Formulation of Category Including Several Noncommutative Geometries

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Contents

Motivation

- Several ways of quantization of classical mechanics or classical field theory are well known.
  - Canonical Quantization,
  - Path Integral,
  - Matrix Regularization, etc.

- However, in quantization of classical gravity, objects that seem unrelated to quantization at first glance appear and they are related to each other.
  - String,
  - Matrix, etc.

$$\begin{tikzcd}
 & \text{Classical mechanics} \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{Gauge theory} \arrow[r, shift left] & \text{String theory} \arrow[l, shift left] & \text{Matrix model} \arrow[l]
\end{tikzcd}$$

- How can we generalize these quantizations? To describe such map, we think category is the most natural tool.

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Preparation for Category Theory

Definition (Category)

A category $\mathscr{C}$ consists of a set of objects $ob(\mathscr{C})$ and a set of morphisms $Mor(\mathscr{C})$. The morphisms satisfy the following conditions:

- There exists identity map.
- There exists composition.
- For all morphisms, these compositions are associative.

Example: Category of Multiplication

$$
\begin{tikzcd}
2 \arrow[r, "\times 3"] \arrow[dr, "\times 6"'] & 6 \arrow[d, "\times 2"] \\
 & 12
\end{tikzcd}
$$

(Identity maps $\times 1$ are omitted.)

Definition (Functor)

A functor is a map between two categories. That is, a functor $F : \mathscr{A} \to \mathscr{B}$ define the following maps:

- $ob(\mathscr{A}) \to ob(\mathscr{B})$,
- $Mor(\mathscr{A}) \to Mor(\mathscr{B})$.

The functor $F : \mathscr{A} \to \mathscr{B}$ keeps the structure of $\mathscr{A}$.

Example: Simple categories

In other words, a functor F gives a subcategory which is the same shape as $\mathscr{A}$ in $\mathscr{B}$.

$\mathscr{A}$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "gf"'] & A' \arrow[d, "g"] \\ & A'' \end{tikzcd}$$

$\mathscr{B}$

$$\begin{tikzcd} B \arrow[r, "f'"] \arrow[d, "h"'] \arrow[dr, "g'f'"] & B' \arrow[d, "g'"] \\ B'' & B''' \end{tikzcd}$$

$F : \mathscr{A} \to \mathscr{B}$

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[dr, "F(gf)"'] & F(A') \arrow[d, "F(g)"] \\ & F(A'') \end{tikzcd}$$

Definition (Natural Transformation)

Let F and G be functors from $\mathscr{A}$ to $\mathscr{B}$. For all objects $A \in ob(\mathscr{A})$, a natural transfomation $\alpha : F \to G$ is a set of morphisms $\{\alpha(A)\}$ between $F(A)$ and $G(A)$.

$\mathscr{A}$ $\qquad\qquad\qquad\qquad$ $\mathscr{B}$

$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd} \quad \xrightarrow[F,G]{} \quad \begin{tikzcd} F(X) \arrow[r, "\alpha(X)"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"'] \\ F(Y) \arrow[r, "\alpha(Y)"'] & G(Y) \end{tikzcd}$$

For all object $X, Y \in ob(\mathscr{A})$ and morphisms $f : X \to Y$, the natural transformation α satisfy the condition:

$$G(f) \circ \alpha(X) = \alpha(Y) \circ F(f).$$

Categorical limit

A categorical limit is the end of a sequence of objects which have some conditions.

Definition (Index Category)

An index category J is a category which does not have structures. That is, J is regarded as a directed graph equipped with composition maps.

Example: Index Category

A directed graph

$$\begin{tikzcd} \cdot \arrow[r] & \cdot \arrow[d] \\ & \cdot \end{tikzcd}$$

Index category J

$$\begin{tikzcd} \cdot \arrow[r] \arrow[dr] & \cdot \arrow[d] \\ & \cdot \end{tikzcd}$$

A set of objects and a set of morphisms in J are just vertices and edges.

Definition (Diagram)

A diagram F of index category J in a category $\mathscr{A}$ is a functor from an index category J to $\mathscr{A}$.

Example: Diagram F of J

Recall that a functor determines a subcategory.

Index category J

$$\begin{tikzcd} \cdot \arrow[r] \arrow[dr] & \cdot \arrow[d] \\ & \cdot \end{tikzcd}$$

$\mathscr{A}$

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "h"'] \arrow[dr, "g'f'"] & A' \arrow[d, "g'"] \\ A'' & A''' \end{tikzcd}$$

$F(J)$

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[dr, "g'f'"'] & A' \arrow[d, "g'"] \\ & A''' \end{tikzcd}$$

Definition (Categorical Limit)

Categorical limit by F of J is the end of a sequence of objects which have morphisms to all objects in $ob(F(J))$.

Example: Categorical Direct Product

J

$A \qquad B$

$F(J)$

$F(A) \qquad F(B)$

The limit by F of J in $\mathscr{A}$

$$\begin{tikzcd}
 & \vdots \arrow[ddl] \arrow[d] \arrow[ddr] & \\
 & X_2 \arrow[dl, "f_2"'] \arrow[d, "h_2"] \arrow[dr, "g_2"] & \\
F(A) & X_1 \arrow[l, "f_1"] \arrow[r, "g_1"'] & F(B)
\end{tikzcd}$$

The end of a sequence of objects $\cdots \to X_2 \to X_1$ is X_1.

Definition of Quantization Category

Definition (Pre-$\mathscr{Q}$ category $\mathscr{C}$)

Let $\mathcal{A}(M)$ be a Poisson algebra $(C^\infty(M), \cdot\ , \{\ ,\ \})$ and RMod be a category of R-module. Pre-$\mathscr{Q}$ category $\mathscr{C}$ is a subcategory of RMod that satisfies the following conditions:

- $\mathcal{A}(M) \in ob(\mathscr{C})$.
- All objects $M_i \in ob(\mathscr{C})$ are Lie algebras.
- There exist morphisms T_k from $\mathcal{A}(M)$ to M_i such that

$$[T_k(f), T_k(g)]_k = i\hbar(T_k)T_k(\{f, g\}) + O(\hbar^{1+\epsilon}(T_k)).$$

Definition (Noncommutatve Character χ)

For all objects $M_i \in ob(\mathscr{C})$, A noncommutatve character $\chi : ob(\mathscr{C}) \to \mathbb{R}$ is defined as

$$\chi(M_i) = \max_{T_i \in \mathscr{C}((A)(M), M_i)} \hbar(T_i).$$

Definition

We define an index category $J^\bullet$ and a diagram $F^\bullet$ for pre-$\mathscr{Q}$ category $\mathscr{C}$. For all objects $M_i \in ob(\mathscr{C})$ up to $\mathcal{A}(M)$, there exists $J^\bullet$ such that

$$\exists T_{ij} \in \mathscr{C}(M_i, M_j),\ \chi(M_i) \leq \chi(M_j) \Leftrightarrow \exists (i,j) \in J^\bullet(i,j).$$

Then diagram $F^\bullet : J^\bullet \to \mathscr{C}$ is defined by

$$ob(\mathscr{C}) \ni i \mapsto F^\bullet(i) = M_i, \quad J^\bullet(i,j) \ni (i,j) \mapsto T_{ij}.$$

These definitions mean that all objects M_i are ordered by $F^\bullet$ of $J^\bullet$ and χ.

Proposition

The pre-$\mathscr{Q}$ category $\mathscr{C}$ has a limit M_∞ for $F^\bullet$ of $J^\bullet$.

Recall that categorical limit by F of J is the end of a sequence of objects which have morphisms to all objects in $ob(F(J))$. In this case, objects in $F^\bullet(J^\bullet)$ are all objects in $ob(\mathscr{C})$ up to $\mathcal{A}(M)$. From the definition of $\mathscr{C}$, $\mathcal{A}(M)$ has morphisms from $\mathcal{A}(M)$ to $\forall M_i \in ob(\mathscr{C})$. So $\mathcal{A}(M)$ is a candidate of a limit always.

Definition (Quantization Category $\mathscr{Q}$)

A quantization category $\mathscr{Q}(\mathscr{C}, J^\bullet, F^\bullet, \chi)$ is defined by satisfying the following conditions:
For all $f, g \in \mathcal{A}(M)$ and a morphism T from $\mathcal{A}(M)$ to limit M_∞,

$Q1$ $T(fg) - T(f)T(g) = 0$

$Q2$ $[T(f), T(g)] - i\hbar(T)T(\{f,g\}) = 0$

$Q1$ and $Q2$ are similar conditions with them in Berezin-Toeplitz quantization or Matrix regularization.

Matrix Regularization

Definition (J. Arnlind, J. Hoppe and G. Huisken (2012))

Let $N_1, N_2, \cdots$ be a strictly increasing sequence of positive integers and $\hbar$ be a real value strictly positive decreasing function such that $\lim_{N\to\infty} N\hbar(N)$ converges. Let T_k be a linear map from $C^\infty(M) \to N_k \times N_k$ Hermitian matrices for $k = 1, 2, \cdots$. If the following conditions are satisfied, then we call this a matrix regularization of (M, ω).

1. $\lim_{k\to\infty} \|T_k(f)\| < \infty$,
2. $\lim_{k\to\infty} \|T_k(fg) - T_k(f)T_k(g)\| = 0$,
3. $\lim_{k\to\infty} \|\frac{1}{i\hbar(N_k)}[T_k(f), T_k(g)] - T_k(\{f, g\})\| = 0$,
4. $\lim_{k\to\infty} 2\pi\hbar(N_k)\mathrm{Tr}T_k(f) = \int_M f\omega$,

Definition (pre-$\mathscr{Q}$ category for Matrix regularization)

Let $\{N_i\}$ be a strictly increasing sequence of $\mathbb{Z}^+$ and Let $\hbar$ be a strictly decreasing function such that $\lim_{i\to\infty} N_i\hbar(N_i)$ converges. A pre-$\mathscr{Q}$ category for Matrix regularization $\mathscr{C}_{MR}$ is defined as follows:

- Set of objects:

$$ob(\mathscr{C}_{MR}) = \{\mathcal{A}(M), Mat_{N_k}\ (k = 1, 2, \cdots), Mat_\infty\},$$

  where Mat_{N_k} is a $N_k \times N_k$ matrix algebra and Mat_∞ is the limit of $k \to \infty$.

- Set of morphisms:

$$\exists! T_i : \mathcal{A}(M) \to Mat_{N_i}, \quad \text{if } N_i \le N_j,\ \exists! T_{ij} : Mat_{N_j} \to Mat_{N_i}$$

  such that

$$T_i = T_{ij} \circ T_j.$$

For $\mathscr{C}_{MR}$, the character χ is given as $\chi(Mat_{N_k}) = \hbar(N_k)$. Then the index category $J^\bullet_{MR}$ and the diagram $F^\bullet_{MR}$ is given by

$J^\bullet_{MR}$

$$\cdots \longrightarrow 3 \xrightarrow[f_{23}]{} 2 \xrightarrow[f_{12}]{} 1$$

$F^\bullet_{MR}(J^\bullet_{MR})$

$$\cdots \longrightarrow F(3) \xrightarrow[F(f_{23})]{} F(2) \xrightarrow[F(f_{12})]{} F(1)$$

equal to

$$\cdots \longrightarrow Mat_{N_3} \xrightarrow[T_{23}]{} Mat_{N_2} \xrightarrow[T_{12}]{} Mat_{N_1}$$

Since morphisms T_{ij} is oriented from the larger N_j to the smaller N_i, the limit by $F^{\bullet}_{MR}$ of $J^{\bullet}_{MR}$ is Mat_{∞} or $\mathcal{A}(M)$. $\mathscr{C}_{MR}$ is given by the following diagram for all i, j. Thus the limit is Mat_{∞}.

$\mathscr{C}_{MR}$

$$
\begin{tikzcd}
 & \mathcal{A}(M) \arrow[dl] \arrow[d, "T"] \arrow[dr] & \\
Mat_{N_i} & Mat_{\infty} \arrow[l, "T_{i\infty}"'] \arrow[r, "T_{j\infty}"'] & Mat_{N_j} \arrow[ll, bend left, "T_{ij}"]
\end{tikzcd}
$$

Theorem

$\mathscr{Q}_{MR} = (\mathscr{C}_{MR}, J^{\bullet}_{MR}, F^{\bullet}_{MR}, \chi)$ is a quantization category for Matrix regularization.

Equivalence of Categories

Quantization categories for Deformation quantization and Pre-quantization can be constructed in analogy with $\mathscr{Q}_{MR}$. We discuss these quantization categories relationships.

Definition (Equivalent Categories)

Recall that the natural transformation $\alpha : F \to G$.

$$\begin{tikzcd} F(X) \arrow[r, "\alpha(X)"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"'] \\ F(Y) \arrow[r, "\alpha(Y)"'] & G(Y) \end{tikzcd}$$

If all $\alpha(\cdot)$ are isomorphisms, a natural transformation α is called natural isomorphism. Let $F_1 : \mathscr{A} \to \mathscr{B}$ and $F_2 : \mathscr{B} \to \mathscr{A}$ be functors. If natural transfromations $\alpha_1 : F_2 \circ F_1 \to I_{\mathscr{A}}$ and $\alpha_2 : F_1 \circ F_2 \to I_{\mathscr{B}}$ are isomorphisms, then $\mathscr{A}$ and $\mathscr{B}$ are called equivalent categories.

Theorem

Let $\mathscr{Q}_{MR}$ and $\mathscr{Q}_{DQ}$ be a quantization category for Matrix regularization and Deformation quantization, respectively. For the limit M_∞, if $\mathcal{A}(M) \simeq M_\infty$ then $\mathscr{Q}_{MR}$ and $\mathscr{Q}_{DQ}$ are equivalent categories.

Theorem

Let $\mathscr{Q}_{PQ}$ be a quantization category for Pre-quantization. Under some conditions, $\mathscr{Q}_{PQ}$ and $\mathscr{Q}_{MR}$ and $\mathscr{Q}_{DQ}$ are equivalent categories.

That is,

$$\mathscr{Q}_{PQ} \simeq \mathscr{Q}_{MR} \simeq \mathscr{Q}_{DQ}.$$

under some conditions.

Conclusions and Discussions

- We define the category $\mathscr{Q}$ as a generalization of some quantizations.
- We show the equivalence of sevaral quantization.
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Thank you for listening.

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