Generalised contact geometry as reduced generalised complex geometry

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Motivation

Given a vector bundle $E \cong TM \oplus \wedge^{i_1} T^*M \oplus \cdots \oplus \wedge^{i_n} T^*M$, there is a natural Leibniz algebroid structure with a prescribed set of symmetries and twists, interpreted as splitting the bundle (Baraglia [arXiv:1101.0856]).

- The Leibniz algebroid (and associated L_∞) structure and symmetries can be described explicitly.
- Additional structure is sometimes present (e.g. $O(d,d)$ and $E_{d(d)}$)
- The additional structure should be compatible with the 'twisting symmetry' enlarging the notion of equivalence.
- Generalised contact structures were studied in the literature but the full set of twists were not used.

Overview

Generalised contact geometry is the odd-dimensional counterpart to Hitchin's generalised contact geometry. Based on a Courant algebroid structure on

$$E \cong TM \oplus \mathbb{R} \oplus \mathbb{R} \oplus T^*M.$$

- **Background on Generalised Contact structures**
  - Iglesias and Wade [arXiv:math/0404519]
  - Poon and Wade [arXiv:0912.5314]
  - Sekiya [arXiv:1212.6064]
  - Aldi and Grandini [arXiv:1312.7471]
- **Extra symmetries and twists**
  - K.W [arXiv:1708.09550]

Emphasis on symmetry and twists

Taking $E \cong TM \oplus \mathbb{R} \oplus \mathbb{R} \oplus T^*M$:

- Gauge transformations are given by

$$\mathrm{Diff}(M) \ltimes (\Omega^2_{\mathrm{cl}}(M), \Omega^1_{\mathrm{cl}}(M), \Omega^1_{\mathrm{cl}}(M)) \qquad \left(vs. \mathrm{Diff}(M) \ltimes \Omega^2_{\mathrm{cl}}(M) \right).$$

$$(B, b, a) \in (\Omega^2_{\mathrm{cl}}(M), \Omega^1_{\mathrm{cl}}(M), \Omega^1_{\mathrm{cl}}(M)).$$

- The twists are given by

$$(H_3, H_2, F) \in (\Omega^3(M), \Omega^2(M), \Omega^2(M)).$$

- The twists describe non-trivial bundles $E \cong TM \oplus \mathbb{R} \oplus \mathbb{R} \oplus T^*M$ and allow a description of non-coorientable structures.

Generalised Complex vs Generalised Contact

Generalised Contact structures can be viewed as an S^1-invariant reduction of generalised complex structures.

Generalised complex geometry	Generalised contact geometry
$TM \oplus T^*M$	$TM \oplus \mathbb{R} \oplus \mathbb{R} \oplus T^*M$
Pure spinor ρ	Mixed pair φ, ψ
Integrability $d_H\rho$	Integrability $d_{H_3,F,H_2}(\varphi,\psi)$
Generalised complex structure	Generalised contact structure
Generalised metric	Generalised metric
Generalised Kähler	Generalised coKähler
T-duality acts Gen. Comp.	T-duality acts Gen. Con.

Generalised Contact algebroid

The *Contact Courant algebroid* is described on the vector bundle $E \cong TM \oplus \mathbb{R} \oplus \mathbb{R} \oplus T^*M$. Given sections $V = (v, f, g, \xi) \in (\Gamma(TM), C^\infty(M), C^\infty(M), \Gamma(T^*M))$:

$$
\begin{aligned}
V_1 \circ_{H_3,H_2,F} V_2 = \Big(&[v_1, v_2], v_1(f_2) - v_2(f_1) - \iota_{v_1}\iota_{v_2}F, v_1(g_2) - v_2(g_1) - \iota_{v_1}\iota_{v_2}H_2, \\
&\mathcal{L}_{v_1}\xi_2 - \iota_{v_2}d\xi_1 - \iota_{v_1}\iota_{v_2}H_3 + g_2 df_1 + f_2 dg_1 \\
&+ f_1\iota_{v_2}H_2 - f_2\iota_{v_1}H_2 + g_1\iota_{v_2}F - g_2\iota_{v_1}F \Big);
\end{aligned}
$$

$$
\begin{aligned}
\langle V_1, V_2 \rangle =& \tfrac{1}{2}(\iota_{v_1}\xi_2 + \iota_{v_2}\xi_1 + f_1 g_2 + g_1 f_2); \\
\rho(V) =& \rho((v, f, g, \xi)) = v.
\end{aligned}
$$

The Leibniz identity for $\circ_{H_3,H_2,F}$ gives

$$
dH_3 + H_2 \wedge F = 0, \quad dH_2 = 0, \quad dF = 0.
$$

Sekiya quadruple

Definition

A *Sekiya quadruple* on an odd-dimensional manifold M is given by the quadruple $(\Phi, e_1, e_2, \lambda) \in (\mathrm{End}(\mathbb{T}M), \Gamma(\mathbb{T}M), \Gamma(\mathbb{T}M), C^\infty(M))$, satisfying the following conditions (where $\mathbb{T}M = TM \oplus T^*M$):

$$\begin{aligned}
\langle e_1, e_1 \rangle = 0 = \langle e_2, e_2 \rangle,\ \ \langle e_1, e_2 \rangle &= \tfrac{1}{2};\\
\Phi^* &= -\Phi;\\
\Phi(e_1) = \lambda e_1,\ \ \Phi(e_2) &= -\lambda e_2;\\
\Phi^2(e) &= -e + 2(1+\lambda^2)(\langle e, e_2 \rangle e_1 + \langle e, e_1 \rangle e_2),\ \ \text{for } e \in \Gamma(\mathbb{T}M).
\end{aligned}$$

Sekiya quadruples can be associated to generalised almost complex structures on $M' = M \times \mathbb{R}$ (or $M' = M \times S^1$):

$$\mathbb{J}_{\mathrm{inv}} = \begin{pmatrix} \Phi & \mu e_1 & \mu e_2 \\ -2\mu\langle e_2, \cdot \rangle & -\lambda & 0 \\ -2\mu\langle e_1, \cdot \rangle & 0 & \lambda \end{pmatrix}, \quad (\mathbb{J}^* = -\mathbb{J},\ \mathbb{J}^2 = -1).$$

Generalised Contact structures

Definition

Let M be an odd-dimensional manifold of dimension m. A *generalised almost contact structure* is a quadruple (L, e_1, e_2, λ), where $L \subset \mathbb{T}M \otimes \mathbb{C}$ is a maximal isotropic subspace $\dim_{\mathbb{R}}(L) = m - 1$, and $e_1, e_2 \in \Gamma(\mathbb{T}M)$, satisfy $\langle e_1, e_1 \rangle = 0, \quad \langle e_2, e_2 \rangle = 0, \quad \langle e_1, e_2 \rangle = \frac{1}{2}$.

- Identifying $+i$-eigenbundle of $\Phi \in \mathrm{End}(\mathbb{T}M)$ with L we have a correspondence between Sekiya quadruples and generalised contact structures.
- A (H_3, H_2, F)-generalised contact structure is an almost generalised contact structure where the $+i$-eigenbundle of $\mathbb{J}_{\mathrm{inv}}$ is involutive with respect to $\circ_{H_3, H_2, F}$.

Mixed pairs

Definition

A *mixed pair* $(\varphi, \psi, e_1, e_2)$ consists of two differential forms $\varphi, \psi \in \Gamma(\Omega^{\mathrm{ev/od}}(M) \otimes \mathbb{C})$ and a choice of two sections $e_1, e_2 \in \Gamma(\mathbb{T}M)$ satisfying

$$(\varphi, \bar{\varphi})_{m-1} \neq 0, \quad (\psi, \bar{\psi})_{m-1} \neq 0, \quad ((\varphi, \psi), (\bar{\varphi}, \bar{\psi}))_{m-1} \neq 0,$$
$$e_1 \cdot \psi = 0, \quad \mu e_1 \cdot \varphi = (1 + i\lambda)\psi, \quad e_2 \cdot \varphi = 0, \quad \mu e_2 \cdot \psi = (1 - i\lambda)\varphi,$$

where $\mu = \sqrt{1 + \lambda^2}$, $\lambda \in C^\infty(M)$.

$$((\varphi_1, \psi_1), (\varphi_2, \psi_2))_{m-1} := (-1)^{|\varphi_1|}(\alpha(\varphi_1) \wedge \psi_2)_{m-1} + (-1)^{|\psi_1|}(\alpha(\psi_1) \wedge \varphi_2)_{m-1},$$

where $\alpha(dx^1 \otimes dx^2 \otimes \cdots \otimes dx^k) = dx^k \otimes dx^{k-1} \otimes \cdots \otimes dx^1$, $m = \dim(M)$, $|\varphi| = k$ for $\varphi \in \Omega^k(M)$, and $(\cdot\ )_{m-1}$ is the projection to $\Omega^{m-1}(M)$.

Integrability

The integrability of a generalised contact structure is encoded in the mixed pair description by the requirement:

$$d_{H_3,H_2,F}(\varphi,\psi) = V \cdot (\varphi,\psi),$$

where

$$\begin{aligned} d_{H_3,H_2,F}(\varphi,\psi) :=& (d\varphi + H_3 \wedge \varphi + F \wedge \psi, H_2 \wedge \varphi - d\psi - H_3 \wedge \psi), \\ V \cdot (\varphi,\psi) =& (v, f, g, \xi) \cdot (\varphi,\psi) = (\iota_v \varphi + \xi \wedge \varphi + f\psi, g\varphi - \iota_v \psi - \xi \wedge \psi), \end{aligned}$$

and (H_3, H_2, F) satisfy

$$dH_3 + H_2 \wedge F = 0, \quad dH_2, \quad dF = 0.$$

Example: Symplectic structure

Let $\omega \in \Gamma(\wedge^2 T^*P)$ be a symplectic form, where $P = M \times S^1$ where S^1 has coordinate t, and $\dim(M) = m = 2n+1$. The symplectic form is S^1-invariant if it admits the decomposition

$$\omega = \theta + dt \wedge \eta \in \Omega^2(P).$$

$$0 \neq \omega^{n+1} = (\theta + dt \wedge \eta)^{n+1} = (n+1)\theta^n \wedge dt \wedge \eta \Rightarrow \eta \wedge \theta^n \neq 0.$$

There exists a Reeb vector field $R \in \Gamma(TM)$ such that $\iota_R \eta = 1$ and $\iota_R \theta = 0$. θ is non-degenerate on $\ker(\eta)$, and $\phi(v) := \iota_v \theta$ $(v \in \ker(\eta))$. The generalised complex structure $\mathbb{J}_\omega$ is reduced to a generalised contact structure $(\Phi, e_1, e_2, \lambda)$:

$$\mathbb{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \quad \Rightarrow \quad \Phi = \begin{pmatrix} 0 & -\theta^{-1} \\ \theta & 0 \end{pmatrix}, \quad e_1 = \eta, \quad e_2 = R, \quad \lambda = 0,$$

$$\rho_{\mathbb{J}_\omega} = e^{i\omega} \quad \Rightarrow \quad \varphi = e^{i\theta}, \quad \psi = \eta \wedge e^{i\theta},$$

where $\mathbb{J}_\omega \in \mathrm{End}(TP \oplus T^*P)$ and $\Phi \in \mathrm{End}(\ker(\eta) \oplus \mathrm{Ann}(R))$.

Example: Complex structure

Consider an almost complex structure $J \in \mathrm{End}(TP)$, on $P = M \times S^1$, and $\dim M = 2n+1$. Given local coordinates, x, for M, and t for S^1, the almost complex structure is S^1-invariant if there exists the decomposition

$$J^I_K \partial_I \otimes dx^K = (\phi^i_k)\partial_i \otimes dx^k + R^i \partial_i \otimes dt + \alpha_k \partial_t \otimes dx^k.$$

The conditions $J^* = -J$ and $J^2 = -1$ give

$$\iota_R \alpha = 1, \quad \Phi(R) = 0 = \phi^*(\alpha), \quad \phi^2(v) = -v + (\iota_v \alpha)R.$$

The generalised almost complex structure $\mathbb{J}_J$ reduces to a generalised almost contact structure $(\Phi, e_1, e_2, \lambda)$:

$$\mathbb{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix} \quad \Rightarrow \quad \Phi = \begin{pmatrix} \phi & 0 \\ 0 & -\phi^* \end{pmatrix}, \quad e_1 = \alpha, \quad e_2 = R, \quad \lambda = 0,$$

$$\rho_J = \Omega_J \quad \Rightarrow \quad \varphi = \Omega_\phi, \quad \psi = \alpha \wedge \Omega_\phi,$$

where $\Omega_J \in \Omega^{2n+2,0}$ is the decomposable top form giving the pure spinor describing $\mathbb{J}_J$, and $\alpha \wedge \Omega_\phi := \Omega_J$.

Symmetries

$$e^{(B,b,a)}V = \Big(v, f + 2\langle v, a\rangle, g + 2\langle v, b\rangle, \xi + \iota_v B - fb - ga - \langle v, a\rangle b - \langle v, b\rangle a\Big).$$

Compatibility with symmetries

$$\langle e^{(B,b,a)}V_1, e^{(B,b,a)}V_2\rangle = \langle V_1, V_2\rangle,$$
$$e^{(B,b,a)}V_1 \circ_{(0,0,0)} e^{(B,b,a)}V_2 = e^{(B,b,a)}(V_1 \circ_{H_3',H_2',F'} V_2),$$

where

$$H_3' = dB + \tfrac{1}{2}(da \wedge b + a \wedge db), \quad H_2' = db, \quad F' = da.$$

Composition of symmetries:

$$(B_2, b_2, a_2) \cdot (B_1, b_1, a_1) = (B_1 + B_2 - \tfrac{1}{2}(b_1 \wedge a_2 + a_1 \wedge b_2), b_1 + b_2, a_1 + a_2),$$

Splitting bundles

$$\begin{tikzcd} 0 \arrow[r] & T^*P \arrow[r, leftrightarrow, "r_a"] & E \arrow[r, "\pi_*"] \arrow[l, bend left, "\sigma^*"] & TP \arrow[r] \arrow[l, bend left, "\sigma"] & 0, \end{tikzcd}$$

Exact Courant algebroid with $H_3(X_1, X_2) = \sigma^*(\sigma(X_1) \circ_{\mathrm{Dorf}} \sigma(X_2))$, $X \in \Gamma(P)$.

$$\begin{tikzcd} S^1 \arrow[r] & P \arrow[d] \\ & M \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & P \times \mathbb{R} \arrow[r] & TP/\mathrm{U}(1) \arrow[r, "\pi_*"] \arrow[l, bend left, "s^*"] & TM \arrow[r] \arrow[l, bend left, "s"] & 0, \end{tikzcd}$$

Atiyah algebroid with $F(v_1, v_2) = s^*([s(v_1), s(v_2)])$, where $v \in \Gamma(TM)$.

Results

- Create examples based on non-coorientable structures (those not defined by a globally defined one-form) by considering $F \neq 0$. H_3, H_2, F are globally defined forms but B, b, a are not in general global forms: Gerbe structure
- Use reduction to define generalised coKähler and generalised coKähler–Einstein structures.
- T-duality maps a generalised coKähler(–Einstein) structure to another generalised coKähler(–Einstein) structure.