Discrete Quantum Systems & Stochastics

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Outline

- **Time evolution problem for non-self adjoint operators**
  Brownian motion: *(Einstein, Langevin,... early 20th century)*.
  Fokker-Planck equ. (PDE) & **Langevin equ. (SDE)**.
- **I**. Connections QM & SDEs: Feynman-Kac formula, Itô vs Statonovich, Girsanov's theorem, geometric phases, SUSY QM..:

  Feynman's path integral.
- **II**. Discrete **Quantum Integrable Systems**: (DNLS) & associated SDEs. Expectation values by solving SDEs. Stochastic transport & heat, Hamilton-Jacobi (KPZ) & viscous Burgers equs.
- Generalizations, quantum field theories. Tensor random fields

Time evolution problem

- `Time evolution problem for non-self adjoint operators`

The PDE

$$\partial_t f(\mathrm{x}, t) = -\Big(\hat{L}_0 + u(\mathrm{x})\Big) f(\mathrm{x}, t),$$

$$\hat{L}_0 = \frac{1}{2}\sum_{i,j=1}^{M} g_{ij}(\mathrm{x})\frac{\partial^2}{\partial \mathrm{x}_i \partial \mathrm{x}_j} + \sum_{j=1}^{M} b_j(\mathrm{x})\frac{\partial}{\partial \mathrm{x}_j},$$

$g(\mathrm{x}) = \sigma(\mathrm{x})\sigma^T(\mathrm{x})$ $M \times M$ diff. matrix; $b(\mathrm{x})$ M-vector: drift; $u(x)$ potential.

- Generalized imaginary time Schrödinger's equation: path integral

Stochastic point of view

- **SDE** the *starting poin*t in stochastic analysis

The SDE

$$dx_t = b(x_t)dt + \sigma(x_t)dw_t,$$

w M-vector, comp.w_j; M-ind. Wiener proc.: $\mathbb{E}(d\mathrm{w}_{ti}) = 0$, $\mathbb{E}(d\mathrm{w}_{ti}d\mathrm{w}_{tj}) = \delta_{ij}dt$, $\mathrm{w}_{t=0} = 0$. w_t, *non-differentiable*!

- SDE yields $\hat{L}_0$ via Ito's formula & prob. measure in *Feynman-Kac*.
- **Opposite viewpoint**: SDE via Feynman's Path integral (time discretization)!

Fokker-Planck (Kolmogorov) equation

- The adjoint operator needed:

The adjoint

$$\hat{L}_0^\dagger f(\mathrm{x}, t) = \frac{1}{2}\sum_{i,j=1}^{M} \frac{\partial^2}{\partial \mathrm{x}_i \partial \mathrm{x}_j}\Big(g_{ij}(\mathrm{x}) f(\mathrm{x}, t)\Big) - \sum_{j=1}^{M} \frac{\partial}{\partial \mathrm{x}_j}\Big(b_j(\mathrm{x}) f(\mathrm{x}, t)\Big).$$

- `Fokker-Planck (Kolmogorov forward) equ.`

$$\partial_{t_1} f(\mathrm{x}, t_1) = \hat{L}_0^\dagger f(\mathrm{x}, t_1)$$

$t_1 \geq t_2$, $f(\mathrm{x}, t_2) = f_{t_2}(\mathrm{x})$.

- `Kolmogorov backward equ.`

$$-\partial_{t_2} f(\mathrm{x}, t_2) = \hat{L}_0 f(\mathrm{x}, t_2)$$

$t_2 \leq t_1$, $f(x, t_1) = f_{t_1}(x)$.

The canonical transform

- Solve Fokker-Planck; g complicates $\to$ (non-Gaussian measure!). Turn g to identity! **Simplified operator**:

$$\hat{L} = \frac{1}{2}\sum_{j=1}^{M}\frac{\partial^2}{\partial \mathrm{y}_j^{\,2}} + \sum_{j=1}^{M}\tilde{b}_j(\mathrm{y})\frac{\partial}{\partial \mathrm{y}_j} + u(\mathrm{y})$$

New set of parameters y_j (change of frame):

$$d\mathrm{y}_i = \sum_j \sigma_{ij}^{-1}(\mathrm{x})\ d\mathrm{x}_j, \qquad \det\sigma \neq 0,$$

- Induced drift

$$\tilde{b}(\mathrm{y}) = \sigma^{-1}(\mathrm{y})\Big(b(\mathrm{y}) - \frac{1}{2}(\nabla_{\mathrm{y}}\sigma^T(\mathrm{y}))^T\Big), \qquad \nabla_{\mathrm{y}} = \Big(\partial_{\mathrm{y}_1}, \dots, \partial_{\mathrm{y}_M}\Big)$$

one first solves for $\mathrm{x} = \mathrm{x}(\mathrm{y})$.

Path integral: identity diffusion matrix

- Start: F-P equ.: $\partial_t f(\mathrm{y}, t) = \hat{L}^\dagger f(\mathrm{y}, t) \to$ propagator.
  Trotter-Suzuki product (e.g. [*B. Simon '05*]),

$$
\begin{aligned}
f(\mathrm{y}, t) &= \int \prod_{j=1}^{M} dy'_j \; K(\mathrm{y}, \mathrm{y}' | t, t') f(\mathrm{y}', t') \\
&= \int \prod_{n=1}^{N} \prod_{j=1}^{M} dy_{jn} \; \prod_{n=1}^{N} K(\mathrm{y}_{n+1}, \mathrm{y}_n | t_{n+1}, t_n) f(\mathrm{y}_1, t_1).
\end{aligned}
$$

$y_1 = y_i, \; y_{N+1} = y_f$.

Path integral: identity diffusion matrix

- $M \times N$ space-time lattice:

$$K(\mathrm{y}_f, \mathrm{y}_i|t) = \int d\mathbf{q}\ \exp\Big[-\sum_{j,n} \frac{\big(\Delta \mathrm{y}_{jn} - \delta \tilde{b}_{jn}(\mathrm{y})\big)^2}{2\delta} + \delta \sum_n u_n(\mathrm{y})\Big]$$

$\delta = t_{n+1} - t_n$, $\Delta \mathrm{y}_n = \mathrm{y}_{n+1} - \mathrm{y}_n$.

- Lebesgue measure:

$$d\mathbf{q} = \frac{1}{(2\pi\delta)^{\frac{NM}{2}}} \prod_{n=2}^{N} \prod_{j=1}^{M} \mathrm{y}_{jn}$$

The propagator

- Expand the square & contin. time limit $(\delta \sim \frac{1}{N} \to 0)$:

$$K(\mathrm{y}_{f,i}|t) = \int d\mathbb{P} \exp\Big[\int \tilde{b}^T(\mathrm{y}_s)d\mathrm{y}_s - \frac{1}{2}\int_0^t \tilde{b}^T(\mathrm{y}_s)\tilde{b}(\mathrm{y}_s)ds + \int_0^t u(\mathrm{y}_s)ds\Big]$$

$$\sum_n \tilde{b}_n^T(\mathrm{y})\Delta\mathrm{y}_n \ \to \ \int \tilde{b}^T(\mathrm{y}_s)d\mathrm{y}_s$$

Key point: $\int \tilde{b}^T(\mathrm{y}_s)d\mathrm{y}_s \ \to$ **Itô's integral** (*forward*)

$$d\mathbb{P} = \lim_{\delta\to 0}\lim_{N\to\infty} \frac{1}{(2\pi\delta)^{\frac{NM}{2}}} \exp\Big[-\sum_{n=1}^{N}\sum_{j=1}^{M}\frac{(\Delta\mathrm{y}_{jn})^2}{2\delta}\Big] \prod_{n=2}^{N}\prod_{j=1}^{M} d\mathrm{y}_{jn}.$$

Discrete time SDE

- Recall

$$K(\mathrm{y}_f,\mathrm{y}_i|t) = \int d\mathbf{q}\ \exp\Big[-\sum_{j,n}\frac{\big(\Delta \mathrm{y}_{jn} - \delta\tilde{b}_{jn}(\mathrm{y})\big)^2}{2\delta} + \delta\sum_n u_n(\mathrm{y})\Big]$$

Alternatively: assume *Wiener paths* (discrete time SDE!):

$$\Delta \mathrm{y}_n - \delta\tilde{b}_n(\mathrm{x}) = \Delta \mathrm{w}_n$$

w_t indep. of y_t, $\mathrm{w}_0 = 0$, then the propagator:

$$K(\mathrm{y}_f,\mathrm{y}_i|t) = \int d\mathbb{M}\ \exp\Big[\int_0^t u(\mathrm{y}_s)ds\Big]$$

$$d\mathbb{M} = \lim_{\delta\to 0}\lim_{N\to\infty}\frac{1}{(2\pi\delta)^{\frac{NM}{2}}}\exp\Big[-\sum_{n=1}^{N}\sum_{j=1}^{M}\frac{(\Delta \mathrm{w}_{jn})^2}{2\delta}\Big]\prod_{n=2}^{N}\prod_{j=1}^{M}d\mathrm{w}_{jn}.$$

Corollaries

- Itô vs Stratonovich calculus

  Discretization scheme (forward vs symmetric).

$$\int \tilde{b}^T(\mathrm{y}_s)d\mathrm{y}_s = \int \tilde{b}^T(\mathrm{y}_s)\circ d\mathrm{y}_s - \frac{1}{2}\int_0^t \nabla_{\mathrm{y}}\tilde{b}(\mathrm{y}_s)ds$$

  Girsanov's theorem...

- Gauge theories; geometric phase

  Via Itô-Stratonovich correspondence:

$$\begin{aligned} K(\mathrm{y}_{f,i}|t) &= \int d\mathbb{P}\exp\Big[\int \tilde{b}^T(\mathrm{y}_s)\circ d\mathrm{y}_s - \frac{1}{2}\int_0^t \nabla_{\mathrm{y}}\tilde{b}(\mathrm{y}_s)ds\Big] \\ &\times \exp\Big[-\frac{1}{2}\int_0^t \tilde{b}^T(\mathrm{y}_s)\tilde{b}(\mathrm{y}_s)ds + \int_0^t u(\mathrm{y}_s)ds\Big]. \end{aligned}$$

  1^{st} term: standard geometric phase, Bohm-Aharonov.

Wiener measure

- $s \in [0,\ t]$, w_s, Wiener's representation of the Brownian path:

$$\mathrm{w}_s = \frac{\mathrm{f}_0}{\sqrt{t}} s + \sqrt{\frac{2}{t}} \sum_{k>0} \frac{\mathrm{f}_k}{\omega_k} \sin \omega_k s, \qquad \omega_k = \frac{2\pi k}{t}.$$

$\mathrm{f}_0 = \frac{\mathrm{w}_t}{\sqrt{t}}$ where f_k, M-vectors: f_{kj}, $j \in \{1,\ 2, \dots, M\}$ normal variables.

- Continuum limit, $N \to \infty$, $\delta \to 0$; $\mathrm{w}(s=0) = 0$, $\mathrm{w}(s=t) = \mathrm{w}_t$:

$$d\mathbb{M} = \frac{e^{-\frac{1}{2t}\mathrm{w}_t^T \mathrm{w}_t}}{(2\pi t)^{\frac{M}{2}}}\ d\mathbb{M}_0$$

$$d\mathbb{M}_0 = \prod_{k\geq 1} \prod_{j=1}^{M} \frac{d\mathrm{f}_{kj}}{\sqrt{2\pi}}\ \exp[-\frac{1}{2}\sum_{k\geq 1}\sum_{j} \mathrm{f}_{kj}^2].$$

Feynman-Kac

- *Solution of Fokker-Planck equation*, **Feynmann-Kac formula**:

$$f(\mathrm{x}_f, t) = \int d\mathrm{x}_i \; K(\mathrm{x}_f, \mathrm{x}_i | t) f_0(\mathrm{x}_i)$$
$$= \int d\mathbb{M} \; e^{\int_0^t u(\mathrm{x}_s) ds} f_0(\mathrm{x}_i)$$

  f_0 initial profile. Integration over final states $\rightarrow$ partition function.

- *Solution of backward Kolmogorov:*

$$f(\mathrm{x}_i, t) = \int d\mathbb{M} \; e^{\int_0^t u(\mathrm{x}_s) ds} f_f(\mathrm{x}_f)$$

Expectation Values

- *AIM*: compute expectation values $\langle \mathcal{O} \rangle$:

$$\langle \mathcal{O}(\mathrm{x}_s) \rangle = \frac{\mathbb{E}_t\Big(\mathcal{O}(\mathrm{x}_s)\ e^{\int_0^t u(\mathrm{x}_s) ds}\Big)}{\mathbb{E}_t\Big(e^{\int_0^t u(\mathrm{x}_s) ds}\Big)}, \qquad 0 \leq s \leq t$$

$$\mathbb{E}_t\Big(\mathcal{O}(\mathrm{x}_s)\Big) = \int d\mathbf{w_t}\ d\mathbb{M}_0\ \frac{e^{-\frac{1}{2t}\mathrm{w}_t^T \mathrm{w}_t}}{(2\pi t)^{\frac{M}{2}}} \mathcal{O}(\mathrm{x}_s).$$

- **Via solutions of SDEs**, alternative to the usual Lagrangian description (paths around the classical solution sim. to stochastic optimal control).

Quantum Integability: DNLS hierarchy

- Focus on DNLS hierarchy [*Kundu & Ragnisco '94*], [*Sklyanin '99*]:

$$I^{(1)} = \frac{1}{2}\sum_{j=1}^{M} z_j Z_j$$

$$I^{(2)} = \frac{1}{2}\sum_{j=1}^{M} z_j^2 Z_j^2 + \sum_{j=1}^{M} \big(c_j z_j - z_{j+1}\big) Z_j$$

$$I^{(3)} = \sum_{j=1}^{M} z_j Z_{j-1} - \sum_{j=1}^{M}(z_j Z_j + z_{j-1} Z_{j-1}) z_j Z_{j-1} + \frac{1}{3}\sum_{j=1}^{M}(z_j Z_j)^3$$

$[z_i,\ Z_j] = -\delta_{ij}$. $I^{(1)}$ M-harmonic oscillators.

Discr. stochastic transport & heat equs

- Focus on $I^{(2,3)}$, DST model [*Eilbeck & Scott '86*] . The map:

$$z_j \mapsto \mathrm{x}_j, \qquad Z_j \mapsto \partial_{\mathrm{x}_j}$$

$$I^{(k)} = \frac{1}{2}\sum_{j=1}^{M} \mathrm{x}_j^2 \partial_{\mathrm{x}_j}^2 - \sum_{j=1}^{M} \Delta^{(k-1)}(\mathrm{x}_{tj})\partial x_j + \ldots, \quad k = 2,\ 3$$

$\mathrm{x}_j = \mathrm{x}_{j+M}$, and $\Delta^{(1)}(x_j) = x_{j+1} - x_j$, $\Delta^{(2)}(x_j) = x_{j+2} - 2x_{j+1} + x_j$.

Discr. stochastic transport & heat equs

- The quantum equations of motion:

$$\frac{dz_j}{dt} = -\Delta^{(k-1)}(z_j) + z_j^2 Z_j, \qquad k = 2,\ 3$$

- The relevant SDEs:

$$d\mathrm{x}_{tj} = -\Delta^{(k-1)}(\mathrm{x}_{tj})dt + \mathrm{x}_{tj}d\mathrm{w}_{tj}, \qquad j \in \{1,\ 2, \ldots, M\}$$

Reduces to M-particle *Black-Scholes* model (*Geom. Brownian*).
Instead of perturbing around classical solution we solve SDEs!

Solving SDEs: integrators

- Introduce integrator factors to solve SDEs (e.g. [*Oksendal '03*]):

$$d\mathrm{x}_{tj} = b_j(\mathrm{x}_t)dt + \mathrm{x}_{tj} d\mathrm{w}_{tj},$$

- Introduce set of integrator factors:

$$\mathcal{F}_j(t) = \exp\Big(- \int_0^t d\mathrm{w}_{sj} + \frac{1}{2}\int_0^t ds\Big)$$

Define $\mathrm{y}_{tj} = \mathcal{F}_j(t)\mathrm{x}_{tj}$:

$$\frac{d\mathrm{y}_t}{dt} = \mathcal{A}^{(k)}(t)\mathrm{y}_t$$

For instance:

$$\mathcal{A}^{(2)}(t) = \sum_{j=1}^{M} \Big(e_{jj} - \mathcal{B}_j^{(2)}(t) e_{jj+1}\Big), \qquad \mathcal{B}_j^{(2)}(t) = \exp\Big(\Delta^{(1)}(\mathrm{x}_{tj})\Big)$$

The usual calculus rules apply.

Solving the SDEs

- Formal **solution** of linear problem (monodromy):

$$\mathrm{y}_t = \mathcal{P}\exp\Big(\int_0^t \mathcal{A}^{(k)}(s)ds\Big)\mathrm{y}_0,$$

$$\mathcal{P}\exp\Big(\int_0^t \mathcal{A}^{(k)}(s)ds\Big) = \sum_{n=0}^{\infty}\int_0^t\int_0^{t_n}..\int_0^{t_2} dt_n...dt_1\mathcal{A}^{(k)}(t_n)...\mathcal{A}^{(k)}(t_1)$$

$t \geq t_n \geq t_{n-1}\ldots \geq t_2.$

- Expansion→ deformed algebras

Stochastic transport & heat equations

- Hamiltonian of DST → Hamiltonian of $1+1$ QFT:

QFT Hamiltonian

$$H^{(k)} = \int dx \ \Big(\frac{1}{2}\varphi^2(x)\hat{\varphi}^2(x) - \partial_x^{k-1}\varphi(x)\hat{\varphi}(x)\Big), \quad k=2,\ 3$$

- SDEs → the stochastic transport ($k=2$) & stochastic heat ($k=3$) equations multiplicative noise:

$$\partial_t\varphi(x,t) = -\partial_x^{k-1}\varphi(x,t) + \varphi(x,t)\dot{W}(x,t)$$

Stochastic Hamilton-Jacobi & Burgers equations

- Stochastic heat equation, mapped to st. Hamilton-Jacobi (1d KPZ) & viscous Burgers (Cole-Hopf transf. [*Corwin '18*]), $\varphi = e^h$, $u = \partial_x h$:

St. Hamilton-Jacobi

$$\partial_t h(x,t) = -\partial_x^2 h(x,t) - (\partial_x h(x,t))^2 + \dot{W}(x,t)$$

- And $u = \partial_x h$:

St. Viscous Burgers

$$\partial_t u(x,t) = -\partial_x^2 u(x,t) - 2u(x,t)\partial_x u(x,t) + \partial_x \dot{W}(x,t).$$

Generalizations/Comments

- Let a tensor random field Y with $\mathrm{Y}_{i_1 i_2 \ldots i_d}$, $d \in \mathbb{N}$:

$$\hat{L}_0 = g_{i_1 \ldots i_d ; j_1 \ldots j_d}(\mathrm{Y}) \frac{\partial^2}{\partial \mathrm{Y}_{i_1 \ldots i_d} \partial \mathrm{Y}_{j_1 \ldots j_d}} + b_{i_1 \ldots i_d}(\mathrm{Y}) \frac{\partial}{\partial \mathrm{Y}_{i_1 \ldots i_d}}, \quad i_k \in \{1, \ldots, M\}.$$

  $g_{i_1 \ldots i_d ; j_1 \ldots j_d}$, $b_{i_1 \ldots i_d}$ tensor diffusion coefficients & drift comp. Generalized Itô's formula; $d\mathrm{w}_{t i_1 \ldots i_d}\ d\mathrm{w}_{t j_1 \ldots j_d} = \delta_{i_1 j_1} \ldots \delta_{i_d j_d} dt$,

- Cont. limit ($M \to \infty$) tensor fields $\to$ continuous random fields depend on t and $\mathbf{x} \in \mathbb{R}^d$, i.e

$$\mathrm{Y}_{t i_1 \ldots i_d} \ \to\ \varphi(\mathbf{x}, t), \qquad \mathrm{w}_{t i_1 \ldots i_d} \ \to\ W(\mathbf{x}, t)$$

- Stochastic quantization, relativistic (*time evolution*). $d = 2$ matrix models... *"Stochastic integrability"* (dressing).

Schematically

- PDEs, $\hat{L}_0$
  $\frac{1}{2}\sum g_{ij}\partial^2_{\mathrm{x}_i\mathrm{x}_j} + \sum b_j(\mathrm{x})\partial_{\mathrm{x}_j} \longrightarrow \frac{1}{2}\sum \partial^2_{\mathrm{y}_j} + \sum \tilde{b}_j(\mathrm{y})\partial_{\mathrm{y}_j}$
  $\longrightarrow \frac{1}{2}\sum D^2_{\mathrm{y}_j}$
- SDEs
  $d\mathrm{x}_t = b(\mathrm{x}_t)dt + \sigma(\mathrm{x}_t)d\mathrm{w}_t \longrightarrow d\mathrm{y}_t = \tilde{b}(\mathrm{y}_t)dt + d\mathrm{w}_t$
  $\longrightarrow d\mathrm{z}_t = d\mathrm{w}_t$

Corollaries

- Super-symmetry & drift

  Consider $\hat{L}$, $\hat{L}^{(s)}$; $\mathcal{D} = K^s - K$ *(SUSY, Atiyah-Singer index theorem)*:

$$\mathcal{D} = 2\int d\hat{\mathbb{M}}\ \exp\Big[\int_0^t u(\mathrm{y}_s)ds\Big]\ \sinh\big(\frac{1}{2}\int_0^t \nabla_{\mathrm{y}_s}\tilde{b}(\mathrm{y}_s)\big)\ ds,$$

$$\frac{d\hat{\mathbb{M}}}{d\mathbb{P}} = \exp\Big[-\frac{1}{2}\int_0^t \tilde{b}^T(\mathrm{y}_s)\tilde{b}(\mathrm{y}_s)ds + \int \tilde{b}^T(\mathrm{y}_s)\circ d\mathrm{y}_s\Big].$$

Remarks

- `Drift as super-potential`
  $\mathrm{D}_{\mathrm{y}_j} = \partial_{\mathrm{y}_j} + \tilde{b}_j(\mathrm{y})$ the covariant derivative:

$$\hat{L} = \frac{1}{2}\sum_j \mathrm{D}^2_{\mathrm{y}_j} + V(\mathrm{y})$$

$$V(\mathrm{y}) = -\frac{1}{2}\tilde{b}^T(\mathrm{y})\tilde{b}(\mathrm{y}) - \frac{1}{2}\nabla_{\mathrm{y}}\tilde{b}(\mathrm{y}) + u(\mathrm{y})$$

  $\tilde{b}$ super-potential produces V; Riccati equation.

- `Equivalent SDEs`
  Generalization of *Lamperti transform* SDE: $d\mathrm{y}_t = b(\mathrm{y}_t)dt + d\mathrm{w}_t$.

Corollaries

- `Super-symmetry & drift`
  Consider $\hat{L}$, $\hat{L}^{(s)}$; $\mathcal{D} = K^s - K$ *(SUSY, Atiyah-Singer index theorem)*:

SUSY QM

$$\mathcal{D} = 2\int d\hat{\mathbb{M}}\ \exp\Big[\int_0^t u(\mathrm{y}_s)ds\Big]\ \sinh\big(\frac{1}{2}\int_0^t \nabla_{\mathrm{y}_s}\tilde{b}(\mathrm{y}_s)\big)\ ds,$$

$$\frac{d\hat{\mathbb{M}}}{d\mathbb{P}} = \exp\Big[-\frac{1}{2}\int_0^t \tilde{b}^T(\mathrm{y}_s)\tilde{b}(\mathrm{y}_s)ds + \int \tilde{b}^T(\mathrm{y}_s)\circ d\mathrm{y}_s\Big].$$

Wiener measure

- Compute the measure:

$$d\mathbb{M} = \lim_{\delta\to 0}\lim_{N\to\infty} \frac{1}{(2\pi\delta)^{\frac{NM}{2}}} \exp\Big[-\frac{1}{2\delta}\sum_{n=1}^{N} \Delta \mathrm{w}_n^T \Delta \mathrm{w}_n\Big] \prod_{j=1}^{M}\prod_{n=2}^{N} d\mathrm{w}_{jn}$$

- $s \in [0,\ t]$, w_s, Wiener's representation of the Brownian path:

$$\mathrm{w}_s = \frac{\mathrm{f}_0}{\sqrt{t}} s + \sqrt{\frac{2}{t}} \sum_{k>0} \frac{\mathrm{f}_k}{\omega_k} \sin \omega_k s, \qquad \omega_k = \frac{2\pi k}{t}.$$

$\mathrm{f}_0 = \frac{\mathrm{w}_t}{\sqrt{t}}$ where $\mathrm{f}_k,\ k \in \{0,\ 1, \dots\}$ M-vectors: $\mathrm{f}_{kj},\ j \in \{1,\ 2, \dots, M\}$ normal variables.

Quantum Integrability & Stochastics

- **Quantum integrability**: (a) existence of quantum Lax pair (b) R-matrix & YBE; charges in involution
- The $(L_n,\ A_n)$ Lax pair, zero curvature condition (semi-discrete):

$$\dot{L}_n(\lambda) = A_{n+1}(\lambda)\ L_n(\lambda) - L_n(\lambda)\ A_n(\lambda)$$

  Quantum EoM analogous to Heisenberg's picture *(Korepin '83, AD & Findlay '17)*

- The transfer matrix:

$$\mathfrak{t}(\lambda) = tr(L_N(\lambda) \dots L_1(\lambda))$$

  Conserved quantities: $\ln t(\lambda) = \sum_k \frac{I^{(k)}}{\lambda^k}\ (\dot{\mathfrak{t}}(\lambda) = 0)$

Solving SDEs

- Solve SDEs via iteration. Boundary term -heat kernel- plays crucial role. Standard: $\hat{L} = \frac{1}{2}\sum_i \partial^2_{\mathrm{x}_i} + u(\mathrm{x})$, SDE

$$d\mathrm{x}_t = d\mathrm{w}_t \ \Rightarrow \ \mathrm{x}_s = \mathrm{x}_0 + \mathrm{w}_s.$$

  Presence of drift: relations between w_t and x_t complicated!

- *Examples*: M-harmonic oscillators, Orenstein-Uhlenbeck, Geometric Brownian...

The DST & Black-Scholes models

- Via the canonical transform:

$$I^{(2)} = \frac{1}{2}\sum_{j=1}^{M} \partial_{\mathrm{y}_j}^2 + \sum_{j=1}^{M}\big(C_j + B_j e^{\mathrm{y}_{j+1}-\mathrm{y}_j}\big)\partial_{\mathrm{y}_j}$$

  and the SDEs are

$$d\mathrm{y}_{tj} = \big(C_j + B_j e^{\mathrm{y}_{tj+1}-\mathrm{y}_{tj}}\big)dt + d\mathrm{w}_{tj}$$

  $B_j = 0$: constant drift (Black-Scholes), solvable.

- Also, associated to *Toda chain* ($\mathcal{H} = I + I^{\dagger}$):

$$\mathcal{H} = \frac{1}{2}\sum_{j=1}^{M} \partial_{\mathrm{y}_j}^2 + \sum_{j=1}^{M}\tilde{B}_j e^{\mathrm{y}_{j+1}-\mathrm{y}_j}$$

Stochastic transport equation

- Cont. limit: $M \to \infty$, $\delta \to 0$ ($\delta \sim \frac{1}{M}$):

$$\mathrm{x}_{tj} \;\to\; \varphi(x,t), \qquad \mathrm{w}_{tj} \;\to\; W(x,t)$$
$$\frac{\mathrm{x}_{tj+1} - \mathrm{x}_{tj}}{\delta} \;\to\; \partial_x \varphi(x,t), \qquad \delta \sum_j f_j \;\to\; \int dx\; f(x)$$

- $\big[\varphi(x),\ \hat{\varphi}(y)\big] = \delta(x-y)$, $\hat{\varphi}(x) = \frac{\partial}{\partial \phi(x)}$. *Brownian sheet* [*Prevôt & Röckner '07*], $(X_t^{(n)},\ Y_t^{(n)})$ indep. Brownian:

$$W(x,t) = \frac{\sqrt{L}}{\pi} \sum_{n \geq 1} \frac{1}{n} \left(X_t^{(n)} \cos \frac{n\pi x}{L} + Y_t^{(n)} \sin \frac{n\pi x}{L} \right)$$