

Formulation of Category Including Several Noncommutative Geometries

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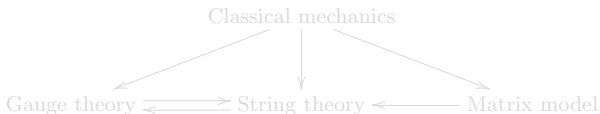
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Motivation

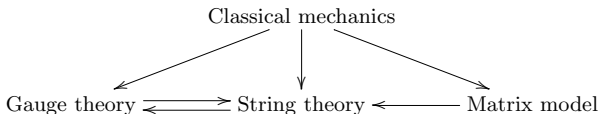
- Several ways of quantization of classical mechanics or classical field theory are well known.
 - Canonical Quantization,
 - Path Integral,
 - Matrix Regularization, etc.
- However, in quantization of classical gravity, objects that seem unrelated to quantization at first glance appear and they are related to each other.
 - String,
 - Matrix, etc.



- How can we generalize these quantizations? To describe such map, we think category is the most natural tool.

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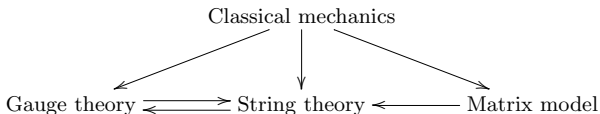
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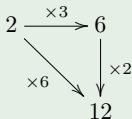
Preparation for Category Theory

Definition (Category)

A category \mathcal{C} consists of a set of objects $ob(\mathcal{C})$ and a set of morphisms $Mor(\mathcal{C})$. The morphisms satisfy the following conditions:

- There exists identity map.
- There exists composition.
- For all morphisms, these compositions are associative.

Example: Category of Multiplication



(Identity maps $\times 1$ are omitted.)

Definition (Functor)

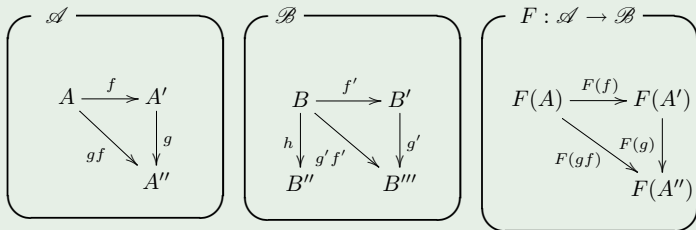
A functor is a map between two categories. That is, a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ define the following maps:

- $ob(\mathcal{A}) \rightarrow ob(\mathcal{B}),$
- $Mor(\mathcal{A}) \rightarrow Mor(\mathcal{B}).$

The functor $F : \mathcal{A} \rightarrow \mathcal{B}$ keeps the structure of \mathcal{A} .

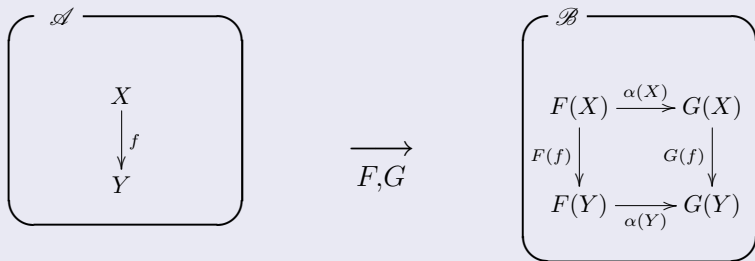
Example: Simple categories

In other words, a functor F gives a subcategory which is the same shape as \mathcal{A} in \mathcal{B} .



Definition (Natural Transformation)

Let F and G be functors from \mathcal{A} to \mathcal{B} . For all objects $A \in ob(\mathcal{A})$, a natural transformation $\alpha : F \rightarrow G$ is a set of morphisms $\{\alpha(A)\}$ between $F(A)$ and $G(A)$.



For all object $X, Y \in ob(\mathcal{A})$ and morphisms $f : X \rightarrow Y$, the natural transformation α satisfy the condition:

$$G(f) \circ \alpha(X) = \alpha(Y) \circ F(f).$$

Categorical limit

A categorical limit is the end of a sequence of objects which have some conditions.

Definition (Index Category)

An index category J is a category which does not have structures. That is, J is regarded as a directed graph equipped with composition maps.

Example: Index Category

A directed graph



Index category J



A set of objects and a set of morphisms in J are just vertices and edges.

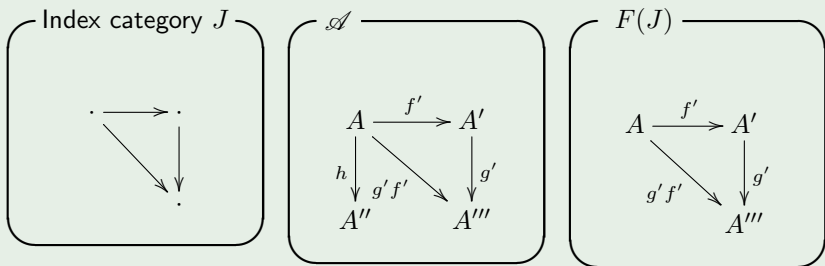


Definition (Diagram)

A diagram F of index category J in a category \mathcal{A} is a functor from an index category J to \mathcal{A} .

Example: Diagram F of J

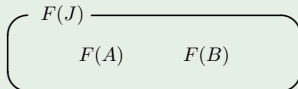
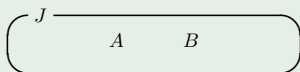
Recall that a functor determines a subcategory.



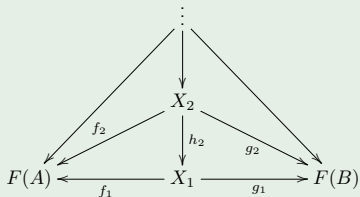
Definition (Categorical Limit)

Categorical limit by F of J is the end of a sequence of objects which have morphisms to all objects in $ob(F(J))$.

Example: Categorical Direct Product



The limit by F of J in \mathcal{A}



The end of a sequence of objects $\cdots \rightarrow X_2 \rightarrow X_1$ is X_1 .



Definition of Quantization Category

Definition (Pre- \mathcal{Q} category \mathcal{C})

Let $\mathcal{A}(M)$ be a Poisson algebra $(C^\infty(M), \cdot, \{, \})$ and $R\text{Mod}$ be a category of R -module. Pre- \mathcal{Q} category \mathcal{C} is a subcategory of $R\text{Mod}$ that satisfies the following conditions:

- $\mathcal{A}(M) \in \text{ob}(\mathcal{C})$.
- All objects $M_i \in \text{ob}(\mathcal{C})$ are Lie algebras.
- There exist morphisms T_k from $\mathcal{A}(M)$ to M_i such that

$$[T_k(f), T_k(g)]_k = i\hbar(T_k)T_k(\{f, g\}) + O(\hbar^{1+\epsilon}(T_k)).$$

Definition (Noncommutative Character χ)

For all objects $M_i \in \text{ob}(\mathcal{C})$, A noncommutative character $\chi : \text{ob}(\mathcal{C}) \rightarrow \mathbb{R}$ is defined as

$$\chi(M_i) = \max_{T_i \in \mathcal{C}((A)(M), M_i)} \hbar(T_i).$$

Definition

We define an index category J^\bullet and a diagram F^\bullet for pre- \mathcal{Q} category \mathcal{C} . For all objects $M_i \in \text{ob}(\mathcal{C})$ up to $\mathcal{A}(M)$, there exists J^\bullet such that

$$\exists T_{ij} \in \mathcal{C}(M_i, M_j), \chi(M_i) \leq \chi(M_j) \Leftrightarrow \exists (i, j) \in J^\bullet(i, j).$$

Then diagram $F^\bullet : J^\bullet \rightarrow \mathcal{C}$ is defined by

$$\text{ob}(\mathcal{C}) \ni i \mapsto F^\bullet(i) = M_i, \quad J^\bullet(i, j) \ni (i, j) \mapsto T_{ij}.$$

These definitions mean that all objects M_i are ordered by F^\bullet of J^\bullet and χ .

Proposition

The pre- \mathcal{Q} category \mathcal{C} has a limit M_∞ for F^\bullet of J^\bullet .

Recall that categorical limit by F of J is the end of a sequence of objects which have morphisms to all objects in $ob(F(J))$. In this case, objects in $F^\bullet(J^\bullet)$ are all objects in $ob(\mathcal{C})$ up to $\mathcal{A}(M)$. From the definition of \mathcal{C} , $\mathcal{A}(M)$ has morphisms from $\mathcal{A}(M)$ to $\forall M_i \in ob(\mathcal{C})$. So $\mathcal{A}(M)$ is a candidate of a limit always.

Definition (Quantization Category \mathcal{Q})

A quantization category $\mathcal{Q}(\mathcal{C}, J^\bullet, F^\bullet, \chi)$ is defined by satisfying the following conditions:

For all $f, g \in \mathcal{A}(M)$ and a morphism T from $\mathcal{A}(M)$ to limit M_∞ ,

$$Q1 \quad T(fg) - T(f)T(g) = 0$$

$$Q2 \quad [T(f), T(g)] - i\hbar(T)T(\{f, g\}) = 0$$

$Q1$ and $Q2$ are similar conditions with them in Berezin-Toeplitz quantization or Matrix regularization.

Matrix Regularization

Definition (J. Arnlind, J. Hoppe and G. Huisken (2012))

Let N_1, N_2, \dots be a strictly increasing sequence of positive integers and \hbar be a real value strictly positive decreasing function such that $\lim_{N \rightarrow \infty} N\hbar(N)$ converges. Let T_k be a linear map from $C^\infty(M) \rightarrow N_k \times N_k$ Hermitian matrices for $k = 1, 2, \dots$. If the following conditions are satisfied, then we call this a matrix regularization of (M, ω) .

- 1 $\lim_{k \rightarrow \infty} \|T_k(f)\| < \infty,$
- 2 $\lim_{k \rightarrow \infty} \|T_k(fg) - T_k(f)T_k(g)\| = 0,$
- 3 $\lim_{k \rightarrow \infty} \left\| \frac{1}{i\hbar(N_k)} [T_k(f), T_k(g)] - T_k(\{f, g\}) \right\| = 0,$
- 4 $\lim_{k \rightarrow \infty} 2\pi\hbar(N_k) \text{Tr} T_k(f) = \int_M f \omega,$

Definition (pre- \mathcal{Q} category for Matrix regularization)

Let $\{N_i\}$ be a strictly increasing sequence of \mathbb{Z}^+ and Let \hbar be a strictly decreasing function such that $\lim_{i \rightarrow \infty} N_i \hbar(N_i)$ converges. A pre- \mathcal{Q} category for Matrix regularization \mathcal{C}_{MR} is defined as follows:

- Set of objects:

$$ob(\mathcal{C}_{MR}) = \{\mathcal{A}(M), Mat_{N_k} \ (k = 1, 2, \dots), Mat_{\infty}\},$$

where Mat_{N_k} is a $N_k \times N_k$ matrix algebra and Mat_{∞} is the limit of $k \rightarrow \infty$.

- Set of morphisms:

$$\exists! T_i : \mathcal{A}(M) \rightarrow Mat_{N_i}, \quad \text{if } N_i \leq N_j, \quad \exists! T_{ij} : Mat_{N_j} \rightarrow Mat_{N_i}$$

such that

$$T_i = T_{ij} \circ T_j.$$

For \mathcal{C}_{MR} , the character χ is given as $\chi(Mat_{N_k}) = \hbar(N_k)$. Then the index category J_{MR}^\bullet and the diagram F_{MR}^\bullet is given by

J_{MR}^\bullet

$$\dots \longrightarrow 3 \xrightarrow{f_{23}} 2 \xrightarrow{f_{12}} 1$$

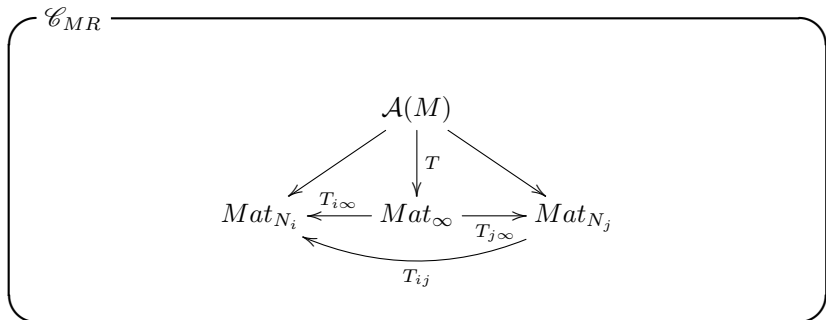
$F_{MR}^\bullet(J_{MR}^\bullet)$

$$\dots \longrightarrow F(3) \xrightarrow{F(f_{23})} F(2) \xrightarrow{F(f_{12})} F(1)$$

equal to

$$\dots \longrightarrow Mat_{N_3} \xrightarrow{T_{23}} Mat_{N_2} \xrightarrow{T_{12}} Mat_{N_1}$$

Since morphisms T_{ij} is oriented from the larger N_j to the smaller N_i , the limit by F_{MR}^\bullet of J_{MR}^\bullet is Mat_∞ or $\mathcal{A}(M)$. \mathcal{C}_{MR} is given by the following diagram for all i, j . Thus the limit is Mat_∞ .



Theorem

$\mathcal{Q}_{MR} = (\mathcal{C}_{MR}, J_{MR}^\bullet, F_{MR}^\bullet, \chi)$ is a quantization category for Matrix regularization.

Equivalence of Categories

Quantization categories for Deformation quantization and Pre-quantization can be constructed in analogy with \mathcal{Q}_{MR} . We discuss these quantization categories relationships.

Definition (Equivalent Categories)

Recall that the natural transformation $\alpha : F \rightarrow G$.

$$\begin{array}{ccc}
 F(X) & \xrightarrow{\alpha(X)} & G(X) \\
 F(f) \downarrow & & G(f) \downarrow \\
 F(Y) & \xrightarrow{\alpha(Y)} & G(Y)
 \end{array}$$

If all $\alpha(\cdot)$ are isomorphisms, a natural transformation α is called natural isomorphism. Let $F_1 : \mathcal{A} \rightarrow \mathcal{B}$ and $F_2 : \mathcal{B} \rightarrow \mathcal{A}$ be functors. If natural transformations $\alpha_1 : F_2 \circ F_1 \rightarrow I_{\mathcal{A}}$ and $\alpha_2 : F_1 \circ F_2 \rightarrow I_{\mathcal{B}}$ are isomorphisms, then \mathcal{A} and \mathcal{B} are called equivalent categories.

Theorem

Let \mathcal{Q}_{MR} and \mathcal{Q}_{DQ} be a quantization category for Matrix regularization and Deformation quantization, respectively. For the limit M_∞ , if $\mathcal{A}(M) \simeq M_\infty$ then \mathcal{Q}_{MR} and \mathcal{Q}_{DQ} are equivalent categories.

Theorem

Let \mathcal{Q}_{PQ} be a quantization category for Pre-quantization. Under some conditions, \mathcal{Q}_{PQ} and \mathcal{Q}_{MR} and \mathcal{Q}_{DQ} are equivalent categories.

That is,

$$\mathcal{Q}_{PQ} \simeq \mathcal{Q}_{MR} \simeq \mathcal{Q}_{DQ}.$$

under some conditions.

Conclusions and Discussions

- We define the category \mathcal{Q} as a generalization of some quantizations.
- We show the equivalence of several quantization.
- However, the quantization category has almost not physical structure. (Not enough to describe physical phenomena.)

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Thank you for listening.

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