^\mu\psi^\nu\psi^\rho$.

Connection and curvature

For $\mathcal{E} \to \mathcal{M}$, according to the definition, we write a connection as

$$\mathcal{Q}_\mathcal{E} = d_\mathcal{M} + \Upsilon^\alpha{}_\beta\, s_\alpha \frac{\partial}{\partial s_\beta} \ , \tag{3}$$

and the curvature $R_{\mathcal{Q}_\mathcal{E}} = \mathcal{Q}^2_\mathcal{E}$ reads

$$R_{\mathcal{Q}_\mathcal{E}} = (d_\mathcal{M}\Upsilon^\alpha{}_\beta + (-)^{|s_\alpha|-|s_\gamma|+1}\Upsilon^\alpha{}_\gamma\Upsilon^\gamma{}_\beta)s_\alpha \frac{\partial}{\partial s_\beta} \ . \tag{4}$$

The curvature for pullback of vector bundles

In case $E \to M$ is an ordinary vector bundle, $\mathcal{E} = p^* E[k]$,

- Degree $|s^\alpha| = k$, $|\Upsilon^\alpha{}_\beta| = 1$.
- Expand $\Upsilon^\alpha{}_\beta = \Gamma_\mu{}^\alpha{}_\beta \, \psi^\mu + V^{\mu\alpha}{}_\beta \, b_\mu$.
- $\Gamma_\mu{}^\alpha{}_\beta$ connection ∇ on E,
  $V \in \Gamma(TM \otimes \mathrm{End} E)$.
  Write $D = \nabla + V$ for cov. derivative.

$$\begin{tikzcd} p^* E[k] \arrow[r, dashed] \arrow[d] & E[k] \arrow[d] \\ T^*[2]T[1]M \arrow[r, "p"] & M \end{tikzcd}$$

The curvature can be expanded as:

$$R_{\mathcal{Q}_\mathcal{E}} = R_\nabla + VV + R^{(1,1)}_{\mathcal{Q}_\mathcal{E}} \, , \tag{5}$$

- $R_\nabla{}^\alpha{}_\beta = (\partial_\mu \Gamma_\nu{}^\alpha{}_\beta - \Gamma_\mu{}^\alpha{}_\gamma \Gamma_\nu{}^\gamma{}_\beta + \frac{1}{2} H_{\rho\mu\nu} V^{\rho\alpha}{}_\beta)\psi^\mu \psi^\nu$
  Contains ordinary curvature and twist.
- $VV^\alpha{}_\beta = -(V^{\mu\alpha}{}_\gamma V^{\nu\gamma}{}_\beta) b_\mu b_\nu$
  Product of endomorphisms.
- $(R^{(1,1)}_{\mathcal{Q}_\mathcal{E}})^\alpha{}_\beta = (\partial_\mu V^{\nu\alpha}{}_\beta + V^{\nu\gamma}{}_\beta \Gamma_\mu{}^\alpha{}_\gamma - \Gamma_\mu{}^\gamma{}_\beta V^{\nu\alpha}{}_\gamma)\psi^\mu b_\nu + \underline{V^{\mu\alpha}{}_\beta p_\mu}$
  Contains the p-dependent term.

Torsion, $\mathcal{E} = p^*\mathcal{M}_1$

In case of Courant algebroids, the fibration of graded manifolds is

$$T^*[2]T[1]M \xrightarrow{p_2} \mathbb{T}[1]M := (T \oplus T^*)[1]M \xrightarrow{p_1} M \, .$$

- $\mathcal{M}_1 = \mathbb{T}[1][M]$.
- $\mathcal{E} = p^*\mathcal{M}_1$, (s^μ, s_μ).
- $\tau = \psi^\mu s_\mu + b_\mu s^\mu$.

$$\begin{tikzcd} p^*\mathbb{T}[1](M) \arrow[r, dashed] \arrow[d] & \mathbb{T}[1](M) \arrow[d] \\ T^*[2]T[1]M \arrow[r, "p"] & M \end{tikzcd}$$

The torsion therefore is

$$T_{\mathcal{Q}_\mathcal{E}} = \mathcal{Q}_\mathcal{E}(\tau) = T_\nabla + T_V + T^{(1,1)}_{\mathcal{Q}_\mathcal{E}} \, , \qquad (6)$$

where T_∇ contains the standard part, T_V is an endomorphism and $T^{(1,1)}_{\mathcal{Q}_\mathcal{E}}$ contains the degree 2 part.

Remarks: Degree 2 and sections in vector bundles

Remarks

- $R_{\mathcal{Q}_\mathcal{E}} = \mathcal{Q}_\mathcal{E}^2$ and $T_{\mathcal{Q}_\mathcal{E}} = \mathcal{Q}_\mathcal{E}(\tau)$ are global objects, but not canonically sections in vector bundles over M: They contain the degree 2 objects, which transform inhomogeneously, w.r.t. coordinate changes.
- Via introducing a connection ∇^K on M, i.e. $\nabla^K \partial_\mu = K^\rho{}_\mu \partial_\rho$ and $K^\rho{}_\mu = K_\nu{}^\rho{}_\mu \, \psi^\nu$ there's an identification

$$T^*[2]T[1]M \simeq T[1]M \oplus T^*[1]M \oplus T^*[2]M$$

- This allows to compare the global definitions to traditional geometric expressions (tensors, ordinary sections in degree shifted bundles).
- With $\tilde{p} := p + K^T b$, i.e. $\tilde{p}_\mu = p_\mu + K_\nu{}^\rho{}_\mu \, \psi^\nu b_\rho$, we separate $R^{(1,1)}_{\mathcal{Q}_\mathcal{E}}$ and $T^{(1,1)}_{\mathcal{Q}_\mathcal{E}}$ into a section containing K and a $\tilde{p}$ part:

$$R_{\mathcal{Q}_\mathcal{E}} = R_K + V^\mu \tilde{p}_\mu \;, \quad T_{\mathcal{Q}_\mathcal{E}} = T_K + \tilde{p}_\mu s^\mu \;. \tag{7}$$

Comparison to traditional geometry

With the connection K it it turns out that for $a, b \in \Gamma(\mathbb{T}M)$

$$R_K(a,b) = [\iota_b^K[\iota_a^K R_{\mathcal{Q}_\mathcal{E}}]] \ , \quad T_K(a,b) = \iota_b^K \iota_a^K (T_{\mathcal{Q}_\mathcal{E}}) \ . \tag{8}$$

with $\iota_a^K = a^\mu \left(\frac{\partial}{\partial\psi^\mu} + K_\mu{}^\rho{}_\nu \, b_\rho \frac{\partial}{\partial p_\nu}\right) + a_\mu \left(\frac{\partial}{\partial b_\mu} - K_\rho{}^\mu{}_\nu \, \psi^\rho \frac{\partial}{\partial p_\nu}\right)$.

We want to compare to the familiar expressions:

$$\begin{aligned} R_D(a,b) &= [D_a, D_b] - D_{[a,b]_C} \ , \\ T_D(a,b) &= D_a b - D_b a - [a,b]_C \ . \end{aligned} \tag{9}$$

Note, for $[a,b]_C$ the Courant bracket these are not not sections in any bundle, they are not meaningful. Our apporach allows to *deduce* the terms to get well defined sections $R_K(a,b)$, $T_K(a,b)$.

Comparison to traditional geometry

Two of our main results are:

Proposition (comparison of curvature)

Let $D = \nabla + V$ be a Courant algebroid connection in a vector bundle $E \to M$, $a = X + \xi$, $b = Y + \eta$ be generalized vectors. The curvature R_K reads

$$R_K(a, b) = R_D(a, b) + V_{\tilde{K}(a,b)} \,, \tag{10}$$

where $\tilde{K}(a, b)$ is defined by

$$\tilde{K}(X + \xi, Y + \eta) = [X + \xi, Y + \eta]_C - [X, Y] + \nabla^K_Y \xi - \nabla^K_X \eta \,. \tag{11}$$

Proposition (comparison of torsion)

Let $D = \nabla + V$ be a Courant algebroid connection in $\mathbb{T}M$, and $a = X + \xi$, $b = Y + \eta$. The torsion T_K reads

$$T_K(a, b) = T_D(a, b) + \tilde{K}(a, b) \,, \tag{12}$$

where $\tilde{K}(a, b)$ is defined by (11).

Discussion and Outlook

- We defined the notion of **connection** $\mathcal{Q}_{\mathcal{E}}$ and **curvature** $R_{\mathcal{Q}_{\mathcal{E}}}$ on graded vector bundles $\mathcal{E} \to \mathcal{M}$ over $\mathcal{Q}$-manifolds $\mathcal{M}$. The curvature is simply $R_{\mathcal{Q}_{\mathcal{E}}} = \mathcal{Q}_{\mathcal{E}}^2$.
- In case of N$\mathcal{Q}$-manifolds, we give a definition of **torsion** as derivative of the tautological section $T_{\mathcal{Q}_{\mathcal{E}}} = \mathcal{Q}_{\mathcal{E}}(\tau)$.
- These expressions are global by construction and give rise to ordinary tensors R_K and T_K with the help of an auxiliary connection K on the body manifold. In particular they suggest a natural way to get Courant algebroid curvature and torsion.
- For an appropriate compatibility condition on K, $R_{\mathcal{Q}_{\mathcal{E}}}$ and $T_{\mathcal{Q}_{\mathcal{E}}}$ give the standard Lie algebroid curvature and torsion if one restricts the bundle $\mathcal{E} \to \mathcal{M}$ to a Dirac structure $\mathcal{L} \subset \mathcal{M}$.

Discussion and Outlook

- Computation of **Ricci** and **scalar curvature** is done in the standard way by taking a trace and then contracting with a generalized metric.
- For Courant algebroids, setting the torsion T_K to zero fixes the auxiliary connection K in terms of Γ and V. Moreover D takes the simple form

$$D = \nabla + V = \begin{pmatrix} \nabla & 0 \\ H & \nabla^* \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix} . \tag{13}$$

- The only remaining freedom is V. Taking $V^{\rho}{}_{\mu\nu} := \alpha g^{\rho\sigma} H_{\sigma\mu\nu}$, we get for the scalar curvature

$$Scal(\nabla, V) = R(\nabla) - \frac{\alpha^2}{2} H^2 , \tag{14}$$

which is interesting for e.g. string theory.