Neutrino mixing analysis based on singular values

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Physically admissible mixing matrices are contractions

Neutrino mixing in the Standard Model

$$\nu_\alpha^{(f)} = (U_{PMNS})_{\alpha i}\nu_i^{(m)}$$

Mixing matrix

$$U_{PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Experimental values of mixing parameters

$$\theta_{12} \in [31.38^\circ, 35.99^\circ], \quad \theta_{23} \in [38.4^\circ, 53.0^\circ],$$
$$\theta_{13} \in [7.99^\circ, 8.91^\circ], \quad \delta \in [0, 2\pi]$$

[www.nu-fit.org]

Full experimental data - interval matrix

$$U_{PMNS} \xrightarrow{experiments} V_{osc}$$

CP Invariant Case

$$\mathrm{V_{osc}} = \begin{pmatrix} 0.799 \div 0.845 & 0.514 \div 0.582 & 0.139 \div 0.155 \\ -0.538 \div -0.408 & 0.414 \div 0.624 & 0.615 \div 0.791 \\ 0.22 \div 0.402 & -0.73 \div -0.567 & 0.595 \div 0.776 \end{pmatrix}$$

$$V_{osc} \xrightarrow{?} BSM$$

Extended mixing - BSM models

Complete mixing

$$\begin{pmatrix} \nu^{(f)} \\ \hat{\nu}^{(f)} \end{pmatrix} = \begin{pmatrix} V_{osc} & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix} \begin{pmatrix} \nu^{(m)} \\ \hat{\nu}^{(m)} \end{pmatrix} \equiv U \begin{pmatrix} \nu^{(m)} \\ \hat{\nu}^{(m)} \end{pmatrix}$$

Observable part

$$\nu_\alpha^{(f)} = \underbrace{(V_{osc})_{\alpha i} \nu_i^{(m)}}_{\text{SM part}} + \underbrace{(V_{lh})_{\alpha j} \hat{\nu}_j^{(m)}}_{\text{BSM part}}$$

Standard approaches to deviation from unitarity

$$\not{U}_{PMNS} \equiv (1 + \eta) N$$

$$\not{U}_{PMNS} \equiv (I - \alpha) U$$

[Antusch, Biggio, Fernandez-Martinez, Gavela and Lopez-Pavon, 2006.]

[Fernandez-Martinez, Gavela, Lopez-Pavon, and Yasuda, 2007.]

[Z.-z. Xing. 2008, 2012]

Our approach: mixing matrix and singular values

Singular values σ_i of a given matrix A are positive square roots of the eigenvalues λ_i of the matrix $AA^\dagger$

$$\sigma_i(A) = \sqrt{\lambda_i(AA^\dagger)}$$

Properties:

- generalization of eigenvalues
- always positive
- stable under perturbations

Unitary matrices

$$UU^\dagger = I = diag(1, 1, ..., 1) \Longrightarrow \text{all singular values equal to 1}$$

Characterization of physical mixing matrices

$$\begin{pmatrix} V_{osc} & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix} \quad ?$$

Contraction

$$\|V_{osc}\| \leq 1$$

Contractions

$$\|A\| \leq 1$$

Operator norm (spectral norm)

$$\|A\| := \sup_{\|x\|=1} \|Ax\| = \sigma_{\max}(A)$$

Contractions as submatrices of the unitary matrix

$$\left\| \begin{pmatrix} V_{osc} & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix} \right\| = 1 \Longrightarrow \| V_{osc} \| \leq 1$$

Statistics of Contractions in V_{osc}

Experimental mixing matrix

$$\mathrm{V_{osc}} = \begin{pmatrix} 0.799 \div 0.845 & 0.514 \div 0.582 & 0.139 \div 0.155 \\ -0.538 \div -0.408 & 0.414 \div 0.624 & 0.615 \div 0.791 \\ 0.22 \div 0.402 & -0.73 \div -0.567 & 0.595 \div 0.776 \end{pmatrix}$$

Contractions: 4 %

Non-physical: 96%

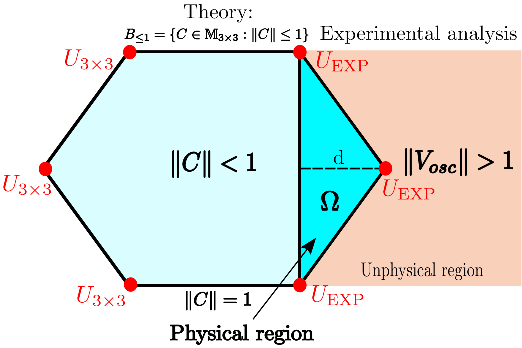
Contractions as a convex combination of unitary matrices

$$V = \sum_{i=1}^{m} \alpha_i U_i, \quad \alpha_i \geq 0 \text{ and } \sum_{\mathrm{i=1}}^{\mathrm{m}} \alpha_1 = 1$$

$$\|V\| = \|\sum_{i=1}^{m} \alpha_i U_i\| \leq \sum_{i=1}^{m} \alpha_i \|U_i\| = 1$$

Physical Region

$$\Omega := \mathrm{conv}(U_{PMNS}) = \{\sum_{i=1}^{m} \alpha_i U_i \mid U_i \in U(3), \alpha_1, ..., \alpha_m \geq 0, \sum_{i=1}^{m} \alpha_i = 1,$$
$$\theta_{12}, \theta_{13}, \theta_{23} \text{ and } \delta \text{ given by experimental values}\}$$



Unitary dilation

Contractions

$$U_{PMNS} \to V_{osc} \xrightarrow{\text{contractions}} \Omega$$

BSM?

$$V \in \Omega \xrightarrow{\text{dilation}} \begin{pmatrix} V & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix} \equiv U \to UU^\dagger = I$$

CS decomposition

$$U \equiv \begin{pmatrix} V & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix} = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} \left(\begin{array}{c|cc} C & -S & 0 \\ \hline S & C & 0 \\ 0 & 0 & I_{m-n} \end{array}\right) \begin{pmatrix} Q_1^\dagger & 0 \\ 0 & Q_2^\dagger \end{pmatrix}$$

where $C \geq 0$ and $S \geq 0$ are diagonal matrices satisfying $C^2 + S^2 = I_n$
$W_1, Q_1 \in M_{n\times n}$ and $W_2, Q_2 \in M_{m\times m}$ are unitary matrices

Unitary dilation: an example

As an illustration let us take two U_{PMNS} matrices

$$U_1 :\ \theta_{12} = 31.38^\circ, \theta_{23} = 38.4^\circ, \theta_{13} = 7.99^\circ,$$
$$U_2 :\ \theta_{12} = 35.99^\circ, \theta_{23} = 52.8^\circ, \theta_{13} = 8.90^\circ,$$

and let us construct a contraction as

$$V = \frac{1}{2}U_1 + \frac{1}{2}U_2,$$

The set of singular values

$$\sigma_1(V) = 1,\ \boldsymbol{\sigma_2(V) = 0.991},\ \boldsymbol{\sigma_3(V) = 0.991}$$

for which we get the following unitary dilation

$$U = \left(\begin{array}{ccc|cc} 0.822411 & 0.548133 & 0.146854 & 0.0169583 & -0.0368511 \\ -0.468394 & 0.520442 & 0.70103 & -0.133845 & 0.0197681 \\ 0.311417 & -0.643236 & 0.686702 & 0.0250273 & 0.130689 \\ \hline -0.0524981 & 0.122242 & -0.0336064 & 0.599485 & 0.788536 \\ -0.0671638 & 0.00403263 & 0.119588 & 0.788536 & -0.599485 \end{array}\right)$$

Alpha parametrization meets singular values

[poster by K. Porwit at Neutrino 2018: https://doi.org/10.5281/zenodo.1300485]

$$\not{U}_{PMNS} \equiv (I - \alpha)U$$

$$\alpha = \begin{pmatrix} \alpha_{ee} & 0 & 0 \\ \alpha_{\mu e} & \alpha_{\mu\mu} & 0 \\ \alpha_{\tau e} & \alpha_{\tau\mu} & \alpha_{\tau\tau} \end{pmatrix}$$

m	$m^2 > EW$	$\Delta m^2 \gtrsim 100eV^2$	$\Delta m^2 \sim 0.1 - 1eV^2$
α_{ee}	$1.3 \cdot 10^{-3}$	$2.4 \cdot 10^{-2}$	$1.0 \cdot 10^{-2}$
$\alpha_{\mu\mu}$	$2.2 \cdot 10^{-4}$	$2.2 \cdot 10^{-2}$	$1.4 \cdot 10^{-2}$
$\alpha_{\tau\tau}$	$2.8 \cdot 10^{-3}$	$1.0 \cdot 10^{-1}$	$1.0 \cdot 10^{-1}$
$\|\alpha_{\mu e}\|$	$6.8 \cdot 10^{-4}$	$2.5 \cdot 10^{-2}$	$1.7 \cdot 10^{-2}$
$\|\alpha_{\tau e}\|$	$2.7 \cdot 10^{-3}$	$6.9 \cdot 10^{-2}$	$4.5 \cdot 10^{-2}$
$\|\alpha_{\tau\mu}\|$	$1.2 \cdot 10^{-3}$	$1.2 \cdot 10^{-2}$	$5.3 \cdot 10^{-2}$

[Blennow, Coloma, Fernandez-Martinez, Hernandez-Garcia, Lopez-Pavon, 2017]

The inverse way of thinking

3+1 scenario:

- $m > EW$: $\sigma(\not{U}_{PMNS}) = \{1, 1, 0.9917\}$
- $\Delta m^2 \gtrsim 100eV^2$: $\sigma(\not{U}_{PMNS}) = \{1, 1, 0.899\}$
- $\Delta m^2 \sim 0.1 - 1eV^2$: $\sigma(\not{U}_{PMNS}) = \{1, 1, 0.888\}$

[Li and Mathias, 2004]

Error estimation

Weyl's inequality

Each element with 0.001 accuracy

$\Downarrow$

0.003 error estimation

Summary

- V_{osc} includes a lot of space for BSM
- Matrix theory and convex geometry help to restrict it into physical space
- Singular values enrich studies beyond unitarity
- Contractions are natural to describe interplay between SM and BSM mixing theories especially to define physical region Ω
- Dilations allow for appropriate construction of complete unitary matrices
- Use of the inverse eigenvalue problem to confront 3+N scenarios with experimental data

Details in
arXiv:1708.09196

Backup slides

Quark Sector

Wolfenstein parametrization

$$s_{12} = \lambda, \quad s_{23} = A\lambda^2, \quad s_{13}e^{i\delta} = A\lambda^3(\rho + i\eta)$$

$$V_{CKM} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(\rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)$$

Experimental values

$$\lambda = 0.22506 \pm 0.00050, \quad A = 0.811 \pm 0.026,$$
$$\bar{\rho} = 0.124^{+0.019}_{-0.018}, \quad \bar{\eta} = 0.356 \pm 0.011,$$

where, $\bar{\rho} = \rho(1 - \lambda^2/2)$ and $\bar{\eta} = \eta(1 - \lambda^2/2)$

Distribution of contractions

All matrices within V_{CKM} are contractions with 2‰ accuracy

$$6\% \text{ of } \|\mathrm{V_{CKM}}\| = 1.002,$$
$$94\% \text{ of } \|\mathrm{V_{CKM}}\| = 1.001$$

$$0.961 \leq \|\mathrm{V_{osc}}\| \leq 1.178$$

Matrix norm

A matrix norm is a function $\|\cdot\|$ from the set of all complex (real matrices) into $\mathbb{R}$ that satisfies the following properties

$$\begin{aligned}
&\|A\| \geq 0 \text{ and } \|\mathrm{A}\| = 0 \Longleftrightarrow \mathrm{A} = 0, \\
&\|\alpha A\| = |\alpha|\|A\|, \alpha \in C, \\
&\|A+B\| \leq \|A\| + \|B\|, \\
&\|AB\| \leq \|A\|\|B\|
\end{aligned}$$

Examples of matrix norms

- spectral norm: $\|A\| = \max_{\|x\|_2=1} \|Ax\|_2 = \sigma_1(A)$
- Frobenius norm: $\|A\|_F = \sqrt{Tr(A^\dagger A)} = \sqrt{\sum_{i,j=1}^n |a_{ij}|^2} = \sqrt{\sum_{i=1}^n \sigma_i^2}$
- maximum absolute column sum norm:
  $\|A\|_1 = \max_{\|x\|_1=1} \|Ax\|_\infty = \max_j \sum_i |a_{ij}|$
- maximum absolute row sum norm:
  $\|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_i \sum_j |a_{ij}|$

Weyl's inequalities

- For eigenvalues:
  Let A and B be $n \times n$ Hermitian matrices. Then

$$\lambda_j(A+B) \leq \lambda_i(A) + \lambda_{j-i+1}(B) \text{ for i} \leq \text{j},$$
$$\lambda_j(A+B) \geq \lambda_i(A) + \lambda_{j-i+n}(B) \text{ for i} \geq \text{j}.$$

- For singular values:
  Let A and B be a $m \times n$ matrices and let $q = \min\{m, n\}$. Then

$$\sigma_j(A+B) \leq \sigma_i(A) + \sigma_{j-i+1}(B) \textit{ for } i \leq j$$

Algorithm

The following steps lead to a contraction settled by U_{PMNS} and then to its unitary dilation of a minimal dimension

1) Select a finite number of unitary matrices U_i, $i = 1, 2, ...m$, within experimentally allowed range of parameters θ_{13}, θ_{23} and δ.
2) Construct a contraction U_{11} as a convex combination of selected matrices U_i

$$V = \sum_{i=1}^{m} \alpha_i U_i, \quad \alpha_1, ..., \alpha_m \geq 0, \quad \sum_{i=1}^{m} \alpha_i = 1.$$

3) Find singular value decomposition of V, i.e.

$$V = W_1 \Sigma Q_1^\dagger$$

where W_1, Q_1 are unitary, Σ is diagonal, and determine number η of singular values strictly less than 1.
4) Use CS decomposition

$$U = \begin{pmatrix} V & V_{lh} \\ V_{hl} & V_{hh} \end{pmatrix} =$$

$$\begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} \left(\begin{array}{cc|c} I_r & 0 & 0 \\ 0 & C & -S \\ \hline 0 & S & C \end{array}\right) \begin{pmatrix} Q_1^\dagger & 0 \\ 0 & Q_2^\dagger \end{pmatrix}$$

to find the unitary dilation $U \in \mathbb{M}_{(3+\eta)\times(3+\eta)}$ of contraction U_{11}.