Bonneau Identities

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Goal

small Goal: presentation of the theoretical background for the following G.Bonneau (1980) and C.P.Martín and D. Sánchez-Ruiz (1999) formulas:

$$
\begin{aligned}
N[\hat{g}_{\mu\nu}\mathcal{O}^{\mu\nu}]\cdot\Gamma_{ren} \;&=\; -\sum_{n=2}^{4}\sum_{r}^{4-n}\sum_{\substack{i_1\ldots i_r\\ 1\leq i_j\leq n}} \\
&\left\{ \text{r.s.p.}\frac{(-i)^r}{r!}\frac{\partial^r}{\partial p_{i_1}^{\mu_1}\ldots\partial p_{i_r}^{\mu_r}}\overline{\langle\tilde{\phi}(p_1)\ldots\tilde{\phi}(p_n)\; N[\check{g}_{\mu\nu}\mathcal{O}^{\mu\nu}]\rangle}^{1PI}\Big|_{p_i\equiv\check{g}\equiv 0}\right\} \\
&N\left[\frac{1}{n!}\prod_{k=1}^{n}\left\{\left(\prod_{\{\alpha/i_\alpha=k\}}\partial_{\mu_\alpha}\right)\phi\right\}\right](x)\cdot\Gamma_{ren} \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
& N[\hat{\Delta}](x) \cdot \Gamma_{ren} \\
= \quad & -\sum_{n=2}^{4} \sum_{\{j_1 \ldots j_n\}} \Bigg[\sum_{r=0}^{\delta(J)} \sum_{\substack{i_1 \ldots i_r \\ 1 \leq i_j \leq n}} \Bigg\{ \frac{(-i)^r}{r!} \frac{\partial^r}{\partial p_{i_1}^{\mu_1} \ldots \partial p_{i_r}^{\mu_r}} (-i\hbar) \mathrm{r.s.p.} \\
& \times \overline{\langle \tilde{\phi}_{j_1}(p_1) \ldots \tilde{\phi}_{j_n}(p_n) N[\check{\Delta}](q = -\sum p_i)}^{1PI}_{K=0} \Bigg|_{p_i=0, \check{g}=0} \Bigg\} \\
& \times N\Bigg[\frac{1}{n!} \prod_{k=n}^{1} \Bigg\{ \Bigg(\prod_{\{\alpha / i_\alpha = k\}} \partial_{\mu_\alpha} \Bigg) \phi_{j_k} \Bigg\} \Bigg](x) \cdot \Gamma_{ren} \\
& + \text{ext. source part} \Bigg] \qquad (2)
\end{aligned}
$$

Notions needed

1. The BM renormalization of graph amplitudes
1.1. Some basic notions on the graphs
1.2. Basic expressions for an amplitude
1.3. Dimensional covariants, normal form and regularization of the amplitude
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1.5. The amplitude using the variables in terms of which the divergent part can be clearly expressed
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2.3. Basic Bonneau identity
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3.2. Slavnov Taylor identities and anomalous insertions
3.3. Bonneau identities for anomalous insertions in the renormalized action

1. The BM renormalization of graph amplitudes

1.1 Some basic notions on the graphs

$\diamondsuit$ graph $G = \{\mathcal{V}^G = \{V_i\}, \mathcal{L}^G = \{\ell_\ell\}, e = \{e_{\ell i}\}\}$

$\diamondsuit$ $M = \# V_i; L = \# \ell_\ell; h_G = \#$ loops

$\diamondsuit$ $e_{\ell i} = \{1$ if $V_i = V_{f_\ell}$; -1 if $V_i = V_{f_\ell}$; 0 otherw.$\}$ $r(e) = M - 1$

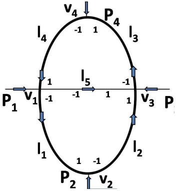
$\diamondsuit$ G's: connected (con.), 1PI, 1PR, non-empty, nontrivial (n.t.), reduced (G/H)(red.), tree, chord, two-tree, 1PI-component of G, subgraph, proper subgraph, non-overlapping subgraphs

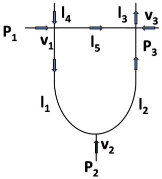
$\diamondsuit$ $\sum_i p_i = 0$: p_E, e_E, A_E, $A(k;k)$, $A(ki;kj)$, $(A_E^{-1})_{ij} = A(ki;kj)/A(k;k)$

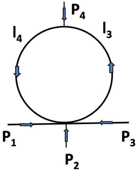
$\diamondsuit$ Symanzik polynomials: $d(\alpha) = A(k;k) \prod_{\ell \in \mathcal{L}_G} = \sum_T \prod_{\ell \notin T} \alpha_\ell$;

$D_{ij}^k = A(ki;kj) \prod_{\ell \in \mathcal{L}_G} = \sum_{T_2} \prod_{\ell \notin T_2} \alpha_\ell$

$\diamondsuit$ k-invariants: $d(\alpha)$, $p_E^T A_E^{-1} p_E$ $(A = e^T \alpha e,\ A_E = e_E^T \alpha e_E)$







Labelled forest

$$p_E \;=\; (p_1 \ldots, p_{k-1} \overset{k}{|} p_{k+1} \ldots p_{M-1}) \qquad e_E = \left(\cdots \overset{k}{\Big|} \ldots \right)$$

Labelled forest: G con.; forest; maximal forest $\mathcal{C}$; $\forall H \subset G$ def. $\mathcal{M} = \{H' | H'_{max} \subset H\}$, $\bar{H} = H/\mathcal{M}$; mapping $\sigma : \mathcal{C} \to \mathcal{L}_G$, $\sigma(H) \in \bar{H}$; labelled forest$= (\mathcal{C}, \sigma)$; joining unique subset of α-space ($\underline{\alpha} = \{\alpha_1, \ldots, \alpha_L\}$)

$$\mathcal{D}(\mathcal{C}, \sigma) \;=\; \{\underline{\alpha} | \alpha_\ell \geq 0 \, \forall \ell, \alpha_\ell \leq \alpha_{\sigma(H)} \text{ for } \ell \in H \in \mathcal{C}\} \qquad (3)$$

Properties:

1. G with n 1PI components $\Rightarrow \mathcal{C} = \bigoplus_{i=1}^{n} \mathcal{C}_i$
2. any maximal forest $\mathcal{C}$ may be labelled
3. $\forall \; H \in G$ there is $1-1$ corresp. $\{\mathcal{C}, \sigma\}_G \leftrightarrow ((\mathcal{C}_1, \sigma_1)_{G/H}, (\mathcal{C}_2, \sigma_2)_H)$
4. Any $\mathcal{C}$ has h_G elements
5. $G - \sigma(\mathcal{C})$ is a tree in G
6. $\bigcup_{(\mathcal{C},\sigma)} \mathcal{D}(C, \sigma)$ covers whole α-interaction region
7. if $(\mathcal{C}_1, \sigma_1) \neq (\mathcal{C}_2, \sigma_2)$ overlap has Lebesgue measure zero.

1.2. Basic expressions for an amplitude

$$
\begin{aligned}
\mathcal{T}_G(\underline{p}) &= (2\pi)^{n/2} \int \exp(i\underline{p}\underline{x})\mathcal{T}_G(\underline{x})d\underline{x} \\
&= \lim_{\epsilon\to 0} \hbar^{h_G-1}(2\pi)^{n/2}\delta(\sum p_i)(i/2)^{-nh_G/2} \int \prod_{\ell\in\mathcal{L}_G} d\alpha_\ell I_\epsilon(\underline{p},\underline{u},\underline{\beta})|_{\underline{u}=0} \\
I_\epsilon(\underline{p},\underline{u},\underline{\alpha}) &= d(\alpha)^{-n/2} \prod_{i\in\mathcal{L}} X_i\left(p_i, -\frac{\partial}{\partial u_\ell}\right) \prod_{\ell\in\mathcal{L}_G} Z_\ell\left(-\frac{\partial}{\partial u_\ell}\right) \exp iW(\underline{p},\underline{u},\underline{\beta}) \\
W(\underline{p},\underline{u},\underline{\alpha}) &= V(\underline{p},\underline{u},\underline{\alpha}) + -\underline{\alpha}(\underline{m^2} - i\epsilon) \\
V(\underline{p},\underline{u},\underline{\alpha}) &= (\underline{p}^T,\underline{u}^T)\begin{pmatrix} 0 & -2e_E^T \\ -2e_E & -4\alpha \end{pmatrix}\begin{pmatrix} \underline{p} \\ \underline{u} \end{pmatrix} \equiv (\underline{p}^T,\underline{u}^T)M\begin{pmatrix} \underline{p} \\ \underline{u} \end{pmatrix} \\
d(\alpha) &= detM/(-4)^L \qquad (4)
\end{aligned}
$$

Z_ℓ are spin structures of propagators; X_i are vertex factors.

Use tHVBM scheme :

1. D dim. objects p_μ, $g_{\mu\nu}$, γ_μ satisfy the same algebra as in 4 dimensions up to $g_{\mu\mu} = D$. Even $g_{\mu\nu}\epsilon_{\nu\rho\sigma\tau} = \epsilon_{\mu\rho\sigma\tau}$ although $\epsilon_{\nu\rho\sigma\tau}$ is pure 4-dim. quantity (as well as γ_5),
2. $D-4$ dim. objects introduced, leading to splitting the D dim. objects to $D-4$ dim. ones and 4 dim. ones e.g.
$g_{\mu\nu} = \hat{g}_{\mu\nu} + (g_{\mu\nu} - \hat{g}_{\mu\nu})$
Contraction of D object with $D-4$ (4) object represents the projection of D object to $D-4(4)$ space, while the contaction of $D-4$ object with 4 object gives zero result.
3. $\epsilon_{\mu\nu\rho\sigma}$ and γ_5 are defined in 4 dim, space, and (anti)commutation rules of γ_5 and γ-matrices as follows

$$\epsilon_{\mu_1\mu_2\mu_3\mu_4}\epsilon_{\nu_1\nu_2\nu_3\nu_4} = -\sum_{\pi\in S_4} sign\pi \prod_{i=1}^{4}(g_{\mu_i\nu_{\pi(i)}} - \hat{g}_{\mu_i\nu_{\pi(i)}}) \qquad (5)$$

$$\mathrm{Tr}(\gamma_5\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}) = \epsilon_{\mu_1\mu_2\mu_3\mu_4}\mathrm{Tr}\,1 \qquad (6)$$

$$\{\gamma_\alpha,\gamma_5\} = \{\hat{\gamma}_\alpha,\gamma_5\} = 2\hat{\gamma}_\alpha\gamma_5 \qquad (7)$$

4. **Normal form (NF)**

4.1 eliminate γ_5 using (6)

4.2 to eliminate Tr use $\{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu}$, $\mathrm{Tr}\gamma_\mu = 0$, $\mathrm{Tr}\,1 = 4$

4.3 antisymetrize product of γ's

4.4 use (5) to remove ϵ-tensors

4.5 eliminate $g_{\mu\nu}$'s, $\hat{g}_{\mu\nu}$'s performing contractions with them

4.6 use $g_{\mu\mu} = D$, $\hat{g}_{\mu\mu} = 4 - D$, $p_\mu q_\mu = p\cdot q$,
$\hat{p}_\mu q_\nu = p_\mu\hat{q}_\mu = \hat{p}_\mu\hat{q}_\mu = \widehat{p\cdot q}$

5. **Regularization**

5.1. Interpreting all covariants in the described algebra, considering the amplitude as formal power series, performing all the derivatives over u_ℓ variables and putting $u_\ell = 0$ and reducing the expressions to their normal form one obtains explicit D-dependence of the amplitude, which can be used to define the counterterms.

5.2 With determined counterterms, one can interpret the amplitude as a 4-dimensional setting all "hatted" objects to zero. The Feynman integrand $I_\epsilon(\underline{p}, \underline{u}, \underline{\alpha})$ becomes a distribution in $M-1$ 4-momenta and amplitude, $\mathcal{T}_{G,\epsilon}(p, D)$ a distribution in M 4-momenta, which corresponds to a meromorphic function in the D-dimensional complex plane having the singularities lying on the manifolds which correspond to subdiagrams H of the graph G (r_ℓ (v_i)is degree of Z_ℓ (V_i) : $\omega_H =$ is degree of divergence)

$$\omega_H(D) \;=\; Dh_H - 2L_H + \sum_\ell r_\ell(+\sum_i v_i) \tag{8}$$

1.4. Renormalized amplitude in terms of labelled forests

Using 6-th property of the labelled forest the amplitude becomes

$$
\begin{aligned}
\mathcal{T}_G &= \lim_{\epsilon\to 0} \mathcal{T}_{G,\epsilon} = \lim_{\epsilon\to 0} \int d\underline{\alpha} I_{G,\epsilon}(\underline{p},\underline{\alpha}) \\
&= \lim_{\epsilon\to 0} \sum_{(\mathcal{C},\sigma)} \int_{\mathcal{D}(\mathcal{C},\sigma)} d\underline{\alpha} I_{G,\epsilon}(\underline{p},\underline{\alpha})
\end{aligned}
\tag{9}
$$

Using the forest formula, assuming the action of C_H on I_G is known, and defining C_H so that, for the forest F, $\int_{\mathcal{D}(\mathcal{C},\sigma)} d\underline{\alpha} \prod_{H\in F}(-C_H) I_{G,\epsilon}(\underline{p},\underline{\alpha}) = 0$ unless $F \subseteq \mathcal{C}$, renormalized amplitude reads ($\mathcal{F} = \{F\}$ is set of all forests of G).

$$
\begin{aligned}
R^G &= \lim_{\epsilon\to 0} R^G_{\nu,\epsilon} = \lim_{\epsilon\to 0} \sum_{(\mathcal{C},\sigma)} \int_{\mathcal{D}(\mathcal{C},\sigma)} d\underline{\alpha} \sum_{F\in\mathcal{F}} \prod_{H\in F} (-C_H) I_{G,\epsilon}(\underline{p},\underline{\alpha}) \\
&= \lim_{\epsilon\to 0} \sum_{(\mathcal{C},\sigma)} \int_{\mathcal{D}(\mathcal{C},\sigma)} d\underline{\alpha} \prod_{H\in\mathcal{C}} (1 - C_H) I_{G,\epsilon}(\underline{p},\underline{\alpha})
\end{aligned}
\tag{10}
$$

1.5. The amplitude using the variables in terms of which the divergent part can be clearly expressed.

Variables α_ℓ are inapropriate for extracting the amplitude singulaties. New variables appropriate for this purpose $\{(\underline{t}, \underline{\beta}) = (t_H, H \in \mathcal{C}), \ell \in \mathcal{L}'_G = \mathcal{L}_G \backslash \sigma(\mathcal{C})\}$ and auxiliary variables ζ_H and ξ_H are introduced

$$\alpha_\ell = \begin{cases} \prod_{H \subset H' \in \mathcal{C}} t^2_{H'} = t^2_H \xi^2_H = \zeta^2_H \text{ if } \ell = \sigma(H), \ H \in \mathcal{C} \\ \beta_\ell \zeta^2_H \text{ if } \ell \in \mathcal{L}'_{\bar{H}}, \ H \in \mathcal{C} \end{cases} \tag{11}$$

and taking $\beta_\ell = 1$ for $\ell \in \sigma(H)$. In terms of new variables the image of the $\mathcal{D}(\mathcal{C}, \sigma)$ reads

$$\mathcal{D}(\mathcal{C}, \sigma) = \{(\underline{t}, \underline{\beta}) | 0 \leq t_G < \infty; 0 \leq t_H \leq \text{ for } H \neq G; \ 0 \leq \beta_\ell \leq \text{ for } \ell \in \mathcal{L}'_G\} \tag{12}$$

1.6. Regularized amplitude in terms of (t, β) variables

Substituting α_ℓ's in terms of (t, β) one obtains $\mathcal{D}(\mathcal{C}, \sigma)$ contributon to $\mathcal{T}_{G,\epsilon}$

$$\int_{\mathcal{D}(\mathcal{C},\sigma)} d\underline{\alpha} I_{G,\epsilon}(\underline{p}, \underline{\alpha}) = \prod_{\ell \in \mathcal{L}'_G} \int_0^1 d\beta_\ell \prod_{H \in \mathcal{C}} \frac{2dt_H}{t_H} t_H^{\nu h_H - \omega_H + r_H} \times Z\Big(-i \frac{\partial}{\partial u_\ell} \Big) g(\underline{p}, \underline{u}, \underline{t}, \underline{\beta}, \nu) \tag{13}$$

$g(\underline{p}, \underline{u}, \underline{t}, \underline{\beta}, \nu)$ beeing an analytic function.
Set of all mappings the $H \to H/\mathcal{M}(\mathcal{H}) = \bar{\mathcal{H}}$ intoduces linear tranformation of $\underline{p}$ momenta to the momenta $q = Rp$ associated with vertices of $\bar{H}$. With accompanied decomposition of $\underline{u} = (\underline{u}_H, \ell \in \mathcal{L}_{\bar{H}}, H \in \mathcal{C})$, and transformed incidence matrix eR^T one can express $V(\underline{p}, \underline{u}, \underline{\alpha})$ in terms of the $\bar{H}$ momenta q:
Introducing scaled dimensionless quantities $\underline{\tilde{q}} = (\underline{\tilde{q}}_H, q_H\zeta_H, H \in \mathcal{C})$, $\underline{\tilde{u}} = (\underline{\tilde{u}}_H = u_H/\zeta_H, H \in \mathcal{C})$, $\underline{\tilde{e}} = \underline{\tilde{e}}_{HH'} = e_{HH'}\zeta_H/\zeta_{H'}$, $H, H' \in \mathcal{C}$ one obtains $V(\underline{p}, \underline{u}, \underline{\alpha})$ in terms of dimensionless quantities

$$V(\underline{p},\underline{u},\underline{\alpha}) = (\underline{\tilde{q}}^T,\underline{\tilde{u}}^T)\tilde{M}^{-1}\begin{pmatrix}\underline{\tilde{q}}\\ \underline{\tilde{u}}\end{pmatrix}$$

$$\tilde{M} = \begin{pmatrix}0 & -2\tilde{e}^T\\ -2\tilde{e} & -4\beta\end{pmatrix}, \beta = (\beta_\ell = \alpha_\ell/\zeta_H^2, H \in \mathcal{C}) \quad (14)$$

Properties of R, eR^T, $d(\alpha)$,

1. R is up to permutations triangular, with diagonal values 1, and $\det R = \pm 1$
2. eR^T has the block diagonal structure with: $e_{HH'} = 0$ for $\bar{H}$ if $H \not\supseteq H'$; $e_{HH'} =$ incidence matrix $e_{\bar{H}}$ if $H = H'$; if $H \supset H'$, $e_{HH'}$ indicates how H' is contained in H.
3. $d(\alpha) = \tilde{d}\prod_{H\in\mathcal{C}}\zeta_H^{2h_{\bar{H}}}$, $\tilde{d} = \tilde{M}/(-4)^L\ \forall\ell \in \mathcal{L}_G$
4. $\tilde{M}$ as $\tilde{M}(\underline{t},\underline{\beta})$ are independent of t_G, and polynomials in remaining variables
5. if for $\mathcal{C}$ $h_{\bar{H}} \leq 1\ \forall H \in \mathcal{C}$ then $\tilde{d} \geq 1$ so $\tilde{d}^{-1}$ and $\tilde{M}^{-1}$ are C^∞ in $(\underline{t},\underline{\beta})$. (e.g. for 1PI G, and $(\mathcal{C},\sigma)$)

Using result (14) one obtains the expression for the $\mathcal{D}(\mathcal{C},\sigma)$ contribution to the regularized amplitude $\mathcal{T}_{G,\epsilon}$, from Eq. (13)

$$\begin{aligned}
&\int_{\mathcal{D}(\mathcal{C},\sigma)} d\underline{\alpha}_{G,\epsilon} I_{G,\epsilon}(\underline{p},\underline{u}) \\
&= \prod_{\ell\in\mathcal{L}'_G}\int_0^1 d\beta_\ell \prod_{H\in\mathcal{C}}\int_0^{\theta_H}\frac{2dt_H}{t_H}\, t_H^{\nu h_H-\omega_H} Z_H\Big(\frac{\partial}{\partial\tilde{u}}\Big) g_\epsilon(\underline{q},\underline{u},\underline{t},\underline{\beta},\nu) \\
&= \prod_{H\in\mathcal{C}}\Big\{\int d\mu_H \zeta_H^{\nu-\bar{\omega}_H} Z_H\Big(-i\frac{\partial}{\partial\tilde{u}}\Big)\Big\} g_\epsilon(\underline{q},\underline{u},\underline{t},\underline{\beta},\nu) \\
\nu &= D-4;\ \omega_H=\omega_{\bar{H}}-\sum_{H'\in\mathcal{M}(H)}\omega_{H'};\ \theta_G=\infty,\ \theta_H=1, H\neq G
\end{aligned}$$

$$\int d\mu_H = \int_0^{\theta_H}\frac{2dt_H}{t_H}\int_0^\infty \prod_{\ell\in\mathcal{L}_{\bar{H}}} d\beta_\ell \qquad (15)$$

Z_H are polynomials in $\underline{t}$, $g_\epsilon(\underline{q},\underline{u},\underline{t},\underline{\beta},\nu)$ is C^∞ in $(\underline{t},\underline{\beta})$, analytic in ν, and exponentially decreasing as $t_G\to\infty$. Therefore all divergences are contained in factors $t_H^{\nu h_H-\omega_H-1}$.

1.7. Treatment of $t_H^{\nu h_H - \omega_H - 1}$ factors

t_H^{λ} may be understood as a distribution (generalized function) with a property it corresponds to a meromorphic function of λ with poles at all nonnegative numbers (ω):

$$\begin{aligned} t^{\nu h - \omega - 1} &= (-)^{\omega} \delta^{(\omega)}(t)/(\omega! \nu h) + \text{ regular at } \nu = 0, \\ \delta^{(\omega)}(t)/\omega! &:= 0 \quad \text{for} \quad \omega < 0 \end{aligned} \tag{16}$$

Using it the t_H integral is splitted into divergent and regular part

$$\begin{aligned} &\int_0^{\theta} dt_H t_H^{\nu h_H - \omega_H - 1} g(\ldots t_H \ldots) \\ &= (\omega_H! \nu h_H)^{-1} \left(\frac{d}{dt_H}\right)^{\omega_H} \Big|_{t_H=0} + \int_0^{\infty} (\ldots)_{reg.} dt_H \end{aligned} \tag{17}$$

$$\begin{aligned} \int_0^{\infty} (\ldots)_{reg.} dt_H &= \int_0^{\infty} dt_H t_H^{\nu h_H - \omega_H - 1} \Big[g(\ldots t_H \ldots) \\ &- \sum_{k=0}^{\omega_H - 1} \frac{t_H^k}{k!} g_{\epsilon}^{(k)}(\ldots t_H = 0 \ldots) - \theta(1 - t_H) \Big(\frac{t_H^{\omega_H}}{\omega_H!} g^{(\omega_H)}(\ldots t_H = 0 \ldots) \Big) \Big] \end{aligned} \tag{18}$$

1.8. Definition of the counterterm, and definition of the renormalized amplitude

1. Algebra $\mathcal{A}_G$ of covariants $\{p_{i,\mu}\}_{i\in\mathcal{V}_G}$, $\{u_{\ell,\mu}\}_{\ell\in\mathcal{L}_G}$,
2. Def. $C(A\in\mathcal{A}_G)$ = sing. part at $n=4$ of $NF(A|_{\underline{u}=0})$
3. Definition of domain for the C_H: $\mathcal{D}_H$:

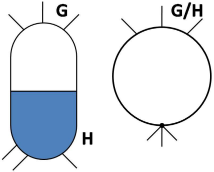
3.1. $\mathcal{G}_G=\{\hat{X}=\{X,F_X,(\mathcal{C},\sigma)_X\}\}$, $H\subset G$, $F_X\in\hat{\mathcal{F}}'=\{$forests of 1PI proper subgraphs of $X\}$, $(\mathcal{C},\sigma)$ a labelled forest for X/F_X.

3.2. $\mathcal{E}_G=\{f:\mathcal{G}_G\to\mathcal{A}_G\}$, $f(\hat{X})=f(\hat{X})(\underline{p},\underline{u})$, $(\underline{p},\underline{u})\in X/F_X$

3.3. $\mathcal{D}_H=\{f\in\mathcal{E}_G:\forall\hat{X}$ of the form $\hat{X}=(H,\emptyset,(\mathcal{C},\sigma)_H)$, $C(f(\hat{X}))$ is a polynomial in $\underline{p}\}$

4. Definition of counterterm C_H:

4.1. Some defs.: $\hat{H}=(H,\emptyset,(\mathcal{C},\sigma)_H)$, $\widehat{X:H}=(X,F_X\cup\{H\},(\mathcal{C},\sigma)_{H:X})$;
$((\mathcal{C},\sigma),(\mathcal{C},\sigma)_{H:X})$ pair of labelled forests for H resp.
$H:X=X/F_X\cup\{H\}$ corresponding to $(\mathcal{C},\sigma)_X$,
$U_X:\underline{q}_H\to\underline{q}_H-e^T_{GH}\partial/\partial\underline{u}^T_G$



4.2. On $\mathcal{D}_H$ the operator C_H is defined by

$$(C_H f)(\hat{X}) = \begin{cases} f(\hat{X}) \text{ if either } X \cap H = \emptyset \text{ or } H \subset H' \text{ for some } H' \in F_X \\ C(f(\hat{H})) \text{ if } H = X \text{ and } F_X = \emptyset \\ U_H(C(f(\hat{H}_0)))f(\widehat{X:H}) \text{ if } H \notin F_X, F_X \cup \{H\} \in \mathcal{F}'_X \\ 0 \text{ else} \end{cases} \tag{19}$$

5. The definition of the renormalized amplitude R_G:
The renormalized amplitude (10) is to be interpreted as

$$\begin{aligned} R_G &= \lim_{\epsilon \to 0} \sum_{(\mathcal{C},\sigma)} \int_{\mathcal{D}(\mathcal{C},\sigma)} d\underline{t}\, d\underline{\beta} \\ &\times \Big[\prod_{H \in \mathcal{C}} (1 - C_H) I_{G,\epsilon}(\underline{p}, \underline{\alpha}) \Big] (G, \emptyset, \{\mathcal{C}, \sigma\})|_{\underline{u}=0} \end{aligned} \tag{20}$$

1. The singular parts of the subdiagrams H:

1.1. The singular part of the H integral is defined as the divergence of H integral after removal of singular parts of all its subdiagrams.

1.2. The singular part of G (t_G integral)

$$\int_0^\infty \frac{dt_G}{t_G} t_G^{\nu h_G} g_\epsilon(\underline{t}, \underline{\beta}, \underline{q}) \tag{21}$$

follows from the assumption that all spin polynomials Z_ℓ are homogeneous in q, m_ℓ and $\sqrt{\epsilon}$ so the only dependence on those variables in $g_\epsilon(\underline{t}, \underline{\beta}, \underline{q})$ comes in combinations $t_G \underline{q}$, $t_G m_\ell$, $t_G\sqrt{\epsilon}$. While for $\omega_G < 0$ there is no singularity, for $\omega_G \geq 0$ the singularity is proportional to $[(\omega_G!)^{-1}(d/dt)^{\omega_G} g_\epsilon]_{t_G=0}$ which is homogeneous polynomial of order ω_G in mentioned three variables.

1.3. In similar way one can treat the reduced diagrams G/H leading to the homogeneous polynomials of order $\omega_G - \omega_H$ ($h_G = h_{G/H} + h_H$, $L_G = L_{G/H} + L_H$).

1.4. Relating the expressions for integrals I_G and $I_{G/H}$ intregrals (proof not shown) one obtains the relation between the expression for the generalized vertex obtained contracting H to the point $\xi_H^{\omega_H} P(q_H)$ and integrals I_G and $I_{G/H}$

$$U_H[\xi_H^{\omega_H} P(\underline{q}_H)] I_{G/H} \quad = \quad [(\omega_H!)^{-1} (d/dt)^{\omega_H} I_G]_{t_H = \underline{\tilde{u}} = 0} \qquad (22)$$

from which follows that the degree of divergence of I_H is ω_H.

2. Alternative expression for the renormalized amplitude

Let $X_0 \in \mathcal{F}'_G$ be any $X_0 \subset \mathcal{C}$. Define

$$X \quad = \quad \{H' \in \mathcal{C} : H' \subset H \text{ for some } H \in X_0\}$$

After performance of all subtractions corresponding to subgraphs $H \in X$, the contribution of the $(\mathcal{C}, \sigma)$ to $R_{G,\epsilon}$ is a sum of the terms of the form

$$
\begin{aligned}
(R^G_{\nu,\epsilon})_{(\mathcal{C},\sigma)_G} &= \prod_{H\in\mathcal{C}\setminus X} \left\{ \int d\mu_H (1-C_H) \zeta_H^{\nu-\omega_{\bar{H}}} Z_H\left(\frac{\partial}{\partial \tilde{u}}\right)\right\} \\
\times & \prod_{H'\in X_0} \left\{ \xi^{-\omega_{H'}} g_{H'}(\xi_H,\nu)\right\} g_{X'}(\underline{\tilde{q}},\underline{\tilde{u}},\underline{t},\underline{\beta},\nu)|_{\underline{\tilde{u}}=0} && (23) \\
\simeq & \prod_{H\in\mathcal{C}\setminus X} \int d\mu_H (1-C_H) \xi^{\nu-\omega_{\bar{H}}} Z_H\left(-\frac{\partial}{\partial \tilde{u}}\right) \\
\times & \prod_{H'\in\mathcal{M}(H)} (\xi_{H'}^{-\omega_{H'}} g_{H'}(\xi_{H'},\nu)) g_{X'}(\underline{\tilde{q}},\underline{\tilde{u}},\underline{t},\underline{\beta},\nu)|_{\underline{\tilde{u}}=0} && (24) \\
= & \left\{ \prod_{H\in\mathcal{C}\setminus H_1} \int d\mu_H (1-C_H)\right\} I^{G/H_1}_{\nu,\epsilon} \bigcirc R^{H_1}_{\nu,\epsilon} && (25) \\
= & \cdots = \left\{ \prod_{H\in\mathcal{C}^i_1} \int d\mu_H (1-C_H)\right\} I^{G/H_i}_{\nu,\epsilon} \bigcirc (R^{H_i}_{\nu,\epsilon})_{(C^i_2,\sigma_2)_{H_i}} && (26) \\
\equiv & (R^{G/H_i}_{\nu,\epsilon})_{(C^i_1,\sigma_1)_{G/H_i}} \bigcirc (R^{H_i}_{\nu,\epsilon})_{(C^i_2,\sigma_2)_{H_i}} && (27)
\end{aligned}
$$

$(\underline{t}, \underline{\beta})$ are scaling variables, $\underline{\tilde{u}}$ are scaled u's for G/X_0 ($\underline{u}_H = 0$ for $H \in X$), $\underline{\tilde{q}}$ are scaled momenta for $\mathcal{C}\backslash X$, $g_H \in J_H^K$ (set of analytic functions) for some K, $g_{X'}$ is element of abstract algebra of covariants analytic at $\nu = 0$ and exponentially decreasing.
$\{H_1, \ldots, H_{\sum_{i=1}^n h_{H_i}}\}$ (n in number of 1PI components of G) is a set of ordered subgraphs (smaller come first). H_1 is the first of them (smallest for 1PI G). $((\mathcal{C}_1^i, \sigma_1^i)_{G/H_i}, (\mathcal{C}_2^i, \sigma_2^i)_{H_i})$ is the pair of labelled forests associated with labelled forest $(\mathcal{C}, \sigma)_G$.

3. Theorem on Renormalizability

The singular part of the dimensionally regularized amplitude for any subgraph H of a graph G consists of poles of order $\leq h_H$, and is a polynomial of degree ω_H in the external momenta of the graph. Singular part vanishes if H is superficially convergent. The amplitudes $R_{G,\epsilon}(\underline{p}, D)$ remaining after performing subtractions corresponding to all 1PI subgraphs are analytic at $D = 4$ at any order of pertubation theory. The limit $\lim_{\epsilon\to 0} R_{G,\epsilon}$ exists in the external momenta and is analytic at $D = 4$, and it represents the renormalized amplitude $R_{G,D}$.

2. The basic Bonneau identity; Trace anomaly

2.1. Definition of $\bar{R}^G_{\nu,\epsilon}$ and inclusion of regularization scale

1. Definition of $\bar{R}^G_{\nu,\epsilon}$

$$\begin{aligned} \bar{R}^G_{\nu,\epsilon} &= \int d\mu_G \Big\{ \prod_{H_i \in \mathcal{C} \setminus G} \int \mu_{H_i}(1 - C_{H_i}) \Big\} I^G_{\nu,\epsilon}(\underline{p}, \underline{t}, \underline{\beta}) \quad (28) \\ R^G_{\nu,\epsilon} &= (1 - C_G)\bar{R}^G_{\nu,\epsilon} \end{aligned}$$

where $I^G_{\nu,\epsilon}(\underline{p}, \underline{t}, \underline{\beta})$ is $I^G_{\nu,\epsilon}(\underline{p}, \underline{\alpha})$ expressed in terms of variables $(\underline{t}, \underline{\beta})$.

2. inclusion of the regularization scale μ

2.1. The regularization scale μ can be included only through the momentum measure, not through the Lagrangian terms (parameters).

2.2. Renormalization theorem remains the same.

2.3. In MS scheme singular part is independent of μ.

1. Lemma 1.
For any meromorphic function of ν, $f(\nu)$ with poles at $\nu = 0$

$$[-\text{p.p.}(\nu f(\nu))] - [\nu \times (\text{-p.p.}(f(\nu)))] = \text{r.s.p.}(f(\nu)), \quad (29)$$
$$[\text{p.p.}(\nu f(\nu))] = [\nu \times (\text{p.p.}(f(\nu)))] - \text{r.s.p.}(f(\nu)) \quad (30)$$

where p.p.$f(\nu)$ is singular part of the Laurent series for $f(\nu)$ near $\nu = 0$ and r.s.p. is the residue of the simple pole (coefficient of $1/\nu$) of $f(\nu)$ at $\nu = 0$.

2. Lemma 2.
2.1. Let G be a 1PI graph and $R^{G}_{\nu,\epsilon}$ its renormalized amplitude.
2.2. Consider the same graph with ν attached to a "special" vertex V of G, G^{ν}, with amplitude and let $R^{G^{\nu}}_{\nu,\epsilon}$ be its amplitude. Let $O_{\delta}(x)$ be the monomial of fields and its derivatives corresponding to the "special" vertex V and δ its canonical dimension. The goal it to evaluate renormalized amplitude of $R^{G^{\nu}}_{\nu,\epsilon}$ and show lemma 2:

Lemma 2.

$$R^{G^{\nu}}_{\nu,\epsilon} - \nu R^{G}_{\nu,\epsilon} = \sum_{\gamma_i} U_{\gamma_i}(\text{r.s.p } \bar{R}^{\gamma_i}_{\nu,\epsilon}) R^{G/\gamma_i}_{\nu,\epsilon} \tag{31}$$

$$\equiv \sum_{\gamma_i} \sum_{(\mathcal{C},\sigma)_{G/\gamma_i}} \prod_{H\in\mathcal{C}} (1 - C_H) U_{\gamma_i}(\text{r.s.p.}(\bar{R}^{\gamma_i}_{\nu,\epsilon})) I^{G/\gamma_i}_{\nu,\epsilon} \tag{32}$$

Here $U_{\gamma_i} : \underline{q}_{\gamma_i} \to \underline{q}_{\gamma_i} - e^{T}_{G\gamma_i}\partial/\partial \underline{u}^{T}_{G}$, is an operator acting on $I^{G/\gamma_i}_{\nu,\epsilon}$

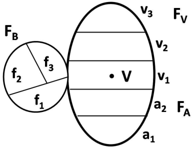
Proof:

1. divide a labelled forest $(\mathcal{C},\sigma)$ into three forests F_V, F_A and F_B:

F_V : $v^i \in F_V \Leftrightarrow v^i \in \mathcal{C}$ and $V \in v^i$

F_A : $a^i \in F_A \Leftrightarrow a^i \in \mathcal{C}, V \notin a^i$ and $a^i \subset v^i$

F_B : $b^i \in F_B \Leftrightarrow b^i \in \mathcal{C}$ and $b^i \notin F_V, F_A$



2. Perform all subtractions associated with the forest F_A, unaware of ν joined to the special vertex V

$$(R_{\nu,\epsilon}^{G^\nu})_{(\mathcal{C},\sigma)} \quad = \quad (R_{\nu,\epsilon}^{G^\nu/a^p})_{(\mathcal{C}_1,\sigma_1)_{G/a^p}} \bigcirc (R_{\nu,\epsilon}^{a^p})_{(\mathcal{C}_2,\sigma_2)_{a^p}} \tag{33}$$

where a^p is largest element of the forest F_A

3. The elements of $v_i \in$ are strictly ordered $v^1 \subset v_2 \subset \cdots \subset v_q \equiv G$. Perform the subtraction $-C_{v_1}$ first, according to that ordering, and using (30)

$$(R_{\nu,\epsilon}^{G^\nu/v^1}) \bigcirc (R_{\nu,\epsilon}^{v^1}) + U_{v_1}(\text{r.s.p.}(\bar{R}_{\nu,\epsilon}))_{(\mathcal{C}_2,\sigma_2)_{v_1}}(R_{\nu,\epsilon}^{G/v_1})_{(\mathcal{C}_1,\sigma_1)_{G/v_1}} \tag{34}$$

4. Perform subtractions associated with elements of F_B, which do not depend on the factor ν associated with special vertex V.

5. Repeat the procedure 3. and 4. for all remaining v_i-s, each giving one additional summand corresponding to the second term in (34).

6. Final result for the $(\mathcal{C},\sigma)$ contribution to $(R^{G^\nu}_{\nu,\epsilon})-(R^{G}_{\nu,\epsilon})$

$$
\begin{aligned}
&(R^{G^\nu}_{\nu,\epsilon})_{(\mathcal{C},\sigma)} - \nu(R^{G}_{\nu,\epsilon})_{(\mathcal{C},\sigma)} \\
&\quad = \sum_{\substack{v^i\in\mathcal{C}\\ V\in v^i}} U_{v^i}(\text{r.s.p.}(\bar{R}^{\nu^i}_{\nu,\epsilon}))_{(\mathcal{C}_2,\sigma_2)_{v_i}} (R^{G/v_1}_{\nu,\epsilon})_{(\mathcal{C}_1,\sigma_1)_{G/v_i}}
\end{aligned}
\tag{35}
$$

Here one-to-one correspondence between $(\mathcal{C},\sigma)$ and $((\mathcal{C}_2,\sigma_2)_{v_i}),(\mathcal{C}_1,\sigma_1))$ is used.

7. Summing over all labeled forests (each γ_i is in one of $(\mathcal{C},\sigma)$) the result can be written as

$$
\begin{aligned}
(R^{G^\nu}_{\nu,\epsilon}) - \nu(R^{G}_{\nu,\epsilon}) &= \sum_{\substack{\gamma^i:1PI\subseteq G\\ V\in\gamma^i}} U_{\gamma_i}\Big(\text{r.s.p.}\sum_{(\mathcal{C}_2,\sigma_2)}(\bar{R}^{\gamma^i}_{\nu,\epsilon})\Big)_{(\mathcal{C}_2,\sigma_2)} \\
&\quad\times \sum_{(\mathcal{C}_1,\sigma_1)}(R^{G/\gamma^i}_{\nu,\epsilon})\Big)_{(\mathcal{C}_1,\sigma_1)}
\end{aligned}
\tag{36}
$$

Note: Through (35) all information on renormalization of $(R^{G^\nu}_{\nu,\epsilon})$ is contained in renormalization of $(R^{G}_{\nu,\epsilon})$ and the expression (36).

2.3. Basic Bonneau identity(ies)

1. From renormalization theorem follows that singular part of the amplitude for γ_i subtracted for all its 1PI subgraphs is a polynomial in the masses and the external momenta of degree ω_{γ_i}. ω_{γ_i} depends on the canonical dimension δ of the special vertex V field monomial O_δ. So, one can reexpress r.s.p.$\bar{R}^{\gamma_i}_{\nu,\epsilon}$ as (p_i are external momenta of the subgraph γ_i with n_i external lines)

$$\bar{R}^{\gamma_i}_{\nu,\epsilon} = \sum_{r=0}^{\omega_{\gamma_i}(n_i,\delta)} \sum_{\substack{\{i_1,\dots,i_r\} \\ 1\leq i_j \leq n_i}} \frac{1}{r!} \left[\frac{\partial^r}{\partial p_{i_1}^{\mu_1} \dots \partial p_{i_r}^{\mu_r}} \text{r.s.p.} \bar{R}^{\gamma^{i}}_{\nu,\epsilon} \right]_{p_i=0} p_{i_1}^{\mu_1} \dots p_{i_r}^{\mu_r} \quad (37)$$

2. Rearranging the sum over subgraph γ_i according to the number n of external lines of subgraphs Γ_i of G containing the special vertex V one obtains

$$(R^{G^{\nu}}_{\nu,\epsilon}) - \nu(R^{G}_{\nu,\epsilon}) = \sum_{n=n_{min}}^{n=n_{max}} \sum_{\substack{\Gamma_i : V\in\Gamma_i\subseteq G \\ n\ ext.\ lines}} \sum_{r=0}^{\omega_{\Gamma_i}(n_i,\delta)} \sum_{\substack{\{i_1,\dots,i_r\} \\ 1\le i_j\le n_i}}$$

$$\left\{ \text{r.s.p.} \frac{\partial^r}{\partial p_{i_1}^{\mu_1}\dots\partial p_{i_r}^{\mu_r}} \bar{R}^{\Gamma^i}_{\nu,\epsilon}\Big|_{pi=0} \right\} \left[U_{\Gamma_i} \left(\frac{1}{r!} p_{i_1}^{\mu_1}\dots p_{i_r}^{\mu_r} \right) R^{G/\Gamma_i}_{\nu,\epsilon} \right] \quad (38)$$

Here $n_{min} \geq 2$ to have 1PI graph; $n_{max} = n_{max}(\delta, \text{theory}) =$ maximal number of external lines for a graphh with insertion $O_\delta(x)$ assuring the divergence of the considered term.

3. The insertion of $\frac{1}{r!} p_{i_1}^{\mu_1}\dots p_{i_r}^{\mu_r}$ leads to the insertion of the normal product

$$\frac{(-i)^r}{r!} N\Big[\prod_{k=1}^{n} \Big[\Big(\prod_{i_\alpha=k} \partial_{\mu_\alpha} \Big) \Phi(x) \Big]\Big] \quad (39)$$

4. Summing over all graphs G contributing to the general Green function containing $\nu O_\delta(x)$, $\langle 0|T(N[\nu O_\delta(x)]X)|0\rangle^{proper}$, in the limit $\nu,\epsilon \to 0$ one obtains

$$\langle 0|T(N[\nu O_\delta(x)]X)|0\rangle^{proper}$$
$$= \sum_{n\geq 2}^{n_{max}(\delta)} \sum_{r=0}^{\omega_O(n,\delta)} \sum_{\substack{\{i_1,\ldots,i_r\}\\ 1\leq i_j\leq n_i}} \Big\{ \frac{(-i)^r}{r!} \frac{\partial^r}{\partial p_{i_1}^{\mu_1}\ldots\partial p_{i_r}^{\mu_r}}$$
$$\text{r.s.p.}\overline{\langle 0|T(N[O_\delta(0)\tilde{\Phi}(p_1)\ldots\tilde{\Phi}(p_n)]X)|0\rangle}^{proper}\big|_{p_i=0}\Big\}$$
$$\times\langle 0|T\Big(N\Big[\frac{1}{n!}\prod_{k=1}^{n}\Big[\Big(\prod_{i_\alpha=k}\partial_{\mu_\alpha}\Big)\Phi\Big](x)\Big]X\Big)|0\rangle^{proper} \qquad (40)$$

The factor $1/n!$ is introduced to cancels $n!$ combinations appearing in evaluation of the matrix element; $\tilde{\Phi}$ is Fourier transform of Φ; bar on the Green function denotes that the overall subtraction has not been done.

5. Since (40) is valid for any X, it is valid on the operator level

$$
\begin{aligned}
& N[\nu O_\delta(x)] \\
&= \sum_{n\geq 2}^{n_{max}(\delta)} \sum_{r=0}^{\omega_O(n,\delta)} \sum_{\substack{\{i_1,\ldots,i_r\} \\ 1\leq i_j\leq n_i}} \Bigg\{ \frac{(-i)^r}{r!} \frac{\partial^r}{\partial p_{i_1}^{\mu_1}\ldots\partial p_{i_r}^{\mu_r}} \\
&\qquad \text{r.s.p.} \overline{\langle 0| T(N[O_\delta(0)\tilde{\Phi}(p_1)\ldots\tilde{\Phi}(p_n)]X)|0\rangle}^{proper} |_{p_i=0} \Bigg\} \\
&\qquad \times N\Bigg[\frac{1}{n!}\prod_{k=1}^{n}\Bigg[\Bigg(\prod_{i_\alpha=k}\partial_{\mu_\alpha}\Bigg)\Phi\Bigg](x)\Bigg] \qquad (41)
\end{aligned}
$$

which is Bonneau identity.

1. Trace anomaly

Normal product N used in Bonneau identity is not Wick normal product, but normal product in sense of Collins, the operation which makes the matrix elements of products of normal ordered sets of fields finite (it includes renormalization and $D \to 4$ limit). For that reason the contraction with the metric tensor does not commute with operator N. As a consequence one obtains nonzero values for

$$g_{\mu\nu}\langle 0|T(N[g_{\mu\nu}P(\phi,\partial\phi)(x)]X)|0\rangle - \langle 0|T(N[g_{\mu\mu}P(\phi,\partial\phi)(x)]X)|0\rangle \quad (42)$$

Let's consider this relation on the operator level in more details

$$\begin{aligned} g_{\mu\nu}N[g_{\mu\nu}P(\phi,\partial\phi)(x)] - N[g_{\mu\mu}P(\phi,\partial\phi)(x)] \\ = \quad N[\nu P(\phi,\partial\phi)(x)] \; = \; N[-\hat{g}_{\mu\mu}P(\phi,\partial\phi)(x)] \end{aligned} \quad (43)$$

This equation represents special form of trace anomaly.

Generalization of this relation is

$$
\begin{aligned}
& g_{\mu\nu} N[O_{\mu\nu\rho\ldots} P(\phi, \partial\phi)(x)] - N[g_{\mu\nu} O_{\mu\nu\rho\ldots}(\phi, \partial\phi)(x)] \\
= \quad & N[-\hat{g}_{\mu\nu} O_{\mu\nu\rho\ldots}(\phi, \partial\phi)(x)] \qquad (44)
\end{aligned}
$$

and represent general form of trace anomaly.

2. Bonneu identity for a trace anomaly

2.1 The general form of the regularized Feynman integrand for any graph H containingspecial vertex $V(O_{\mu\nu})(x)$ is (p_i are external momenta for the graph H)

$$
I_{\nu,\epsilon}^{H^{O_{\mu\nu}}} = \sum_{i,j \in v_H} p_\mu^i p_\nu^j M_{ij}^H + g_{\mu\nu} M^H \qquad (45)
$$

2.2 The corresponding matrix element for the contracted special vertex with $\Delta = -\hat{g}_{\mu\nu} O_{\mu\nu}$ is

$$
I_{\nu,\epsilon}^{H^{\Delta}} = - \sum_{i,j \in v_H} \widehat{p^i \cdot p^j} M_{ij}^H + \nu M^H \qquad (46)
$$

Using the same methods as for the finding the basic Bonneau identites, one finds the following relation for the contribution of the labelled forest $(\mathcal{C}, \sigma)$ to the $R^{G^{\Delta}}_{\nu,\epsilon} - (-\hat{g}_{\mu\nu}) R^{G^{O_{\mu\nu}}}_{\nu,\epsilon}$

$$\begin{aligned} &(R^{G^{\Delta}}_{\nu,\epsilon})_{(\mathcal{C},\sigma)} - (-\hat{g}_{\mu\nu})(R^{G^{O_{\mu\nu}}}_{\nu,\epsilon})_{(\mathcal{C},\sigma)} \\ &= \sum_{\gamma_i \in O, V \in v_i} U_{v_i} (\text{r.s.p.} \bar{M}^{v_i^{O_{\mu\nu}}}_{\nu,\epsilon})_{(C_2,\sigma_2)_{v_i}} (R^{G/\gamma_i}_{\nu,\epsilon})_{(C_1,\sigma_1)_{v_i}} \end{aligned} \quad (47)$$

The only difference in the results for $(R^{G^{\Delta}}_{\nu,\epsilon})_{(\mathcal{C},\sigma)} - (-\hat{g}_{\mu\nu})(R^{G^{O_{\mu\nu}}}_{\nu,\epsilon})_{(\mathcal{C},\sigma)}$ in Eqs. (35) (47) is the presence of r.s.p.$(\bar{M}^{v_i^{O_{\mu\nu}}}_{\nu,\epsilon})$ in place of the complete one r.s.p.$(\bar{R}^{v_i^{O_{\mu\nu}}}_{\nu,\epsilon})$. To characterize the operator whose insertion gives $\bar{M}^{v_i^{O_{\mu\nu}}}_{\nu,\epsilon}$ one introduces a new symmetric metric tensor $\check{g}_{\mu\nu}$ with properties

$$\check{g}_{\mu\mu} = 1, \quad \check{g}_{\mu\nu} g_{\nu\rho} = \check{g}_{\mu\nu} \hat{g}_{\nu\rho} = \check{g}_{\mu\rho} \tag{48}$$

The insertion $N[\check{\Delta}] \equiv N[\check{g}_{\mu\nu} O_{\mu\nu}] \equiv \check{g}_{\mu\nu} N[O_{\mu\nu}]$ is well defined and gives

$$\bar{R}^{v_i^{\check{\Delta}}}_{\nu,\epsilon} \equiv \sum_{i,j \in v_{v_1}} p_i \overset{\vee}{\cdot} p_j \; \bar{M}^{v_i^{O_{\mu\nu}}}_{ij} + \bar{M}^{v_i^{O_{\mu\nu}}} \tag{49}$$

Comparing with (47) one obtains

$$\text{r.s.p. } \bar{M}^{\gamma_i^{O_{\mu\nu}}}_{\nu,\epsilon} \equiv \text{r.s.p. } \bar{R}^{\gamma_i^{\check{\Delta}}}_{\nu,\epsilon}|_{\check{g}=0} \tag{50}$$

Therefore, as in the case of simple Bonneau identity

$$\begin{aligned} &(R^{G^{\Delta}}_{\nu,\epsilon}) - (-\hat{g}_{\mu\nu})(R^{G^{O_{\mu\nu}}}_{\nu,\epsilon}) \\ &= \sum_{\gamma_i, V \in \gamma_i} U_{\gamma_i}\left(\text{r.s.p.}\bar{R}^{\gamma_i^{\check{\Delta}}}_{\nu,\epsilon}|_{\check{g}=0}\right)(R^{G/\gamma_i}_{\nu,\epsilon}) \end{aligned} \tag{51}$$

The remaining steps are the same as for the basic Bonneau identity and lead to the the G graph contribution

$$(R_{\nu,\epsilon}^{G^{\Delta}}) - (-\hat{g}_{\mu\nu})(R_{\nu,\epsilon}^{G^{O^{\mu\nu}}}) = \sum_{n_{min}=2}^{n_{max}=4} \sum_{\substack{\Gamma_i : V \in \Gamma_i \subseteq G \\ n \ ext. \ lines}} \sum_{r=0}^{r=4-n} \sum_{\substack{\{i_1,\ldots,i_r\} \\ 1 \leq i_j \leq n}}$$

$$\left[\left[U_{\Gamma_i} \left(\frac{1}{r!} p_{i_1}^{\mu_1} \ldots p_{i_r}^{\mu_r} \right) R_{\nu,\epsilon}^{G/\Gamma_i} \right] \left\{ \text{r.s.p.} \frac{\partial^r}{\partial p_{i_1}^{\mu_1} \ldots \partial p_{i_r}^{\mu_r}} \bar{R}_{\nu,\epsilon}^{\Gamma^i} \Big|_{pi=0, \check{g}=0} \right\} \right. \tag{52}$$

Finally, summing over all graphs G contributing to $\langle 0 | T(N[-\hat{g}_{\mu\nu} O_{\mu\nu}](x) X) | 0 \rangle^{prop}$ in the limit $\nu, \epsilon \to 0$ one obtains

$$\langle 0 | T(N[-\hat{g}_{\mu\nu} O_{\mu\nu}](x) X) | 0 \rangle^{prop} = \sum_{n_{min}=2}^{n_{max}=4} \sum_{r=0}^{r=4-n} \sum_{\substack{\{i_1,\ldots,i_r\} \\ 1 \leq i_j \leq n}} \left[\text{r.s.p.} \left(\frac{(-i)^r}{r!} \right. \right.$$

$$\left. \left. \frac{\partial^r}{\partial p_{i_1}^{\mu_1} \ldots \partial p_{i_r}^{\mu_r}} \overline{\langle 0 | T(N[\check{g}_{\mu\nu} O_{\mu\nu}](0) \tilde{\phi}(p_1) \ldots \tilde{\phi}(p_n)) | 0 \rangle^{prop}} |_{p_i = \check{g} = 0} \right) \right]$$

$$\times \langle 0 | T \left(N \left[\frac{1}{n!} \prod_{k=1}^{n} \left(\prod_{i_\alpha = k} \partial_{\mu_\alpha} \right) \phi \right] (x) X \right) | 0 \rangle^{prop} \tag{53}$$

Which is another form of Eq. (2).

3. Bonnau identities and Slavnov Taylor identities

3.1. Regularized action principle

Regularized action principle states the following three equations hold in dimensionally regularized theories:

1. Arbitrary polynomial variations of the quantized fields ϕ, $\delta\phi(x) = \delta\theta(x)P(\phi(x))$ leave dimensionally invariant dimensionally regularized the generating functional for the Green functions Z_{DReg} and

$$\delta Z_{DReg}[J,K] \equiv \left\langle \delta(S_{free} + S_{INT}) exp\left\{ \frac{i}{\hbar} S_{INT}[\phi, J, K, \lambda] \right\} \right\rangle_0 \;=\; 0 \quad (54)$$

where (ϕ, λ, J and K represent fields, couplings, sources for the fields and sources for their BRST tranformations)

$$
\begin{aligned}
S_{free}[\phi;\lambda] &= S_{0\ free}[\phi;\lambda] \\
S_{int}[\phi,K,\lambda] &= S_{0\ int}[\phi,K,\lambda] + S_{sct}^{(n)}[\phi,K,\lambda] + S_{fct}^{(n)}[\phi,K,\lambda] \\
S_{INT}[\phi,J,K,\lambda] &= S_{int}[\phi,K,\lambda] + \int d^D x J_i(x)\phi_i(x) \qquad (55) \\
S_{DReg}^{(n)} &= S_{free}[\phi,\lambda] + S_{int}[\phi,K,\lambda] \qquad (56) \\
Z_{DReg}[J,K,\lambda] &= \int \mathcal{D}\phi \exp\left\{\frac{i}{\hbar}\left(S_{DReg}^{(n)} + \int J_i\phi_i\right)\right\} \qquad (57)
\end{aligned}
$$

2. Variations of external fields $E(x) \equiv (K(x), J(x))$ give

$$
\left\langle \frac{\delta S_{INT}}{\delta E(x)} \exp\left\{S_{INT}[\phi,J,K,\lambda]\right\}\right\rangle_0 = -i\hbar \frac{\delta Z_{DReg}[J,K,\lambda]}{\delta E(x)} \qquad (58)
$$

3. Variation of parameters give

$$
\left\langle \frac{\delta(S_{free}+S_{INT})}{\delta\lambda} \exp\left\{S_{INT}[\phi,J,K,\lambda]\right\}\right\rangle_0 = -i\hbar \frac{\delta Z_{DReg}[J,K,\lambda]}{\delta\lambda} \qquad (59)
$$

3.2. Slavnov Taylor identities and anomalous insertions

Although Quantum action principle assures that the action of the Slavnov operator (SS) on the renormalized effective action Γ_{ren} (which lives in 4 dimensions) gives zero result up to the anomalies (or (here BRST) symmetry breaking terms) it is more convenient to use the regularized action principle to compute the action of the Slavnov operator (SS_D) on theregularized action (Γ_{DReg}) in D dimensions and use this result to find the corresponding result for renormalized action defined as

$$LIM_{D\to 4}\Gamma_{DReg}[\phi,\Phi;K;\mu] \quad = \quad \Gamma_{ren}[\phi,\Phi;K;\mu] \tag{60}$$

Action of the Slavnov operator on the regularized effective action gives (proof not shown)

$$\begin{aligned} SS_D(\Gamma_{DReg}) &\equiv \int d^Dx(s_D\phi)\frac{\delta\Gamma_{DReg}}{\delta\phi} + \frac{\delta\Gamma_{DReg}}{\delta K_\Phi}\frac{\delta\Gamma_{DReg}}{\delta\Phi} \\ &= \Delta\cdot\Gamma_{DReg} + \Delta_{ct}\cdot\Gamma_{DReg} + \int d^Dx\left[\frac{\delta S_{ct}^{(n)}}{\delta K_\Phi(x)}\cdot\Gamma_{DReg}\right]\frac{\delta\Gamma_{DReg}}{\delta\Phi(x)} \end{aligned} \tag{61}$$

The three insertion operators Δ, Δ_{ct} and $\delta S_{ct}^{(n)}/\delta K_{\Phi}(x)$ are sources of spurious and essential anomalies. If they exist in the theory essential anomalies cannot be removed and in principle the theory cannot be renormalized. The spurious anomalies can always be removed by the renormalization procedure. Here we will deal with the spurious anomalies. If the theory does not have the essential anomalies the Slavnov operator in 4 dimensions gives zero result. This situation is similar to the one with the trace anomaly we studied before. The operators Δ, Δ_{ct} and $\delta S_{ct}^{(n)}/\delta K_{\Phi}(x)$ may be considered as an effect of dimensional regularization, that is as evanescent operators. The same procedure as done for the trace anomaly can be performed and it leads to the Bonneau identites given in the expression for Bonneau identities (2) what is our second goal:

$$
\begin{aligned}
& N[\hat{\Delta}](x)\cdot\Gamma_{ren} \\
= \quad & -\sum_{n=2}^{4}\sum_{\{j_1\dots j_n\}}\Bigg[\sum_{r=0}^{\delta(J)}\sum_{\substack{i_1\dots i_r\\ 1\leq i_j\leq n}}\Bigg\{\frac{(-i)^r}{r!}\frac{\partial^r}{\partial p_{i_1}^{\mu_1}\dots\partial p_{i_r}^{\mu_r}}(-i\hbar)\text{r.s.p.} \\
& \times\overline{\langle\tilde{\phi}_{j_1}(p_1)\dots\tilde{\phi}_{j_n}(p_n)N[\check{\Delta}](q=-\sum p_i)}^{1PI}_{K=0}\Bigg|_{p_i=0,\check{g}=0}\Bigg\} \\
& \times N\Bigg[\frac{1}{n!}\prod_{k=n}^{1}\Bigg\{\Bigg(\prod_{\{\alpha/i_\alpha=k\}}\partial_{\mu_\alpha}\Bigg)\phi_{j_k}\Bigg\}\Bigg](x)\cdot\Gamma_{ren} \\
& +\text{ext. source part}\Bigg]
\end{aligned} \tag{62}
$$

The more specific details on this last topic both from the theoretical and evaluation point of view will be given in the next talk by Hèrmes Bélusca.

thank you very much

$\epsilon\upsilon\chi\alpha\rho\iota\sigma\tau\omega$ $\pi o\lambda\upsilon$