L_∞ and ExFT $E_{7(7)}$

[1807.06028, YC, Codina, Marques]

Yago Cagnacci[1]

[1]Instituto de Astronomía y Física del Espacio-University of Buenos Aires

Dualities and Generalized Geometries
Corfu, 14th September 2018

Outline

Introduction

L_∞ algebras have first appeared in different contexts:

- Higher spin gauge theories [Berends, Burgers, van Dam, 1985]
- Closed String Field Theory [Zwiebach, 1993]
- Mathematics: Strong Homotopy algebras [Lada, Stasheff, 1993]

Introduction

L_∞ algebras have first appeared in different contexts:

- Higher spin gauge theories [Berends, Burgers, van Dam, 1985]
- Closed String Field Theory [Zwiebach, 1993]
- Mathematics: Strong Homotopy algebras [Lada, Stasheff, 1993]

More recently:

- Field theories [Hohm, Zwiebach, 2017]: YM, DFT, ...
- $\mathcal{W}$ algebras, Quantum L_∞ algebras [Bluemenhagen, Fuchs, Traube, 2017]
- Gauge algebras [Hohm, Kupriyanov, Lüst, Traube, 2017]
- Seiberg-Witten maps and L_∞ quasi-ismorphisms [Blumenhagen, Brinkmann, Kupriyanov, Traube, 2018]
- Extended geometry [Cederwall, Palmkvist, 2018]
- Tensor hierarchies [Arvanitakis, 2018]

Outline

L_∞ introduction - Algebraic structure

Graded vector space: $X = \bigoplus_n X_n \,, \quad n \in \mathbb{Z}$.

Subspaces X_n of degree n.

Multilinear products: $\ell_k : X^{\otimes k} \to X$,

with degree:
$\deg(\ell_k(x_1, x_2, ..., x_k)) = k - 2 + \sum_{i=1}^{k} \deg(x_i)$.

Graded commutative:

$$\ell_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \; = \; (-1)^{\sigma} \epsilon(\sigma; x)\, \ell_k(x_1, \ldots \,, x_k) \;,$$

Koszul sign: $x_1 \wedge \ldots \wedge x_k = \epsilon(\sigma; x) \; x_{\sigma(1)} \wedge \ldots \wedge \; x_{\sigma(k)}$,

$$x_i \wedge x_j \; = \; (-1)^{x_i \, x_j} \, x_j \wedge x_i \,, \quad \forall i, j \;.$$

Example: $\ell_2(x_1, x_2) \; = \; (-1)^{1 + x_1 x_2} \, \ell_2(x_2, x_1)$,

L_∞ introduction - Algebraic structure

L_∞ relations:

n=1: $\ell_1(\ell_1(x)) \;=\; 0\;.$

Nilpotent product

n=2: $\ell_1(\ell_2(x_1,x_2)) \;=\; \ell_2(\ell_1(x_1),x_2)+(-1)^{x_1}\ell_2(x_1,\ell_1(x_2))\;.$

Leibniz rule

n=3:
$$\begin{aligned}0 = {}& \ell_2(\ell_2(x_1,x_2),x_3)+(-1)^{(x_1+x_2)x_3}\ell_2(\ell_2(x_3,x_1),x_2)\\ &+(-1)^{(x_2+x_3)x_1}\ell_2(\ell_2(x_2,x_3),x_1)\\ &+\ell_1(\ell_3(x_1,x_2,x_3))+\ell_3(\ell_1(x_1),x_2,x_3)\\ &+(-1)^{x_1}\ell_3(x_1,\ell_1(x_2),x_3)\\ &+(-1)^{x_1+x_2}\ell_3(x_1,x_2,\ell_1(x_3))\;.\end{aligned}$$

"Generalised Jacobi" identity

n=4: ...

L_∞ introduction - Field Theory

...	X_0	X_{-1}	X_{-2}	...
...	Gauge Parameters ζ	Fields Ψ	EOM $\mathcal{F}$	...

Graded subspaces and the elements they contain.

L_∞ introduction - Field Theory

...	X_0	X_{-1}	X_{-2}	...
...	Gauge Parameters ζ	Fields Ψ	EOM $\mathcal{F}$	...

Graded subspaces and the elements they contain.

Gauge transformations:

$$\delta_\zeta \Psi = \ell_1(\zeta) + \ell_2(\zeta, \Psi) - \frac{1}{2}\ell_3(\zeta, \Psi^2) - \dots ,$$

define the brackets $\ell_{n+1}(\zeta, \Psi^n)$

Ex: YM

$\delta_\zeta A_\mu = \partial_\mu \zeta + [\zeta, A_\mu]$

L_∞ introduction - Field Theory

...	X_0	X_{-1}	X_{-2}	...
...	Gauge Parameters ζ	Fields Ψ	EOM $\mathcal{F}$	...

Graded subspaces and the elements they contain.

Gauge transformations:

$$\delta_\zeta \Psi = \ell_1(\zeta) + \ell_2(\zeta, \Psi) - \frac{1}{2}\ell_3(\zeta, \Psi^2) - \dots ,$$

define the brackets $\ell_{n+1}(\zeta, \Psi^n)$

Ex: YM

$\delta_\zeta A_\mu = \partial_\mu \zeta + [\zeta, A_\mu]$

EOM:

$$\mathcal{F}(\Psi) = \ \ell_1(\Psi) - \tfrac{1}{2}\ell_2(\Psi^2) - \tfrac{1}{3!}\ell_3(\Psi^3) + \tfrac{1}{4!}\ell_4(\Psi^4) + \cdots$$

define the brackets $\ell_n(\Psi^n)$

L_∞ introduction - Field Theory

Gauge algebra

$$[\delta_{\zeta_1}, \delta_{\zeta_2}]\Psi = \delta_{-\mathbf{C}(\zeta_1,\zeta_2,\boldsymbol{\Psi})}\Psi + \delta^T_{\zeta_1,\zeta_2}\ \Psi$$

with

$$\mathbf{C}(\zeta_1, \zeta_2, \boldsymbol{\Psi}) \equiv \sum_{n\geq 0} \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\ \ell_{n+2}(\zeta_1, \zeta_2, \Psi^n)\ ,$$

$$\delta^T_{\zeta_1,\zeta_2}\Psi\ \equiv \sum_{n\geq 0} \frac{1}{n!}(-1)^{\frac{(n-2)(n-3)}{2}}\ \ell_{n+3}(\zeta_1, \zeta_2, \mathcal{F}, \Psi^n)\ .$$

Simple case: $\quad [\delta_{\zeta_1}, \delta_{\zeta_2}]\Psi = \delta_{-\ell_2(\zeta_1,\zeta_2)}\Psi$

L_∞ - Gauge structure

[H,K,L,T 1709.10004]: sufficient conditions for a gauge structure to fit within L_∞

- algebra $(V, [\cdot,\cdot])$,
- $\mathrm{Jac}(v_1, v_2, v_3) = \zeta_{trivial}$
- $[\zeta_{trivial}, V] = \zeta'_{trivial}$,

$$X_2 \xrightarrow{\ell_1 = \iota} X_1 \xrightarrow{\ell_1 = \mathcal{D}} X_0 \ ,$$

$$X_0 = V, \qquad X_1 = U, \qquad X_2 = \mathrm{Ker}(\mathcal{D}),$$

Linear map $\mathcal{D} : U \to V$ and $\zeta_{trivial} \in \mathrm{Im}(\mathcal{D})$

L_∞ - Gauge structure + Fields

One can extend this result to include fields (but not their dynamics)

$$X_2 \xrightarrow{\ell_1=\iota} X_1 \xrightarrow{\ell_1=\mathcal{D}} X_0 \xrightarrow{\ell_1=f} X_{-1}$$

$$X_{-1} = \text{fields}, \qquad X_0 = V, \qquad X_1 = U, \qquad X_2 = \mathrm{Ker}(\mathcal{D}),$$

with $f : X_0 \to X_{-1}$.

$$\delta_\zeta \Psi = \underbrace{\ell_1(\zeta)}_{f(\zeta)} + \ell_2(\zeta, \Psi) - \frac{1}{2}\ell_3(\zeta, \Psi^2) - \ldots \ ,$$

Outline

ExFT and Tensor Hierarchies

ExFT manifest the duality symmetries *prior* to compactification.

ExFT:

- Extended generalized spacetime
  - External space-time
  - Internal space-time: coordinates transforming in a fundamental representation of the duality group

ExFT and Tensor Hierarchies

ExFT manifest the duality symmetries *prior* to compactification.

ExFT:

- Extended generalized spacetime
  - External space-time
  - Internal space-time: coordinates transforming in a fundamental representation of the duality group
- "New" differential geometry in the internal space:
  - Generalized Lie derivative $\widehat{\mathcal{L}}_{\Lambda}$
  - E-bracket
  - 'section constraint'

ExFT and Tensor Hierarchy

External derivatives ∂_μ of tensor are not covariant wrt $\widehat{\mathcal{L}}$
$\Rightarrow$ cov. derivative: $\mathcal{D}_\mu = \partial_\mu - \widehat{\mathcal{L}}_{A_\mu}$

Naive field strength: $F_{\mu\nu} \; = \; 2\partial_{[\mu}A_{\nu]} - [A_\mu, A_\nu]_E.$

Gauge transf: $\delta F_{\mu\nu}$ Non-covariant!

Introduce field $B_{\mu\nu}$ and define $\mathcal{F}_{\mu\nu} = F_{\mu\nu} + \hat{\partial} B_{\mu\nu}$

Now: $\delta_\Lambda \mathcal{F}_{\mu\nu} = \widehat{\mathcal{L}}_\Lambda \mathcal{F}_{\mu\nu}$ It's covariant!

We start over:
Field strength for $B_{\mu\nu}$: $H_{\mu\nu\rho}$ non-covariant $\Rightarrow$ New field $C_{\mu\nu\rho}$...

Outline

ExFT and Tensor Hierarchies

EFT $E_{7(7)}$:

- Invariant tensors: symplectic metric Ω_{MN}, generators $(t_\alpha)_M{}^N$
- S.C.: $\Omega^{MN}\,\partial_M \otimes \partial_N \cdots = 0$, $(t_\alpha)^{MN}\,\partial_M \otimes \partial_N \cdots = 0$ $(t^\alpha)^{MN}\,T_M\,T_N = (t^\alpha)^{MN}\,T_M\,\partial_N \cdots = \Omega^{MN}\,T_M \otimes T_N = \Omega^{MN}\,T_M\,\partial_N \cdots = 0$
- Fields of the (projected) hierarchy: A_μ^M, $B_{\mu\nu M}$
  $\mathcal{F}_{\mu\nu}{}^M = 2\partial_{[\mu}A_{\nu]}{}^M + [A_\mu, A_\nu]^M_{(E)} + \Omega^{MN}B_{\mu\nu N}$,
  $B_{\mu\nu M} = 12(t^\alpha)_M{}^N\partial_N\tilde{B}_{\mu\nu\alpha} + \frac{1}{2}\tilde{B}_{\mu\nu M}$
- We will only discuss the tensor hierarchy sector and ignore other fields such as the graviton and scalars.

ExFT and L_∞

Gauge parameters: $\widehat{\Lambda}^M, \quad \widehat{\Xi}_{\mu M}$

Gauge transformations:

$$\delta A_\mu{}^M = \mathcal{D}_\mu \widehat{\Lambda}^M + \Omega^{MN} \widehat{\Xi}_{\mu N}$$

$$\delta B_{\mu\nu M} = 2\mathcal{D}_{[\mu} \widetilde{\widehat{\widehat{\Xi}}}_{\nu]M} - 2\left\{\widehat{\Lambda}, \mathcal{F}_{\mu\nu}\right\}_M + 2\left\{A_{[\mu}, \delta A_{\nu]}\right\}_M$$

Gauge algebra: Field dependent

Jacobiator: not a trivial parameter

ExFT and L_∞

Gauge parameters: $\widehat{\Lambda}^M, \quad \widehat{\Xi}_{\mu M}$

Gauge transformations:

$$\delta A_\mu{}^M = \mathcal{D}_\mu \widehat{\Lambda}^M + \Omega^{MN} \widehat{\Xi}_{\mu N}$$

$$\delta B_{\mu\nu M} = 2\mathcal{D}_{[\mu} \widetilde{\widehat{\widehat{\Xi}}}_{\nu]M} - 2\left\{\widehat{\Lambda}, \mathcal{F}_{\mu\nu}\right\}_M + 2\left\{A_{[\mu}, \delta A_{\nu]}\right\}_M$$

Gauge algebra: Field dependent

Jacobiator: not a trivial parameter

$\Rightarrow$ Field redefinitions

Field independent gauge algebra

Jacobiator is a trivial parameter

ExFT and L_∞

We have an L_∞ structure for the gauge algebra plus fields (A_μ^M, $B_{\mu\nu M}$)

What about the dynamics?

We incorporate the products coming from the EOM'S

$$\mathcal{F}(\Psi) = \ \ell_1(\Psi) - \frac{1}{2}\ell_2(\Psi^2) - \frac{1}{3!}\ell_3(\Psi^3) + \frac{1}{4!}\ell_4(\Psi^4) + \cdots$$

The EOM's mix all the fields, not just (A_μ^M, $B_{\mu\nu M}$)

ExFT and L_∞

We have an L_∞ structure for the gauge algebra plus fields (A_μ^M, $B_{\mu\nu M}$)

What about the dynamics?

We incorporate the products coming from the EOM'S

$$\mathcal{F}(\Psi) = \ \ell_1(\Psi) - \tfrac{1}{2}\ell_2(\Psi^2) - \tfrac{1}{3!}\ell_3(\Psi^3) + \tfrac{1}{4!}\ell_4(\Psi^4) + \cdots$$

The EOM's mix all the fields, not just (A_μ^M, $B_{\mu\nu M}$)

The L_∞ relations do not work: $\underbrace{\ell_1(\underbrace{\ell_1(\zeta)}_{\in X_{-1}})}_{\in X_{-2}} \neq 0$

KK-DFT as a Tensor Hierarchy and L_∞

We need ALL fields and their gauge transformations:

Ex.: $\delta_\Lambda \mathcal{M}_{PQ} = \widehat{\mathcal{L}}_\Lambda \mathcal{M}_{PQ} = \underbrace{\widehat{\mathcal{L}}_\Lambda \overline{\mathcal{M}}_{PQ}}_{\ell_1(\Lambda)_{PQ}} + \underbrace{\widehat{\mathcal{L}}_\Lambda m_{PQ}}_{\ell_2(\Lambda, m)_{PQ}} + \ldots$

$\delta_\Lambda d = \ldots$

These modify the product $\ell_1(\zeta)$ and we get

$$\ell_1(\ell_1(\Lambda))^\mu{}_M = \widehat{\mathcal{L}}_\Lambda \left(\Delta A^\mu{}_M\right)_{bckg} = 0,$$

So we can have a L_∞ structure for the fields of the tensor hierarchy by modifying some products and using EOM for background fields

Outline

Summary

- We explicitly found the L_∞ structure of the $E_{7(7)}$ ExFT gauge algebra and included fields of the (projected) tensor hierarchy up to the 3 form.
- To take into account the dynamics it is necessary to include all fields.
- We also analyzed the gauged supergravities in 4d and find that there is an underlying L_∞ structures.
- It would be nice to "bootstrap" ExFT from L_∞. [Lust, Blumenhagen, Kupryanov,Brunner, Traube, Brinkmann]